* BEC-BCS-CROSSOVER!

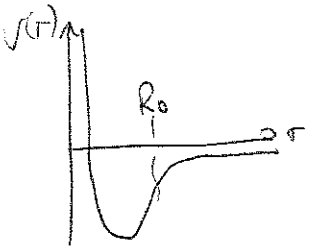
* TWO-BODY COLLISIONS IN SOME MORE DETAIL

* We have just seen that $T_c \ll T_F$. Even for $k_F a = -0.2$ one has $T_c \sim 10^{-4} T_F$. This value is actually typically much smaller than the oscillator energy. As such hence BCS seems extremely difficult (if not hopeless) to achieve.

However one may increase a by Feshbach resonances (p. 97) and set $T_c \sim T_F$, although for $k_F a > 1$ the BCS approach fails.

Before starting with the discussion of the Fermi gas at a Feshbach resonance, we should recall some ideas we saw already on interparticle interactions (p. 86) and learn some new concepts.

In the following we shall consider that the range R_0 of the interatomic potential fulfills $R_0 \ll \lambda_T = \sqrt{2\pi\hbar^2/(m k_B T)}$ and that $k_F R_0 \ll 1$. Under these conditions the main contribution comes from s-wave scattering between different particles.



As in our discussion of p. 86 the problem reduces to the solution of the Schrödinger equation for the relative motion. Proceeding as in p. 89 but keeping terms up to order k^2 (k = relative momentum) we obtain the s-wave scattering amplitude:

$$f_0(k) = \frac{-1}{1/a - k^2 R^*/2 + ik}$$

where R^* acts as the effective range of the interactions.

(Note: in p. 85 we kept only the zeroth order $\rightarrow f_0(k) = -a$ momentum-independent)

(Note II: R^* is typically of the order of R_0, but not always as e.g. close to the so-called narrow Feshbach resonances)

* At this point it's worthy to introduce some ideas

* Unitarity limit: when $|a| \to \infty$, and $k \ll |R^*|^{-1}$, then

$$f_0(k) = i/k \longrightarrow \text{independent of the interaction!}$$

This regime is extremely interesting, since being independent of the particular interaction it is universal! Results found in cold gases are hence applicable to other Fermi systems in e.g. astrophysics or nuclear physics. We will discuss some properties at unitarity in a moment.

* Dimers: For $a > 0$ ($a \gg R_0$) there's a bound state (a dimer) possible. This is a shallow dimer with binding energy

$$\epsilon_b = \frac{-\hbar^2}{2m_r a^2}$$

doesn't depend on the details of the potential. $m_r \equiv$ reduced mass $= \frac{m}{2}$ for identical atoms

See that when a grows, then ϵ_b gets smaller, and hence the dimer is less bound, i.e. bigger.

* Pseudopotential: This is an idea that we already introduced in p. 71. We consider an effective potential $V_{eff}(\vec{r})$ instead of the (more complicated) original potential $V(\vec{r})$.

- In many applications one typically introduces a contact pseudopotential of the form:

$$V_{eff}(\vec{r}) = g\,\delta(\vec{r})\,\frac{\partial}{\partial r}(r\cdot)$$ ← This means that one puts the wavefunction here

Note: the regularization $\frac{\partial}{\partial r}(r\cdot)$ was introduced by Huang and Yang in the 50's to cure the ultraviolet divergence arising from the vanishing range of the pseudopotential.)

As always $g = \frac{4\pi\hbar^2 a}{m}$ (or in general $\frac{2\pi\hbar^2 a}{m_r}$ for different atoms)

This potential has obviously $R_0 = 0$, and leads to

$$f_0(k) = \frac{-1}{1/a + ik}$$

For $a > 0$, it allows for a bound state with binding ϵ_b as before, and with wavefunction

$$\psi_b(r) = \frac{e^{-r/a}}{\sqrt{2\pi a}\, r}$$ (see the calculation in the next page)

Note that the typical width goes like a, and hence gets very large (as already mentioned) at a Feshbach resonance.

(Note: actually the effect of the pseudopotential in the Schrödinger equation may be mimicked by a proper boundary condition at $r=0$ as shown by Bethe and Peierls:

$$\left(\frac{d(r\psi)/dr}{r\psi}\right)_{r=0} = -\frac{1}{a} \qquad \left[\psi(r\to 0) \sim \frac{1}{r} - \frac{1}{a}\right]$$)

* Extra note: The bound state for $a>0$

* We want to solve the Schrödinger equation

$$E\psi = \frac{-\hbar^2}{2m}\nabla^2\psi + V(r)\psi$$

where $V(r) \rightarrow$ interatomic potential with s-wave scatt. length a.

We substitute $V(r)\psi \rightarrow \frac{4\pi\hbar^2 a}{m}\delta(\vec{r})\frac{\partial}{\partial r}(r\psi)$

This is equivalent to a Bethe-Peierls boundary condition

$$\left[\frac{\frac{d}{dr}(r\psi)}{r\psi}\right]_{r=0} = -\frac{1}{a}$$

So, we remove $V(r)$ and impose the Bethe-Peierls boundary condition.

let $\epsilon = \frac{2m|E|}{\hbar^2} \rightarrow \epsilon\psi = -\nabla^2\psi$

We look for radially symmetric solutions $\rightarrow \epsilon\psi = -\frac{d^2}{dr^2}\psi - \frac{2}{r}\frac{d}{dr}\psi$

let $\psi = \frac{1}{r}u \rightarrow \epsilon u = -\frac{d^2}{dr^2}u \rightarrow u = Ae^{i\sqrt{\epsilon}r} + Be^{-i\sqrt{\epsilon}r}$

and $\psi = \frac{1}{r}Ae^{i\sqrt{\epsilon}r} + \frac{1}{r}Be^{-i\sqrt{\epsilon}r}$

We look for bound solutions, i.e. $E=-|E|$, and $\psi = \frac{A}{r}e^{-|\epsilon|^{1/2}r}$

We impose the Bethe-Peierls boundary condition

$$\Rightarrow \frac{d/dr\,(r\psi)}{r\psi} = \frac{-|\epsilon|^{1/2}\,\frac{A}{r}e^{-|\epsilon|^{1/2}r}}{Ae^{-|\epsilon|^{1/2}r}} = -|\epsilon|^{1/2} = -\frac{1}{a}$$

Note that this just has a solution for $a>0$ $\rightarrow$ for $a<0$ there's no bound solution! (two-body)

Hence $|\epsilon|^{1/2} = \frac{1}{a} \rightarrow \varepsilon = -\frac{1}{a^2} \rightarrow \boxed{E = \frac{-\hbar^2}{2ma^2}}$ $\rightarrow$ binding energy

and $\psi = \frac{A}{r}e^{-r/a} \rightarrow$ Normalizing $\int|\psi|^2 d^3r = 4\pi A^2\int_0^\infty e^{-2r/a}\,dr$

$$= 2\pi a A^2 = 1$$

Hence $\boxed{\psi(r) = \frac{e^{-r/a}}{\sqrt{2\pi a}\,r}}$ $\rightarrow$ wave function of the bound state

Finally,

* Recall from p. (98) that we may modify the scattering length by means of a magnetic field using the Feshbach resonances $a = a_{bare}\left[1 - \frac{\Delta B}{B - B_0}\right]$

Note: Typically $k|R^*| \ll 1$ at the resonance (broad resonances) although for the so-called narrow resonances $k|R^*| \gtrsim 1$. The analysis of narrow resonances is more complicated.)

By means of Feshbach resonances we may change the sign and absolute value of a. It is especially interesting to see what happens when starting from $a<0$ and small $|a|$ to see what happens when we cross the resonance towards $a>0$ values. One would naively expect to reach simply a Fermi gas with $a>0$. But this is not what happens, ~~because~~ due to the formation of dimers at $a>0$ discussed before. These dimers have a bosonic nature (they are composed by 2 fermions), and hence for sufficiently low temperature they condense. Hence we will have a transition from a weakly-interacting Fermi gas (which may enter into BCS superfluidity as discussed above) and a molecular BEC.

This transition receives the name BCS-BEC crossover and it has been the subject of an enormous amount of experimental and theoretical research in the last few years.

• BEC of DIMERS

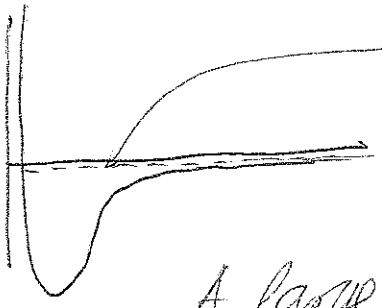
• Let's have first a look to the physics of dimers. These composite bosons interact with each other and with single atoms.

• By means of the so-called zero-range approximation [I won't enter in details here, for more details see PRL 93, 090404 (2004)] one may obtain that:

* Atom-dimer scatt. length: $a_{ad} \simeq 1.18\, a$

* Dimer-dimer scatt. length: $a_{dd} \simeq 0.60\, a$

• The weakly-bound dimers formed near a Feshbach resonance are molecules in the highest ro-vibrational mode.



Hence they seem to be extremely fragile objects. In principle, due to collisions, they can fall into deeper bound states (of size $\sim R_0$). A large kinetic energy ($\sim \hbar^2/mR_0^2$) would be released and the colliding atoms would be released. That's certainly too bad, and it's actually what happens with boson-boson molecules. However, with fermion-fermion molecules the physics is remarkably different (fortunately!)

. Atom-dimer collisions are produced when the three particles approach at a distance $\sim R_0$. However the probability to have this configuration is very largely suppressed due to the Pauli exclusion principle (since two fermions are identical).

* We may describe the relaxation process with the rate equation

$$\dot{n}_a = -\alpha_{ad}\, n_a n_d$$

($\dot{n}_a$: atom loss rate; n_a: density of atoms; n_d: density of dimers)

* A proper calculation leads to the result

$$\alpha_{ad} \propto \left(\frac{\hbar R_0}{m}\right)\left(\frac{R_0}{a}\right)^{3.33}$$

* Dimer-dimer collisions lead to a relaxation

$$\dot{n}_d = -\alpha_{dd}\, n_d^2$$

where $\alpha_{dd} \propto \left(\frac{\hbar R_0}{m}\right)\left(\frac{R_0}{a}\right)^{2.55}$

* The crucial point here is that both α_{dd} and α_{ad} decay with growing $|a|$. Amazingly this means that the less bound is the pair the more stable it is!

* Just as a side remark, A similar theory shows that for fermions with different masses $m_1 > m_2$, $\alpha_{dd} \propto a^{-s}$ has $s>0$ for $m_1/m_2 < 12.3$.

* The behavior of the dilute ($k_F|a| \ll 1$) gas of dimers (the so-called BEC limit) is properly described by the BEC theory that we have discussed in previous lectures. In particular we may easily calculate the

critical temperature (T_c) for condensation. A straightforward calculation shows that in free space

$$T_c \simeq 0.218\, T_F$$

This shows that the BEC of dimers (and the superfluidity) occurs for $T \sim T_F$, at temperatures $\gg T_{BCS}$.
We get hence a sort of <u>high-T_c superfluidity</u>.

• <u>UNITARITY LIMIT</u>

- A much more difficult problem is that of $k_F|a| \gtrsim 1$.

In this case a > interparticle distance

but interparticle distance $\gg R_0$

We have hence the unusual situation of a gas which is <u>both dilute and strongly-interacting</u>.

Two (non trivial) questions arise here

. Is the gas stable? Or does it collapse?

. Does it exhibit superfluidity as the BEC and BCS sides?

Theoretical and experimental results give a clear indication that the gas is indeed stable and superfluid (below a given temperature of course).

- As already mentioned the limit $k_F|a| \to \infty$ is the so-called <u>unitarity-limit</u> and it's of particular relevance. Let's see

very briefly some important properties at unitarity.

- As we already mentioned in p. (225) this regime is characterized by the universal (potential-independent) behavior of the scattering amplitude: $f_0(k) = \frac{i}{k}$

As the scattering length drops out of the problem, the only relevant length scales are k_F^{-1} and λ_T (≡ thermal de-Broglie wavelength). Hence all thermodynamic quantities become universal functions of E_F and T/T_F.

An important example of this universality is provided by the chemical potential at $T=0$:

$$\boxed{\mu = (1+\beta) E_F}$$

where β is a universal dimensionless parameter. This is a crucial relation, it tells us that the chemical potential of the ~~interacting~~ unitary gas has basically the same density dependence as for the ideal one. This has important consequences for density profiles, released energy, collective frequencies, etc.

The value of β has been calculated using Quantum Monte Carlo

$$\boxed{\beta \simeq -0.58}$$

Note that $\beta < 0$, and hence that unitary interactions are attractive.

The superfluid gap (at $T=0$) scales also with E_F at unitarity

$$\boxed{\Delta_{sg} \simeq 0.50\, E_F}$$

- At finite temperature the most relevant point is of course the determination of T_c, which is also expected to be proportional to T_F: $T_c = \alpha T_F$

This value α is not fully clear but should be around 0.2 ± 0.05

- Hence, contrary to the weakly-interacting case, at unitarity the gas becomes superfluid at a much higher temperature, which this time can be achieved experimentally!

- Trapped Fermi gas at unitarity

- As we have already mentioned the universal relation

$$\mu = (1+\beta)\epsilon_F$$

has crucial consequences. Let's see what happens for the case of a harmonically-trapped gas. We shall employ the local density approximation.

- Recall that for a uniform gas with spin 1/2 the Fermi energy is of the form (p. 84):

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \quad \text{where} \quad n = n_\uparrow + n_\downarrow$$

- Then at unitarity

$$\mu[n] = (1+\beta)\frac{\hbar^2}{2m}[3\pi^2 n]^{2/3}$$

- We employ (as in p. 210) the local density approximation introducing a local chemical potential

$$\mu[n(\vec{r})] = \mu_0 - V_{ext}(\vec{r})$$

* In our case we consider for simplicity a isotropic trap

$$V_{ext}(\vec{r}) = \frac{1}{2} m\omega^2 r^2$$

Hence

$$(1+\beta)\frac{\hbar^2}{2m}\left(3\pi^2 n(\vec{r})\right)^{2/3} = \mu_0 - \frac{1}{2} m\omega^2 r^2$$

which leads to:

$$n(\vec{r}) = \frac{1}{3\pi^2}\left[\left(\frac{2m}{\hbar^2}\right)\left(\frac{\mu_0}{1+\beta}\right)\right]^{3/2}\left[1 - \frac{m\omega^2}{2\mu_0} r^2\right]^{3/2}$$

$$= n_0\left(1 - (r/R)^2\right)^{3/2}$$

where $n_0 = \frac{1}{3\pi^2}\left[\left(\frac{2m}{\hbar^2}\right)\frac{\mu_0}{1+\beta}\right]^{3/2}$ and $R^2 = \frac{2\mu_0}{m\omega^2}$

(Note we get a similar density as in p. 211, as we should)

- We normalize

$$N = \int d^3r\, n(\vec{r}) = 4\pi n_0 R^3 \underbrace{\int_0^1 r^2 dr (1-r^2)^{3/2}}_{\pi/32} = \frac{\pi^2}{8} n_0 R^3$$

Then: $\frac{\pi^2}{8}\cdot\frac{1}{3\pi^2}\left[\left(\frac{2m}{\hbar^2}\right)\frac{\mu_0}{1+\beta}\right]^{3/2}\left(\frac{2\mu_0}{m\omega^2}\right)^{3/2} = N$

which leads to

$$\mu_0 = \frac{\hbar\omega}{2}(24N)^{1/3}(1+\beta)^{1/2}$$

and hence $R^2 = \frac{2\mu_0}{m\omega^2} = l_{HO}^2 (24N)^{1/3}(1+\beta)^{1/2}$

$$\Rightarrow R = \underbrace{l_{HO}(24N)^{1/6}}_{\text{\& } R_{ideal}}(1+\beta)^{1/4} \xrightarrow{\beta \simeq -0.58} \boxed{R_{UNITARITY} \simeq R_{IDEAL}\cdot 0.8}$$

The Thomas-Fermi radius is hence reduced at unitarity (we expected that from an attractive gas!)

* Actually one may employ the in-situ measurement of the cloud radius to obtain an experimental estimation of β. Different experiment ~~[illegible]~~ with Lithium and Potassium lead to a value ~ -0.55 compatible with the theoretical results. Actually the fact that Lithium and Potassium lead to the same β is an excellent indication of the universality at unitarity.

* Other way to determine β is by mean of the released energy $\xrightarrow{\text{at unitarity}}$ $E_{released}^{UNIT.} = E_{released}^{IDEAL}(1+\beta)^{1/2}$. Measurements of the released energy give again $\beta \sim -0.59$

• SUPERFLUIDITY

. As we have already discussed superfluidity is both possible for the BEC side (due to the bosonic superfluidity of the dimer BEC) and at the BCS side (due to Cooper pairing). However, as we mentioned already deep at the BCS side ($k_F|a| \ll 1$) T_c/T_F is exponentially small and can't be reached experimentally.

. However at the BCS–BEC crossover around unitarity, high-T_c superfluidity ($T_c/T_F \sim 0.2$) ~~[illegible]~~ may be achieved experimentally. We can't review all experiments and theory devoted recently to this issue, but let me just

Comment very briefly on the observation of quantized vortices. As we already discussed (p. 139) quantized vortices constitute one of the most important consequences of superfluidity. Recent experiments have confirmed their existence in ultracold Fermi gases along the BCS-BEC crossover both at the BCS, and BEC side and at unitarity! (Ketterle's group, 2005).

This is indeed a very remarkable observation which shows that the gas at the crossover is both stable and superfluid.

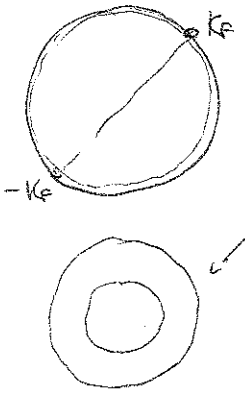
Unfortunately we can't review here the whole body of results on BCS-BEC crossover, a fascinating field indeed which has attracted both a huge theoretical and experimental interest recently.

For more on the BCS-BEC crossover have a look into the review of Giorgini, Pitaevskii and Stringari, cond-mat/0706.3360.

• UNBALANCED FERMI MIXTURES

• Our previous discussion on Fermi superfluidity was based on the assumption that both spin components have the same number of atoms. In the following we will discuss (necessarily very briefly) what happens when $N_\uparrow \neq N_\downarrow$ (another interesting possibility is the case of different species such that $m_\uparrow \neq m_\downarrow$, but we won't study this case here).

• Remember that BCS superfluidity arises from the pairing of particles of different spin occupying states with ~~different~~ opposite momenta close to the Fermi surface.



This mechanism is inhibited in the presence of a spin imbalance, since the Fermi surfaces of the two components don't coincide and pairs of zero total momentum are difficult to form.

Eventually, if the gap between the two Fermi surfaces is too large, superfluidity is broken and the system undergoes a quantum phase transition towards a normal state.

• There's hence a critical value of the polarization

$$P = \frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}$$

. The existence of this critical P_c is clear since

for $P = 0 \rightarrow$ at $T=0 \rightarrow$ superfluidity

$P = 1 \rightarrow$ at $T=0 \rightarrow$ ideal (and hence normal) fully polarized Fermi gas.

• THE CLOGSTON-CHANDRASEKHAR LIMIT

* In the following we shall discuss this transition in some more detail. We'll discuss first the homogeneous case. We consider a different chemical potential for both components $\mu_\uparrow$ and $\mu_\downarrow$, and define $\mu^* \equiv \frac{\mu_\uparrow - \mu_\downarrow}{2}$. This difference of chemical potential creates an imbalance $n_\uparrow \neq n_\downarrow$.

* We may then generalized the gap equation introduced in p. (220). There we assumed $n_\uparrow(\vec{p}) = n_\downarrow(\vec{p}) = n(\vec{p})$. Now we assume imbalance. Hence the gap equation becomes

$$1 = \frac{-g}{(2\pi\hbar)^3} \int d^3p \, \frac{\left[1 - n_\uparrow(\vec{p},T) - n_\downarrow(\vec{p},T)\right]}{2\xi(\vec{p})}$$

where $$n_\uparrow(\vec{p},T) = \frac{1}{e^{[\xi(\vec{p})+\mu^*]/k_BT} - 1}$$

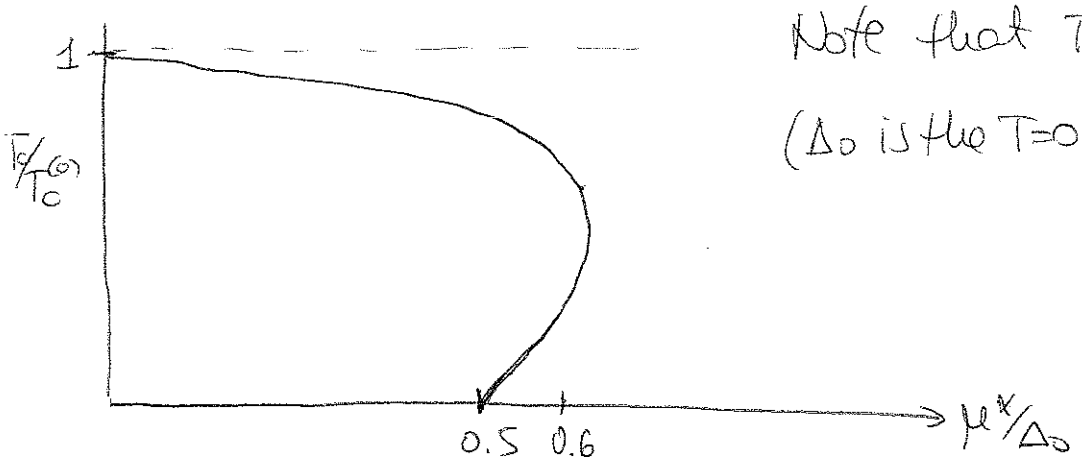
$$n_\downarrow(\vec{p},T) = \frac{1}{e^{[\xi(\vec{p})-\mu^*]/k_BT} - 1}$$

$$\xi(\vec{p}) = \sqrt{\eta(\vec{p})^2 + \Delta^2} \quad \text{as before.}$$

where $\eta(\vec{p}) = p^2/2m - \mu$ with $\mu = \frac{\mu_\uparrow + \mu_\downarrow}{2}$

* For $\mu^* = 0$ we recover the results we already know, and in particular we find the critical temperature $T_c^{(0)}$ of p. (223).

* For other values of μ^* we obtain a graph for T_c.



Note that $T_c = 0$ for $\mu^* = \Delta_0/2$

(Δ_0 is the $T=0$ gap for $\mu^* = 0$)

- Along all this line we have (as for $\mu^* = 0$) a second-order phase transition (i.e. the gap opens smoothly from zero).

* However, Clogston and Chandrasekhar realized in the 60's that actually the situation is different, because actually at $T=0$ there's a 1st order phase transition (at which the gap opens abruptly).

- The reason for that is that at zero temperature and fixed $\mu_\uparrow$ and $\mu_\downarrow$ the stable phase is that with the minimal thermodynamic potential

$$G = E - \mu_\uparrow n_\uparrow - \mu_\downarrow n_\downarrow = E - \mu(n_\uparrow + n_\downarrow) - \mu^*(n_\uparrow - n_\downarrow)$$

Note that in the previous discussion we "forgot" the extra $-\mu^*(n_\uparrow - n_\downarrow)$ term.

- One must hence calculate G_S (G in the superfluid state) and G_N (G in the normal state). The phase transition

occurs when $G_S = G_N$.

* Without too much calculations let's just point the result:

$$G_S - G_N = -N_0 \left(\frac{\Delta_0^2}{2} - \mu^{*2} \right)$$

($N_0 \equiv \frac{m p_F}{2\pi^2 \hbar^3}$ → density of states at the Fermi surface)

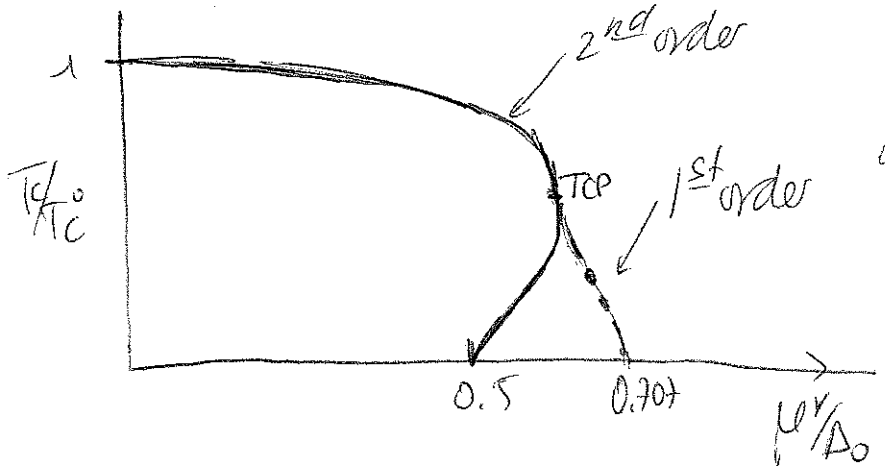
Note
(for more details of this calculation you may have a look e.g. to the volume IX of Landau-Lifshitz).

* Then, the critical μ^* at $T=0$ is

$$\mu_c^* = \frac{\Delta_0}{\sqrt{2}}$$

For this value there's a first order phase transition when coming from large μ^*, since the gap jumps discontinuously. Note that $\Delta_0/\sqrt{2} > \Delta_0/2$ and hence coming from large μ^* the 1st order phase-transition occurs first.

* For other temperatures we get a phase diagram:



Note that when crossing the 1st order line supercooling is possible (we have a coexistence like in all first order phase trans.) but beyond the 2nd order line the normal phase is absolutely unstable with respect to the BCS state.

* Note that there's a tricritical point (TCP). Over it there's just second-order phase transition.

* Hence at $T=0$ there's just normal phase for $\mu^* > \mu_c^* = \Delta_0/\sqrt{2}$.

This corresponds, for the (deep) BCS regime to a very small polarization

$$P_c = \frac{3}{\sqrt{8}} \left(\frac{2}{e}\right)^{7/3} e^{-\pi/2k_F a}$$

* Hence for the deep BCS a very slight imbalance kills superfluidity even at $T=0$. This is still another good reason to look for Feshbach resonances!

I will briefly discuss in a moment the case at unitarity but before doing that let's consider another possible and more exotic superfluid phase which may occur in partially polarized Fermi gases.

---

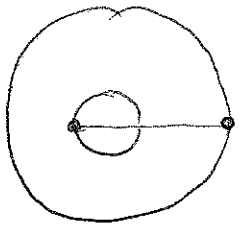
* THE FULDE-FERREL-LARKIN-OVCHINNIKOV (FFLO) PHASE

* In the standard BCS phase it's energetically more favourable to have all pairs with total momentum $\vec{q}=0$, or in other words to have a pairing gap Δ which is translationally invariant.

* This is however not so clear when there's partial polarization. Fulde and Ferrel, and Larkin and Ovchinnikov found that at $T=0$ and large μ^* that a transition to a superfluid with an order parameter with $\vec{q} \neq 0$ may be energetically more advantageous. This result is quite remarkable since it

constitutes a spontaneous breaking of translational invariance. The superfluid phases with $\vec{q} \neq 0$ are called FFLO phases.

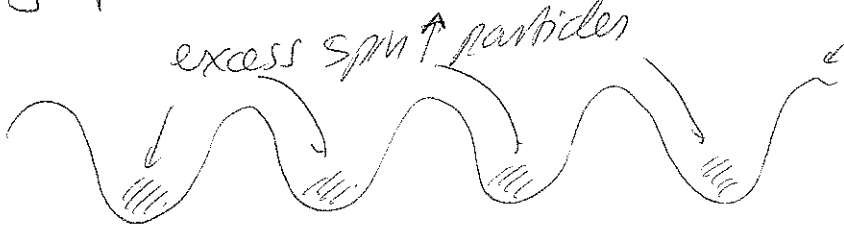
- The qualitative idea of the FFLO phases is rather easy to grasp. If both Fermi surfaces are the same, the kinetic energy of the pairs is minimized by taking $\vec{k}_F$ and $-\vec{k}_F$, and hence $\vec{q} = 0$. However when $\mu_\uparrow \neq \mu_\downarrow$ the two Fermi surfaces are not identical, and it isn't possible to take two exactly opposite momenta at the respective Fermi surfaces.



- Taking Cooper pairs with momenta $q \sim k_{F\uparrow} - k_{F\downarrow} \neq 0$ minimizes the kinetic energy.

The corresponding order parameter will then have a form: $\Delta(x) \propto \cos qx$

The excess (say spin up) atoms are concentrated near the zeros of $\Delta(x)$



- One can show that in the BCS limit and at $T=0$ there's a transition to FFLO at $\mu^* = 0.754\,\Delta_0$ (with $q = 2.40\,\mu^*/v_F$)

(Note: this implies that the order parameter has a modulation with wavelength of the order of the Cooper pair size).

* In terms of polarization this means the critical

$$P_c^{FFLO} = 1.13\ \Delta_0/E_F$$

whereas $P_c^{BCS\text{-}BEC} = 1.06\ \frac{\Delta_0}{E_F}$

Hence for the deep BCS limit the FFLO transition is favoured.

* However, as we have mentioned already, for the deep BCS limit P_c is in any case exceedingly small. For a practical situation we must see what happens in the unitarity limit. Recent experiments on spin imbalanced trapped Fermi gases show that FFLO plays no role, and that the relevant physics is just given by a spatial phase separation between ~~a balanced~~ an unpolarized superfluid and a polarized normal phase.

• IMBALANCED FERMI GASES AT UNITARITY

* I won't discuss at this point all details of the theory at unitarity but simply the major results.

* At unitarity for $\boxed{P > P_c = 0.39}$ the stable configuration is the uniform normal phase. For $P < 0.39$ there's a phase separation between a normal phase (with polarization 0.39) and a superfluid unpolarized phase.

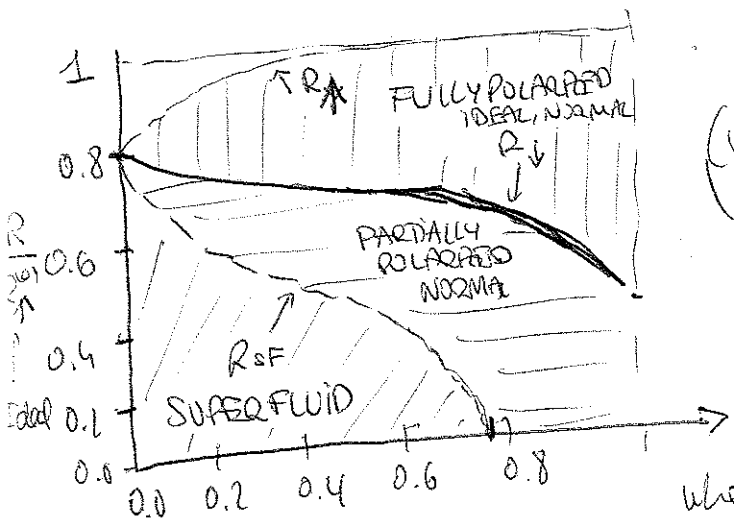
* The first order phase transition occurs at $\mu_c^* \simeq 0.81\,\Delta_0$.
* Note that the values of μ_c^* and P_c are quite large, compared to the exceedingly small values for the deep BCS limit.

- These results obtained for the homogeneous gas may be employed (via local density approximation) to obtain the results for a harmonically trapped gas: $\mu_\sigma(\vec{r}) = \mu_\sigma^0 - V(\vec{r})$.
- For small $N_\downarrow \ll N_\uparrow$ only the normal state is present.
- When at the center the local polarization reaches the critical value 0.39 a superfluid core starts to nucleate with a polarized normal gas outside the superfluid. The radius R_{SF} of the superfluid is determined by the equilibrium between the two phases (SF and normal). Then there's a radius $R_\downarrow$ where the population of $\downarrow$ vanishes, and $R_\uparrow > R_\downarrow$ where the population of $\uparrow$ vanishes. There outer layer $R_\downarrow < r < R_\uparrow$ is obviously a fully polarized (and hence ideal) gas.

* This graphic summarizes the results. Note that the superfluid core disappears for

$$P > P_c^{trap} = 0.77$$

(which must be compared with 0.39 in the untrapped case)



- This is in very good agreement with recent experiments (Ketterle group, 2006)

(Note: although LDA seems not fully correct to describe experiments at Rice where surface tension effects seem to play a role)

* With this we will finish our (unfortunately!) very short discussion on fermions.

We did not cover some very important topics, including in particular the physics of fermions in optical lattices. There you may describe a binary mixture by a Fermi-Hubbard model, which then allows at T=0 for 2 ground-state phases, a Mott-insulator and a metallic phase. The Mott insulator phase appears for low tunneling t and temperatures $T \ll U$ (= on-site interaction). For this phase density fluctuations are suppressed, but the spin entropy is large. For $T < t^2/U$ one gets in 3D samples a Neél order, since one reaches a Heisenberg antiferromagnet.

There's a lot of activity on-going nowadays on lattice fermions, in particular in what concerns new ways of reducing entropy, and on the search of the Mott-Neél phase. We won't review this here. For more details see e.g. the review of Antoine Georges (cond-mat/0702122), the review of I. Bloch et al., and the review of Lewenstein et al.

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* With this we finish our ~~lectures~~ on cold gases which have brought us from quantum optics to "hard-core" condensed-matter physics. I hope that you have got at least a feeling of the richness and the multidisciplinary character of this research field. There's of course much much more, so if you are interested you may have a look to the recommended bibliography.