* EXCITATIONS OF A BEC

* Up to now we have mostly discussed the ground-state properties of condensates (with the exception of our discussion about the Bogoliubov spectrum for the uniform BECs). In this lecture we shall discuss the excitation of a condensate. The excitations carry actually some key information about the condensate (e.g. its stability) and hence are very important to understand.

* In the following we shall consider first the case of small-amplitude oscillations, where the changes in space and time of the order parameter ψ with respect to the stationary configuration are small.
We shall discuss first the general formalism of these oscillations, based on the quasi-classical GPE theory. A second-quantization formalism may be employed as well (we shall comment on that later in this lecture).

* Our starting point is hence the GPE:

$$i\hbar \frac{\partial}{\partial t}\psi(\vec{r},t) = \frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V_{ext}(\vec{r})\psi(\vec{r},t) + g|\psi(\vec{r},t)|^2\psi(\vec{r},t)$$

The small oscillations of the system ~~[illegible]~~ around equilibrium can be investigated by writing

$$\psi(\vec{r},t) = [\underbrace{\psi_0(\vec{r})}_{\text{stationary solution}} + \underbrace{\theta(\vec{r},t)}_{\text{perturbation}}]e^{-i\mu t/\hbar}$$

where θ is a small quantity for which we look for solutions of the form: $\theta(\vec{r},t) = \sum_i u_i(\vec{r})e^{-i\omega_i t} + v_i^*(\vec{r})e^{+i\omega_i^* t}$

where ω_i is the frequency of the oscillations.

Note: We choose this form following our discussion of p. 101 where we introduced the Bogoliubov transformation for the homogeneous case. Actually the previous expression is nothing but such a transformation again)

* We insert this ansatz into the GPE. Up to first order in θ

$$i\hbar \frac{\partial}{\partial t}\psi = i\hbar\left[\frac{\partial}{\partial t}\theta - i\mu(\psi_0+\theta)\right]e^{-i\mu t/\hbar}$$

$$= \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r})\right\}[\psi_0+\theta]e^{-i\mu t/\hbar}$$

$$+ g\left[2|\psi_0|^2\theta + \psi_0^2\theta^*\right]e^{-i\mu t/\hbar}$$

and then, using the time-independent GPE for ψ_0:

$$i\hbar\frac{\partial}{\partial t}\theta = \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu\right\}\theta + 2g|\psi_0|^2\theta + g\psi_0^2\theta^*$$

Since this equation doesn't mix terms with different i in the expansion of θ, we consider in the following the single-mode case $\theta = u(\vec{r})e^{-i\omega t} + v^*(\vec{r})e^{i\omega^* t}$

Then:

$$i\hbar\left[-i\omega u e^{-i\omega t} + i\omega^* v^* e^{i\omega^* t}\right]$$

$$= \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + 2g|\psi_0|^2\right\}\left[u e^{-i\omega t} + v^* e^{i\omega^* t}\right]$$

$$+ g\psi_0^2 u^* e^{i\omega^* t} + g\psi_0^2 v e^{-i\omega t}$$

grouping terms with $e^{-i\omega t}$ and $e^{i\omega^* t}$ we get

$$\hbar\omega\, u = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + 2g|\psi_0|^2\right\}u + g\psi_0^2 v$$

$$-\hbar\omega^* v^* = \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + 2g|\psi_0|^2\right\}v + g\psi_0^2 u^*$$

We may then write the so-called <u>Bogoliubov-de Gennes eqs.</u>

$$\boxed{\begin{aligned}\hbar\omega\, u &= \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + 2g|\psi_0|^2\right\}u + g\psi_0^2 v \\ -\hbar\omega\, v &= \left\{\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - \mu + 2g|\psi_0|^2\right\}v + g(\psi_0^*)^2 u\end{aligned}}$$

(BdG Eqs)

* These equations are certainly very important, and are the key for the analysis of small-amplitude oscillations in a condensate. Before we employ these equations for the particular case of harmonically trapped BECs, let's have a look to <u>some general features</u>:

1) Let's consider $V_{ext}(\vec{r})=0 \longrightarrow$ Uniform gas.

Then $\psi_0 = \sqrt{n}$, $\mu = gn$

For a uniform gas ~~momentum is~~ a good quantum number.

We the look for solutions $u(\vec{r}) = u e^{i\vec{k}\vec{r}}$ and $v(\vec{r}) = v e^{i\vec{k}\vec{r}}$.

The BdG eqs are then:

$$\left.\begin{aligned} \hbar\omega u &= \left(\frac{\hbar^2 k^2}{2m} + gn\right)u + gn v \\ -\hbar\omega v &= gn u + \left(\frac{\hbar^2 k^2}{2m} + gn\right) v \end{aligned}\right\} \quad \hbar\omega \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \frac{\hbar^2 k^2}{2m} + gn & gn \\ -gn & -\left(\frac{\hbar^2 k^2}{2m}\right) - gn \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

It's easy to see that ~~this~~ leads to

$$(\hbar\omega)^2 = \left(\frac{\hbar^2 k^2}{2m}\right)^2 + \left(\frac{gn}{m}\right)(\hbar^2 k^2) = \frac{\hbar^2 k^2}{2m}\left[\frac{\hbar^2 k^2}{2m} + 2gn\right]$$

i.e. we recover the Bogoliubov spectrum we already found using a 2^{nd} quantization formalism in p. 102).

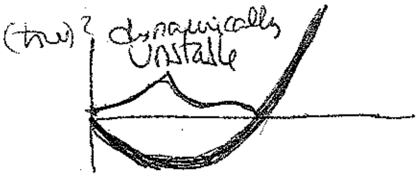
2) By taking suitable combinations of the BdG eqs we get

$$(\omega_i - \omega_i^*) \int d^3 r \left[|u_i|^2 - |v_i|^2\right] = 0$$

Hence unless $\int d^3 r\, |u_i|^2 = \int d^3 r\, |v_i|^2$ the BdG eqs only admit real-frequency solutions. Actually the occurrence of a complex frequency is associated with a <u>dynamical instability</u> of the system. This is clear because this means that the perturbation θ will grow like $\sim e^{\mathrm{Im}(\omega)t}$, i.e. θ will explode, and the perturbation won't be small any more.

• Dynamical instability occurs in many different situations. For example, let's consider a uniform solution, and let's switch to $a<0$. The Bogoliubov spectrum would then be:

$$(\hbar\omega)^2 = \frac{\hbar^2k^2}{2m}\left[\frac{\hbar^2k^2}{2m} - 2|g|n\right] \Rightarrow$$



Clearly there's a regime of low momenta $\hbar k$ for which the frequencies are imaginary, i.e. the uniform solution is clearly dynamically unstable.

This dynamical instability occurs at low momenta (this is why is called phonon-instability) and it's the responsible of the collapse of condensates of $a<0$ (p. 119).

In the following we just consider dynamically stable situations ($\omega \in \mathbb{R}$).

3) Another important property is that 2 solutions $\omega_i \neq \omega_j$ satisfy orthogonality

$$\boxed{\int (u_i^* u_j - v_i^* v_j)\, d^3r = 0}$$

(Note: this may be also found easily by proper combinations of the BdG eqs.)

4) Note that for each solution (u_i, v_i) with frequency ω_i, there's another solution (v_i^*, u_i^*) with frequency $-\omega_i$.

(Note: before in the homogeneous discussion we got actually 2 solutions $\hbar\omega = \pm\left[\frac{\hbar^2k^2}{2m}\left(\frac{\hbar^2k^2}{2m} + 2gn\right)\right]^{1/2}$)

It's important to note that both solutions describe exactly the same physical oscillation (as one can easily see from our original ansatz for the perturbation θ.

(Note: if ω can be imaginary, then there are two solutions ω and $-\omega^*$. One can also see clearly that $-\omega$ and ω^* are solutions, i.e. one always gets solutions that explode and solutions that contract.)

5) Note that the BdG eqs. always allow a solution $\omega = 0$, with $u = \alpha \psi_0$, $v = -\alpha \psi_0^*$, where α is a c-number. Then $\psi(\vec{r}) = \psi_0(\vec{r}) \cdot e^{i\Phi} e^{-i\mu t/\hbar}$ (after re-normalization)

Actually this means, as we know, that we can add any phase to the BEC and we get again a solution.

Remember that the condensate breaks the $U(1)$ symmetry of the Hamiltonian. The $\omega = 0$ mode (Goldstone mode) is just the consequence of this symmetry breaking.

6) It's interesting to have a look to the energy gain when excitations are added to the stationary solution.

let's consider the grand canonical energy:

$$E' = \int \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + V_{ext}(\vec{r}) |\psi|^2 + \frac{g}{2} |\psi|^4 - \mu |\psi|^2 \right) d^3r$$

We then take $\psi = (\psi_0 + \theta) e^{-i\mu t/\hbar}$, and Taylor expand. It's easy to see that the linear term in θ vanishes (the prefactor is zero due to the GPE). Hence only quadratic terms in θ survive. Without entering into all details, one can show that

$$E' = \underbrace{E_0'}_{\text{equilibrium value}} + \underbrace{E^{(2)}}_{\text{contribution of the excitations.}}$$

where: $$E^{(2)} = \sum_i \underbrace{\left[\int d^3r \left[|u_i|^2 - |v_i|^2 \right] \right]}_{\text{number of excitations carrying energy } \hbar\omega_i} \hbar\omega_i$$

(Note: the quantity $\int d^3r (|u_i|^2 - |v_i|^2)$ must hence be positive. This allows us to choose from all solutions of the BdG eqs the ones which are physically relevant (remember that for every ω there's always a $-\omega$, but one goes with (u,v) and the other with (v^*, u^*). So one gives $\int(|u|^2-|v|^2) > 0$, and the other < 0 (!!)))

* Note that the quantity $\int d^3r\, [|u_i|^2 - |v_i|^2]\, \hbar\omega_i$ must be positive for each mode in order to ensure stability. The occurrence of solutions for which the above quantity is negative is a direct signature of an energetic instability, also know as <u>thermodynamical instability</u>. It reveals that ψ_0 does not correspond to a minimal energy configuration (as we originally assumed).

This instability should not be confused to the previously discussed <u>dynamical instability</u> (associated to imaginary ω_i). The thermodynamical instability can only destabilize the system in the presence of dissipation, which drives the system towards configuration of lower energy.

7) It's also quite interesting to relate the amplitudes u and v of the excitation with the <u>fluctuation of the density and the phase of the condensate</u>.

Remember that ψ is a c-number with an amplitude and a phase (p. 108)

$$\psi = \sqrt{n}\, e^{iS}$$

Let $\left.\begin{array}{l} n = n_0 + \delta n \\ S = S_0 + \delta S \end{array}\right\}$ $\psi = \sqrt{n_0 + \delta n}\; e^{i(S_0 + \delta S)} \underset{\text{Taylor}}{\simeq} \sqrt{n_0}\, e^{iS_0}\left[1 + \frac{\delta n}{2n_0} + i\,\delta S\right]$

Then $\psi \cong \psi_0 + \frac{\psi_0}{2n_0}\,\delta n + i\,\psi_0\,\delta S.$

Let $\delta n = \delta n_+ e^{-i\omega t} + \delta n_- e^{i\omega t}$
$\quad\; \delta S = \delta S_+ e^{-i\omega t} + \delta S_- e^{i\omega t}$

in the same spirit as our discussion of the Bogoliubov eqs.

* Note however that n and S are real numbers, and so must be δn and δS. This forces that $\delta n_- = \delta n_+^*$ and $\delta S_- = \delta S_+^*$ (you can easily check that).

Hence

$$\psi \simeq \psi_0 + \left[\frac{\psi_0}{2n_0}\delta n_+ + i\delta S_+\right]e^{-i\omega t} + \left[\frac{\psi_0}{2n_0}\delta n_+^* + i\delta S_+^*\right]e^{i\omega t}$$

On the other hand

$$\psi \simeq \psi_0 + u\,e^{-i\omega t} + v^* e^{i\omega t}$$

Hence
$$u = \frac{\psi_0}{2n_0}\delta n_+ + i\,\delta S_+\psi_0$$
$$v = \frac{\psi_0}{2n_0}\delta n_+ - i\,\delta S_+\psi_0$$

Note hence that

$f_+ = u+v = \frac{\psi_0}{n_0}\delta n_+ \xrightarrow{f_+} \cancel{is}$ related with the <u>density fluctuations</u>

$f_- = u-v = 2i\psi_0\delta S_+ \longrightarrow f_-$ is related with the <u>phase fluctuations</u>

* This is a <u>particularly important</u> identification!

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* The solution of the BdG eqs provides the eigenfreqs of the ~~many body~~ small oscillation of the system. However the interpretation of these solutions in terms of the eigenstates of the many-body Hamiltonian requires a quantization procedure (this is quite easy and we will do it in a moment).

However the semi-classical analysis (without 2nd quantization) discussed above is many times useful, when for example one induces externally the excitation by macroscopically modulating the condensate.

* Quantization of the elementary excitations

* The discussion of the previous section may be easily quantized.
We start with the ψ^4 Hamiltonian

$$\hat{H} = \int d^3r\ \hat{\psi}^+(\vec{r}) \left\{ \frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) + \frac{g}{2}\hat{\psi}^+(\vec{r})\hat{\psi}(\vec{r}) \right\} \hat{\psi}(\vec{r})$$

We perturb the condensate solution:

$$\hat{\psi}(\vec{r},t) = \left[\psi_0(\vec{r}) + \hat{\theta}(\vec{r},t) \right] e^{-i\mu t/\hbar}$$

(Note that we have done here a Bogoliubov approximation, remember p. 107)

now the fluctuations retain their operator character

As before we take

$$\hat{\theta}(\vec{r},t) = \sum_i \left\{ u_i(\vec{r})\,\hat{b}_i\,e^{-i\omega_i t} + v_i^*(\vec{r})\,\hat{b}_i^+ e^{i\omega_i t} \right\}$$

where $u_i(\vec{r})$, $v_i(\vec{r})$ and ω_i obey the above discussed BdG equations.

* Actually using the orthogonality of the solutions of the BdG equations,
and imposing additionally the normalization

$$\int d^3r \left[|u_i(\vec{r})|^2 - |v_i(\vec{r})|^2 \right] = 1$$

we obtain the expression of the Hamiltonian as that of an
<u>independent gas of quasiparticles</u>

$$\hat{H} - \mu\hat{N} = E_0' + \hat{H}^{(2)}$$

where $\boxed{\hat{H}^{(2)} = \sum_i \hbar\omega_i \hat{b}_i^+ \hat{b}_i}$ (compare with p. 140)

* This completes the quantization of the excitations, which is quite crucial to understand the quantum depletion of the condensate
(Note: We already employed the 2nd quantized formalism in our discussion of the Bogoliubov spectrum (p. 107)).

* Collective oscillations of ~~an~~ harmonically-trapped BEC

* In p. 143 we derived the BdG eqs. that allow us to evaluate the small-amplitude excitations of a BEC in an external potential $V_{ext}(\vec{r})$.

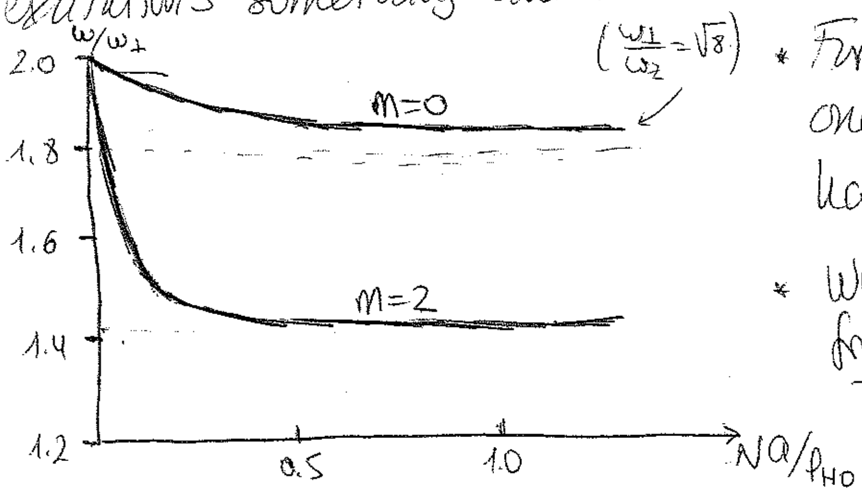
In this section we'll consider the experimentally relevant situation of an harmonic confinement

$$V_{ext}(\vec{r}) = \frac{1}{2} m\omega_x^2 x^2 + \frac{1}{2} m\omega_y^2 y^2 + \frac{1}{2} m\omega_z^2 z^2$$

* In the presence of an inhomogeneous confinement the momentum is obviously not a good quantum number. For spherical traps ($\omega_x = \omega_y = \omega_z$) the natural quantum numbers are the angular momentum (ℓ) and its third component (m). For axisymmetric traps (the usual case) m is still a good number (but not ℓ).

* Typically one has to solve the BdG eqs numerically (although we may obtain some analytical results in the Thomas-Fermi regime, as we will see in a moment).

For a disc-shape trap $\omega_x = \omega_y = \omega_\perp < \omega_z$ one gets for the lowest excitations something like this:



* For the non-interacting case ($a=0$) one recovers $\omega = 2\omega_\perp$, i.e. the harmonic oscillator result.

* When a grows there's a large deviation from the ideal result. In the Thomas-Fermi regime (p. 113) $Na/\ell_{HO} \gg 1$ the curves approach an asymptotic value. (we will come back to this point in a second).

* Special attention demands the so-called <u>dipole mode</u>.
This oscillation corresponds to the motion of the center of mass, which oscillates with the harmonic frequency (or frequencies if not spherically symmetric). Note that the interactions don't affect this mode because the center-of-mass motion is exactly decoupled from the internal degrees of freedom.

Actually, this fact is very useful for checking the trap harmonicity, as well as for determining to a high accuracy the trapping frequencies (Note: remember from our discussion of atomic traps (p. 32) that the relation between trap parameters and trap frequencies may be rather intricated, and hence the dipole mode is in this sense experimentally very very useful!!).

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* In the following we concentrate in the <u>Thomas-Fermi limit</u>. We recall that in this limit the condensate fulfills the hydrodynamic equations (p. 113)

$$\left.\begin{aligned} &\frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) = 0 \\ &m\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left[\frac{1}{2}mv^2 + V_{ext} + gn\right] = 0 \end{aligned}\right\} \text{ where } \begin{aligned} &n = |\psi|^2 \text{ and} \\ &\vec{v} = \frac{\hbar}{m}\vec{\nabla}S \quad \text{with } \psi = |\psi| e^{iS} \end{aligned}$$

* let $\delta n = n - n_0$ the change in the density profile with respect to the equilibrium configuration n_0 (which will be the corresponding Thomas-Fermi inverted paraboloid). After linearization one obtains (I left you that as an exercise) the time-dependent equation for δn:

$$\boxed{\frac{\partial^2}{\partial t^2}\delta n = \vec{\nabla}\cdot\left[c_s^2(\vec{r})\,\vec{\nabla}\delta n\right]}$$

where $c_s(\vec{r})$ is the <u>local sound velocity</u>, i.e. $mc_s^2(\vec{r}) = \mu - V_{ext}(\vec{r})$.

(Note: In deriving the previous equations we assume smooth density variations for both the ground state and the excitations, i.e. for both we can neglect the quantum pressure as we did in p. 115)

* Note that in the uniform case $\frac{\partial^2}{\partial t^2}\delta n = c_s^2\, \nabla^2\delta n$, i.e. we recover a wave-equation $\longrightarrow$ sound waves propagating with the sound velocity c_s
$\longrightarrow$ so the phonons, as we already know!

* Sound waves can also propagate in inhomogeneous media provided that their wave-number q is such that $qL \gg 1$ (with L the condensate size) and $q\xi \ll 1$, i.e. provided that the wave fits inside the BEC and that the quantum pressure is still negligible.

* Here we are not so interested about this short-wavelength excitations, but rather in excitations corresponding to a motion involving the whole BEC. These solutions have frequencies of the order of the oscillation frequencies and correspond to the modes presented in p. 150.

. Let's consider first a spherical trap ($\omega_x = \omega_y = \omega_z = \omega$). We can define the Thomas-Fermi radius $R \Rightarrow$ such that $R^2 = \frac{2\mu}{m\omega^2}$ (p. 117).

We may look for solutions of the form (in $0<r<R$)

$$\delta n(\vec{r}) = \underbrace{P_\ell^{2n_r}\left(\frac{r}{R}\right)} r^\ell Y_{\ell m}(\theta,\phi)$$

{Polynomials of degree $2n_r$ containing only even powers} the quantum number n_r provides the number of nodes of the radial wavefunction.

We may take the ansatz $\delta n_{mode}(\vec{r}) e^{i\omega_{mode} t}$ into the linearized equation of p. 151 and integrate over the spatial degrees of freedom to obtain the analytical result:

$$\boxed{\omega(n_r, \ell) = \omega\left[2n_r^2 + 2n_r\ell + 3n_r + \ell\right]^{1/2}}$$

. This result must be compared with the ideal gas in a harmonic trap

$$\omega(n_r, \ell) = \omega\left[2n_r + \ell\right]$$

The difference is spectacular!!

* The surface excitations are characterized by $n_r = 0$. Then

$$\left.\begin{array}{ll} \omega_\ell = \omega\sqrt{\ell} & \text{(Thomas-Fermi)} \\ \omega_\ell = \omega\ell & \text{(ideal gas)} \end{array}\right\}$$

Surface modes with $\ell = 2$ are actually related to vortex nucleation as we commented in p. 141

Note: the critical frequency for exciting this mode is given by $\Omega_{cr} = \frac{\epsilon_\ell}{\ell} = \frac{\omega}{\sqrt{\ell}} \to \omega/\sqrt{2}$ which is what has been observed in the vortex nucleation

* For the dipole mode ($n_r = 0, l = 1$) one gets that both the ideal gas and the Thomas-Fermi solutions give $\omega(0,1) = \omega$, as we already discussed in p. 151.

* For the compressional modes ($n_r \neq 0$) the lowest solution is the monopole oscillation (breathing mode), characterized by $n_r = 1, l = 0$

$\omega_{TF}(1,0) = \sqrt{5}\,\omega$

$\omega_{IDEAL}(1,0) = 2\omega$


⇐ breathing mode

* Note that for a fixed $N\frac{a}{l_{HO}}$, the predictions of the Thomas-Fermi approximation are worse when one increases n_r and l. This is clear because the larger the energy, the lower the associated wavelength, and the more important is the quantum pressure. Typically only excitations with energy well below μ are well described by the Thomas-Fermi approximation.

* Note that the Thomas-Fermi regime is quite remarkable. In spite of the fact that one has relatively strong interactions, the lowest-lying excitations don't depend any more on the interactions themselves!! This is somehow "magic" and deserves some discussion.

In the homogeneous case the dispersion law $\omega = cq$ obviously depends on interactions, since $c = \sqrt{gn/m}$. Suppose that we consider a box potential (of side R), then $q \sim$ multiples of π/R, but still ω is interaction dependent.

In an harmonic trap $\longrightarrow$ R = TF-radius $\overset{p.\,110}{\sim} \left(N\frac{a}{l_{HO}}\right)^{1/5}\left(\frac{\hbar}{m\omega}\right)^{1/2}$

and $c = \sqrt{\mu/m} = (Na/l_{HO})^{1/5}\left(\frac{\hbar\omega}{m}\right)^{1/2}$.

Then $\omega = cq \sim c/R \sim \omega$ and the interaction disappear!!

* Isn't it beautiful??

* The results can be generalized to axisymmetric traps, where only m is a good quantum number. Explicit results are possible in some particular cases.

  • Quadrupole solutions: $\delta n = r^2 Y_{2m}(\theta,\phi)$ satisfy the hydrodynamic equation of p. 151 for $m = \pm 2$ and $m = \pm 1$

$$\omega^2(\ell=2, m=\pm 2) = 2\omega_\perp^2$$

$$\omega^2(\ell=2, m=\pm 1) = \omega_\perp^2 + \omega_z^2$$

The quadrupole $\ell=2, m=0$ couples with the breathing mode $\ell=0, m=0$

$$\omega^2(m=0) = 2\omega_\perp^2 + \frac{3}{2}\omega_z^2 \mp \frac{1}{2}\sqrt{9\omega_z^4 - 16\omega_z^2\omega_\perp^2 + 16\omega_\perp^4}$$

* The modes $m=2$ and $m=0$ are depicted in p. 150 from the ideal value $(2\omega_\perp)$ to the Thomas-Fermi values.

* The theory may be extended to triaxially-deformed potentials, cigar-shaped traps, disc-shape traps, etc. We won't review all these cases here.

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* Large-amplitude oscillations. Self-similar solutions

* Up to now we just considered small-amplitude oscillations that we could treat with the linearized BdG eqs (p. 143) or the equivalent hydrodynamic equation of p. 151. Large-amplitude oscillation go out of the linear regime and must be treated in general by a simulation of the time-dependent GPE.

When the system is in the Thomas-Fermi regime, the time-dependent GPE reduces to the hydrodynamic equations (p. 113). These eqs. may be employed to investigate non-linear phenomena in a simplified way.

* For example, we may consider what happens if the trap frequencies are changed at $t=0$ from ω_{i0} into ω_i (with $i=x,y,z$).
One can easily prove that the hydrodynamic eqs. admit a class of analytical solutions having a self similar density profile:

$$n(\vec{r},t) = n_0(t)\left\{1 - \frac{x^2}{R_x(t)^2} - \frac{y^2}{R_y(t)^2} - \frac{z^2}{R_z(t)^2}\right\}$$

in the region where $n(\vec{r},t)>0$ and zero elsewhere. The velocity field is instead parametrized as:

$$\vec{v}(\vec{r},t) = \frac{1}{2}\vec{\nabla}\left\{\alpha_x(t)x^2 + \alpha_y(t)y^2 + \alpha_z(t)z^2\right\}$$

* The Thomas-Fermi radii scale as

$$R_i(t) = R_i\, b_i(t)$$

where $R_i^2 = \dfrac{2\mu}{m\omega_{i0}^2}$ and $b_i(t)$ are the scaling parameters, which fulfil $b_i(t=0)=1$.

* The parameter $n_0(t)$ is determined by normalization.

(Note: in principle one may add xy, xz, yz terms for both the density and the phase. This would be important in the description of rotating BECs, but we won't consider that here).

* From the continuity equation $\partial n/\partial t + \vec{\nabla}(n\vec{v})=0$, one gets the relation between α_i and b_i:

$$\boxed{\alpha_i = \dot{b}_i/b_i}$$

From the Euler equation $m\frac{\partial}{\partial t}\vec{v} + \vec{\nabla}\left(\frac{mv^2}{2} + V_{ext} + gn\right)=0$ we get the equation for the scaling parameters:

$$\boxed{\ddot{b}_i + \underbrace{\omega_i^2 b_i}_{\text{trap harmonic confinement}} - \underbrace{\frac{\omega_{0i}^2}{b_i b_x b_y b_z}}_{\text{atom-atom interactions}} = 0}$$

← These are 3 coupled eqs. for the scaling parameters. They may be employed also in the nonlinear regime.

* We may use these eqs. to recover some of the small-amplitude oscillations. Let $\omega_{0i} = \omega_i$, and $b_i = 1 + \epsilon_i$, $\epsilon_i \ll 1$ (small amplitude)

Then $\xrightarrow[\downarrow]{\text{Taylor}}$ $\ddot{\epsilon}_i + [3\epsilon_i + \epsilon_j + \epsilon_k]\omega_i^2 = 0 \qquad j \neq k \neq i$

Hence:

$$\frac{d^2}{dt^2}\begin{pmatrix}\epsilon_x \\ \epsilon_y \\ \epsilon_z\end{pmatrix} = -\begin{bmatrix} 3\omega_x^2 & \omega_x^2 & \omega_x^2 \\ \omega_y^2 & 3\omega_y^2 & \omega_y^2 \\ \omega_z^2 & \omega_z^2 & 3\omega_z^2 \end{bmatrix}\begin{pmatrix}\epsilon_x \\ \epsilon_y \\ \epsilon_z\end{pmatrix}$$

The eigenvalues of the matrix give the collective freqs. Ω_ν^2.

· E.g. for spherically symmetric traps one easily gets two degenerate modes with $\Omega = \sqrt{2}\,\omega$, and one mode with $\Omega = \sqrt{5}\,\omega$, which we can identify with $n_r = 0, l = 2$ (quadrupole) and $n_r = 1, l = 0$ (breathing) in our discussion of p. 152 and p. 153.

* Interestingly we may also employ the self-similar eqs. for the scaling parameters b_i to analyze a very relevant problem in experiments, namely the condensate expansion. In that case, we consider $\omega_i = 0$ after $t = 0$. Then the eqs. reduce to:

$$\ddot{b}_i = \frac{\omega_{0i}^2}{b_i b_x b_y b_z}$$

* let's consider the simplest case of an axially-symmetric trap $\omega_{0x} = \omega_{0y} = \omega_\perp$; $\omega_{0z} = \lambda\omega_\perp$

- Then $b_x = b_y = b_\perp$.
- We consider the dimensionless time $\tau = \omega_\perp t$. Then, the eqs. reduce to:

$$\begin{cases} \dfrac{d^2}{d\tau^2} b_\perp = \dfrac{1}{b_\perp^3 b_z} \\ \dfrac{d^2}{d\tau^2} b_z = \dfrac{\lambda^2}{b_\perp^2 b_z^2} \end{cases}$$

an interesting quantity to monitor is the aspect ratio, given by the ratio of the Thomas-Fermi radii:

$$\eta(t) = \frac{R_\perp(t)}{Z(t)} = \frac{R_\perp(0)}{Z(0)} \frac{b_\perp(\tau)}{b_z(\tau)} = \lambda \frac{b_\perp(\tau)}{b_z(\tau)}$$

This quantity shows a typical inversion during the time-of-flight (i.e. during the expansion), e.g. from $\eta(t=0) < 1$ one gets $\eta(t\to\infty) > 1$. This may be analytically recovered for $\lambda \ll 1$. In that case one may obtain:

$$b_\perp(\tau) = \sqrt{1+\tau^2}$$

$$b_z(\tau) = 1 + \lambda^2 \left\{ \tau \,\mathrm{atan}\,\tau - \ln\sqrt{1+\tau^2} \right\} \simeq 1 \quad \text{for not too large times.}$$

Note that for $\lambda \ll 1 \to \omega_z \ll \omega_\perp$, and we have that the BEC cloud expands very fast in x,y and very slow along z. The reason is pretty obvious. In the strongly confined directions (xy) the BEC wavefunction is strongly compressed (localized), the kinetic energy is hence very large (it's proportional to $\partial^2_{x,y}$). The opposite is true along z. This explains the very different expansion dynamics. (Note: however, the interactions play of course a role in the initial stages of the dynamics, when the density is still large. An ideal gas calculation won't explain the expansion of the BEC properly!)

Note that $\lim_{\tau\to\infty} \left(\frac{R_\perp}{Z}\right) = \frac{2}{\pi\lambda}$. Hence η goes from λ into $\frac{2}{\pi\lambda}$ during the time of flight, which is the above mentioned ~~[illegible]~~ inversion.

• Solitons

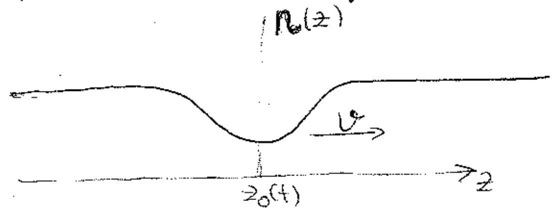
* We have already seen that due to the interparticle interactions the BEC physics is inherently non-linear. Indeed the GPE equation is a particular example of the non-linear Schrödinger equation, one of the paradigms of nonlinear physics, which is most notably encountered in non-linear optics (the GPE is mathematically equivalent to the equation describing a local Kerr medium with instantaneous response).

* The non-linearity leads to remarkable phenomena in condensates (with fascinating links to other fields of physics). Among them we should mention atomic four-wave mixing (3 clouds are prepared and non-linearity leads to a fourth one) and BEC collapse (or Bose-nova, induced by attractive interactions). Here we won't review these phenomena in all detail (for more information see the book of Pierre Meystre on Atom optics), but rather we'll concentrate in a particularly remarkable effect induced by the non-linearity, namely the existence of solitons.

* Solitary waves, or solitons, are localized disturbances which propagate without change of form. Soliton solutions exist for a number of non-linear equations (e.g. the Korteweg-de Vries equation describing the properties of shallow water waves), and they also exist as solutions of the (one-dimensional) GPE.

* Solitons preserve their form because the effects of non-linearity compensate those of dispersion, i.e. interaction energy and kinetic energy balance each other. It shouldn't be a surprise then that the typical size of the solitons (at least the non-moving ones) is of the order of the healing length ξ (remember that from p. 104 the healing length is the length associated with the interaction energy).

* Let's consider first a spatially uniform BEC with repulsive interactions. In this case the solitons correspond to a localized density modulation (actually a density depression) moving in the medium at constant velocity and without changing its shape. These solitons are called dark solitons.



We will consider in the following a one-dimensional case, i.e. the gas is assumed as being strongly-confined in the transversal directions (this demands that the interaction energy gn must be much smaller than the typical energy of the transversal confinement). We will discuss later what happens for higher dimensions.

* Hence the order parameter ψ depends only on the coordinate z, actually through the combination $z-vt$, where v is the velocity of the soliton depression.

* We start with the GPE

$$i\hbar \frac{\partial}{\partial t}\psi(z,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + g|\psi(z,t)|^2\right]\psi(z,t)$$

Let $\psi(z,t) = \sqrt{n}\, f(z,t)\, e^{-i\mu t}$ [$f(z,t)$ will characterize the soliton distortion]

Then:

$$i\hbar \cancel{\sqrt{n}\, e^{-i\mu t/\hbar}}\left[\frac{\partial f}{\partial t} - i\frac{\mu}{\hbar}f\right] = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}f + gn|f|^2 f\right\}\cancel{\sqrt{n}\, e^{-i\mu t/\hbar}}$$

Then $$i\hbar\frac{\partial f}{\partial t} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + gn|f|^2 - \mu\right\}f$$

Let $\eta = \dfrac{z-vt}{\xi}$, where ξ is the healing length. Since f must be a function only of η, then we can substitute

$$\frac{\partial f}{\partial t} = \frac{-v}{\xi}\frac{df}{d\eta}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{\xi^2}\frac{d^2 f}{d\eta^2}$$

* This equation coincides with the continuity equation if

$f_2 = \sqrt{2}\,U = \frac{v}{c_s}$ (this is easy to check, try it!)

We get then an equation for the real part f_1:

$$\sqrt{2}\,\frac{df_1}{d\eta} = 1 - \frac{v^2}{c_s^2} - f_1^2$$

This equation has an analytical solution of the form

$$f_1[\eta] = \sqrt{1 - v^2/c^2}\,\tanh\left[\frac{\eta}{\sqrt{2}}\sqrt{1 - v^2/c_s^2}\right]$$

* Re-introducing the initial physical variables we get the solution:

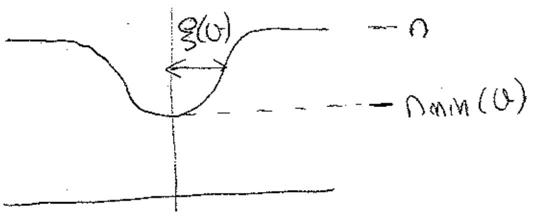
$$\psi(z,t) = \sqrt{n}\left\{ i\,\frac{v}{c_s} + \sqrt{1 - v^2/c^2}\,\tanh\left[\frac{(z - vt)}{\sqrt{2}\,\xi}\sqrt{1 - v^2/c_s^2}\right]\right\} e^{-i\mu t/\hbar}$$

The density profile is hence of the form:

$$n(z,t) = |\psi(z,t)|^2 = n\left\{1 - \left[1 - \frac{v^2}{c_s^2}\right]\mathrm{sech}^2\left[\frac{(z - vt)}{\sqrt{2}\,\xi}\sqrt{1 - \frac{v^2}{c_s^2}}\right]\right\}$$

This solution presents a minimal density $n_{min}^{(v)} = n\,\frac{v^2}{c_s^2}$ at the soliton center $z - vt = 0$.

The soliton width is given by $\xi(v) = \frac{\xi}{\sqrt{1 - v^2/c_s^2}}$



* Hence both the soliton depth and its width are related with each other, and with the velocity v.
* The faster is the soliton, the shallower it is and the wider it is as well. At $v = c_s$ the soliton disappears.

* The soliton phase (apart from the usual $i\mu t/\hbar$ part) is of the form:

$$\phi(z,v,t) = -\arctan\left\{\left(\frac{c_s^2}{v^2}-1\right)^{1/2}\tanh\left[\frac{z-vt}{\sqrt{2}\,\xi}\sqrt{1-v^2/c^2}\right]\right\}$$

One may easily see that for $z-vt \to -\infty$

$$\phi = -\arccos\frac{v}{c_s}$$

and for $z-vt \to +\infty$

$$\phi = +\arccos\frac{v}{c_s}$$

Hence the phase of the wave function undergoes a finite change

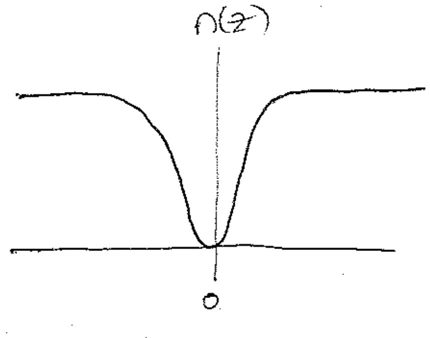
$$\boxed{\Delta\phi = 2\arccos\frac{v}{c_s}}$$

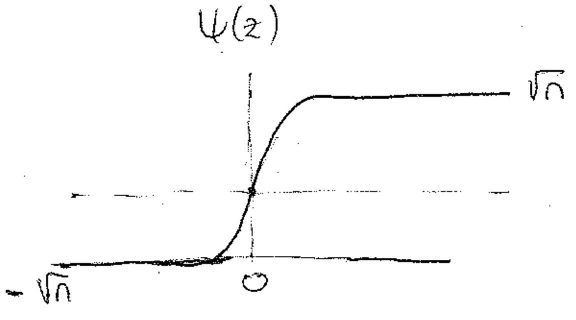
as z varies from $-\infty$ to $+\infty$

* Note that only stationary solitons ($v=0$) have a vanishing density in the center. This is why they are called "black" solitons, whereas for $v\neq 0$ the central density is not zero, and the solitons are called grey solitons. Note that for a black soliton

$$\psi(z,t) = \sqrt{n}\, e^{-i\mu t/\hbar}\tanh\left[\frac{z}{\sqrt{2}\,\xi}\right]$$

and hence the wavefunction (apart from the $e^{-i\mu t/\hbar}$) remains real and odd with a phase change $\Delta\phi = \pi$.





* It's rather interesting to have a look to the energy of the soliton. As we did for the case of vortices (p. 137) we may calculate the energy of the soliton per unit surface by taking the difference between the grand canonical energies in the presence and in the absence of the soliton. The resulting expression is:

recall that $\sqrt{2}\frac{df_1}{d\eta} = 1 - \frac{v^2}{c_s^2} - f_1^2$

$$\mathcal{E} = \int_{-\infty}^{\infty} \left[\frac{\hbar^2}{2m}\left|\frac{d}{dz}\psi\right|^2 + \frac{g}{2}\left[|\psi|^2 - n\right]\right] dz =$$

$$= \frac{4}{3}\hbar c_s n \left[1 - \frac{v^2}{c_s^2}\right]^{3/2} \overset{v \ll c_s}{\simeq} \frac{4}{3}\hbar c n - 2\hbar c_s n \frac{v^2}{c_s^2}$$

$$= \frac{4}{3}\hbar c n + \frac{1}{2}\left[\frac{-4\hbar n}{c_s}\right] v^2$$

Hence, for small $v \ll c_s$ the soliton behaves like a particle with negative mass $\boxed{m_S = -\frac{4\hbar n}{c_s}}$, in accordance with the fact that the density modulation actually corresponds to a "hole"

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* For the case of attractive interactions, the kinetic energy of the condensate can be balanced by the nonlinearity (which is now negative) yielding non-spreading wavepackets. These other solitonic solutions of the 1D GPE are called bright solitons. It's easy to check that the time-independent GPE is solved by the wave function:

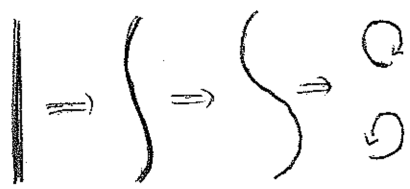
$$\psi(z) = \sqrt{n}\ \mathrm{sech}\left(\frac{z}{\sqrt{2}\xi}\right) \qquad \text{with } \xi = \frac{\hbar}{\sqrt{2m|g|n}}$$

This solution has a chemical potential $\mu = -\frac{1}{2}|g|n$, i.e. a negative (finite) chemical potential. Hence this wavepacket is self-bound (Note that $\mu < 0$ means that the wavepacket can't disperse, because in that case μ would go to zero)

* Let's just discuss some final remarks about solitons in BEC.

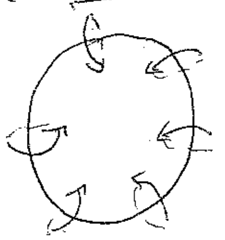
• The solitonic solutions discussed above are only stable in one-dimensional geometries. In higher dimensions they are unstable against perturbations along the transversal dimensions. This may be studied by taking the soliton solution ψ and perturb it using the Bogoliubov-de Gennes formalism discussed in p. 142. One then sees that transversal excitations with momenta $q < 1/\xi$ are dynamically unstable (recall the discussion of p. 145), i.e. wavelengths $\lambda > \xi$. One then needs to cut-off these wavelengths. This is done by a sufficiently large transversal harmonic confinement (say of frequency $\omega_\perp$) such that the associated oscillator length $l_{HO} < \xi$. This in turn implies $gn < \hbar\omega_\perp$, which is the 1D condition.

* For the case of dark solitons this instability is translated into a bending of the dark front, as in the figure, which for obvious reasons is called <u>snake-instability</u>.



In 2D (as in the figure) the soliton breaks into vortex-antivortex pairs, whereas in 3D it breaks into <u>vortex rings</u>.

<u>Note</u>: a <u>vortex ring</u> is given by a closed vortex line



* For the case of bright solitons the dynamical instability leads typically to collapse (Recall our discussion of p. 121).

* Note also that due to its negative mass the dark soliton velocity increases when its energy decreases. Hence dissipation (due to e.g. collisions with thermal excitations, which become possible due to the violation of integrability due to the presence of inhomogeneity both axially and transversally) leads to the acceleration of the soliton. When the dark soliton reaches $v \to c$ it dissappears into an emission of phonons.

* Note also that if the soliton is in a potential which varies significantly within the soliton width (e.g. if the soliton meets an obstacle) then it will emit sound in much the same way as a charged particle emits electromagnetic radiation when accelerated.

* With this we will finish our discussion on BEC solitons. There is much more about them (they are indeed a hot topic in BEC physics) but a complete review will delay us too much. For more information see especially the book of P. Meystre of Atom optics.