• ULTRACOLD FERMI GASES

· Up to now we have mainly considered the physics of ultracold bosons. In these last lectures we shall explore (necessarily very briefly) some important aspects of the physics of ultracold fermions.

· In p. 82 we had a quick look to the ideal homogeneous Fermi gases, where we introduced the important idea of Fermi energy and Fermi degeneracy. In these last lectures we shall first consider the ideal Fermi gas in a harmonic trap (which is the experimentally relevant case), and then we shall discuss some important effects introduced by the interatomic interactions, most importantly the idea of Cooper pairing.

· Before starting with the discussion of Fermi gases in harmonic traps, it's worthy to point at this stage some important differences between bosons and fermions.

· One of the most important differences concerns the interaction properties. As we already mentioned in p. 88 Pauli exclusion forbids the s-wave collision of 2 identical fermions. As a result there's no s-wave scattering between atoms in the same hyperfine state (what is called a polarized Fermi gas). Since p-wave scattering is much weaker at very low temperatures, then a polarized Fermi gas may be consider to a good approximation an ideal gas.

· The absence of ellastic collisions prevents evaporative cooling in the way discussed in p. 42. In this sense the experimental achievement of quantum degenerate Fermi gases cannot be based the same route as that for bosons. Typically one employs the so-called sympathetic cooling. In this technique one has a Bose-Fermi mixture. Pauli exclusion doesn't preclude s-wave scattering between different species (or spin components) and then re-thermalization may become possible. This technique led in 1999 to the first degenerate Fermi gas (Jin's group at JILA).

- IDEAL FERMI GAS IN A HARMONIC TRAP

* In the following we consider a gas of $N \gg 1$ fermions in a harmonic trap (of frequencies $\omega_{x,y,z}$).
* We may easily evaluate the Fermi energy introducing a semiclassical description based on a local-density-approximation (LDA) for the Fermi distribution function.

Homogeneous

$$f(\vec{p}) = \frac{1}{e^{\beta\left[\frac{p^2}{2m} - \mu\right]} + 1}$$

$\Longrightarrow$

Trapped gas (LDA)

$$f(\vec{r},\vec{p}) = \frac{1}{e^{\beta\left[\frac{p^2}{2m} + \underbrace{V_{ext}(\vec{r}) - \mu}_{\text{local chemical potential}}\right]} + 1}$$

where $V_{ext}(\vec{r}) = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$

. We impose the normalization

using the density of states of a 3D harmonic oscillator ($\propto E^2$)

$\omega_{ho} = \sqrt[3]{\omega_x \omega_y \omega_z}$

$$N = \frac{1}{(2\pi\hbar)^3}\int d^3r \int d^3p \, f(\vec{r},\vec{p}) = \frac{1}{2(\hbar\omega_{ho})^3}\int_0^\infty \frac{E^2 dE}{e^{\beta(E-\mu)} + 1}$$

At $T=0$ ($\beta \to \infty$): $N = \frac{1}{2(\hbar\omega_{ho})^3}\frac{E_F^3}{3} \longrightarrow \boxed{E_F = k_B T_F = (6N)^{1/3}\hbar\omega_{ho}}$

$E_F = \mu(T=0)$

Fermi energy of fermions in a 3D harmonic oscillator

· This result may be easily obtained by summing levels:

$$N = \sum_{n=0}^{n_F} \sum_{n_x=0}^{n} \sum_{n_y=0}^{n-n_x} \sum_{n_z=0}^{n-n_x-n_y} 1 = \sum_{n=0}^{n_F} \frac{1}{2}(n+1)(n+2) = \frac{1}{6}(n_F+1)(n_F+2)(n_F+3)$$

For $N \gg 1 \longrightarrow N \simeq \dfrac{n_F^3}{6}$

Since $E_n = \hbar\omega(3/2 + n) \overset{n\gg1}{\cong} \hbar\omega n$

$$\Bigg\} \quad E_F = \hbar\omega n_F \simeq \hbar\omega (6N)^{1/3}$$

[Note: Remember (p. 79) that for a harmonically-trapped Bose gas, $k_B T_C \cong 0.94\, \hbar\omega_{ho} N^{1/3}$. Hence $k_B T_F$ and $k_B T_C$ have the same dependence $N\omega_{ho}^3$. This is not surprising because as we mentioned in p. 82 and p. 73 quantum degeneracy occurs when $n\lambda_T^3 \sim 1$, and for a thermal gas (gaussian distribution) $n \sim N (k_B T/m\omega_{ho}^2)^{-3/2} \longrightarrow n\lambda_T^3 \sim N\omega_{ho}^3$]

* Let's consider for simplicity an axisymmetric trap ($\omega_x = \omega_y = \omega_\perp$).

We may easily find the density distribution at zero temperature

$$n_0(\vec{r}) = \int f(\vec{r},\vec{p}) \frac{d^3p}{(2\pi\hbar)^3} = \frac{8N}{\pi^2 R_\perp^2 Z}\left[1 - \frac{r_\perp^2}{R_\perp^2} - \frac{z^2}{Z^2}\right]^{3/2}$$ (as long as it is ≥ 0)

where the radial and axial widths are defined as

$$E_F = \frac{m\omega_\perp^2}{2} R_\perp^2 = \frac{m\omega_z^2}{2} Z^2$$

[Note: This resembles the Thomas-Fermi profile of a BEC (p. 117) but the power is now 3/2 and not 1.]

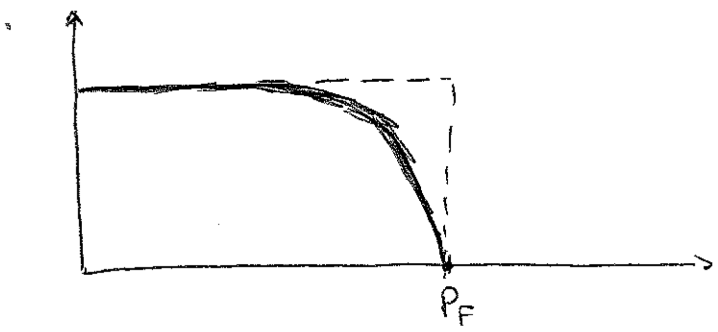
. The momentum distribution at $T=0$ is

$$n_0(\vec{p}) = \int d^3r\, f(\vec{r},\vec{p}) = \frac{8N}{\pi^2 p_F^3}\left(1 - p^2/p_F^2\right)^{3/2}$$ (as long as it is ≥ 0)

where $E_F = p_F^2/2M$ defines the Fermi momentum p_F.

- Remember that for a uniform Fermi gas (p. 83)

$$n(\vec{p}) = \frac{3N}{4\pi p_F^3}\,\Theta(1 - p^2/p_F^2)$$



* Note that the harmonic oscillator trapping leads to a smoothing of the momentum distribution.

. Note also that $p_F = \hbar\,(6\pi^2 n(0))^{1/3}$ ⟶ as for a homogeneous gas of density $n(0)$ (p. 83).

Note that the shape of the profiles doesn't look dramatically different in coordinate space compared to that of a BEC in the Thomas-Fermi regime. In both cases the widths increase with N, as $R = \ell_{HO}\left(\frac{15a}{\ell_{HO}}N\right)^{1/5}$ for the interacting BEC and as $R = \ell_{HO}\,(48N)^{1/6}$ in the ideal Fermi gas.

Note, however, that whereas for bosons the chemical potential is given by the interactions, here the chemical potential is given by the Pauli exclusion).

. The momentum space is on the contrary very different. Note that $n(\vec{p})$ is isotropic, contrary to what happens in the Bose gas, where it (inversely) maps the distribution in coordinate space. This is the consequence of the semiclassical behavior for $N \gg 1$.

Note also that the momentum width in a Thomas-Fermi BEC scales as $\hbar/R$ and decreases with increasing N (remember that if N increases, the width increases due to repulsion, and hence the momentum width decreases). In a Fermi gas, the momentum

width scales as $m\omega_{ho}R$ (p 211) and hence increases with N (!!).
This is so because in a BEC $\Delta p \Delta R \sim \hbar$, but for a Fermi gas $\Delta p \Delta R \sim E_F/\hbar\omega_{ho}$

. The enhancement of the kinetic effect in Fermi gases has important consequences.

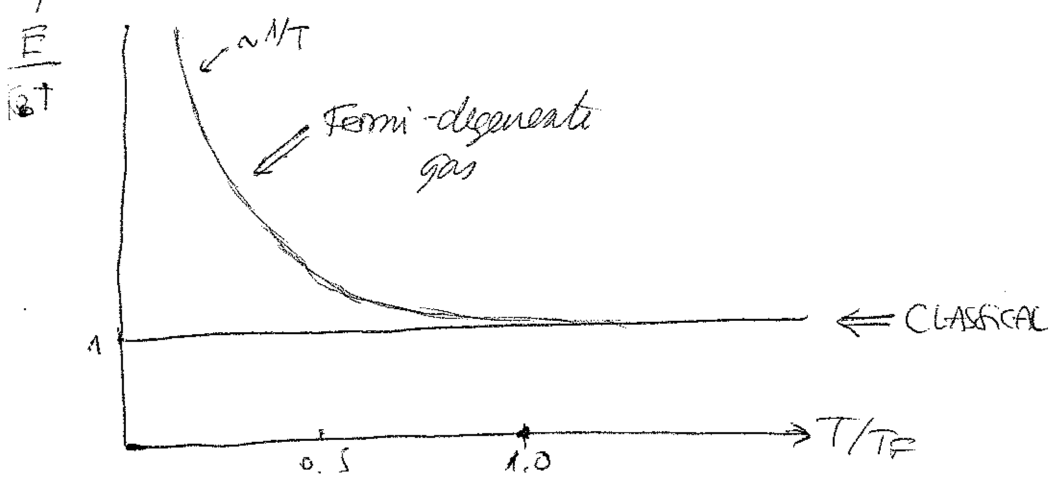
Remember that for an ideal gas the virial theorem tells us that $\langle E_{kin}\rangle = \langle E_{pot}\rangle$, and then $E = 2\langle E_{kin}\rangle$. When we switched-off the trap we measure the released energy (which is of course the kinetic energy). Hence $E_{released} = E/2$. In this way we may easily measure the energy of the Fermi gas

$$E = \frac{1}{2(\hbar\omega)^3}\int_0^\infty \frac{(E')^3\, dE'}{e^{\beta(E'-E_F)}+1} \overset{T=0}{=} \frac{3}{4} N E_F$$

Then at $T=0$ the released energy per particle doesn't go to zero but to $E/N = \frac{3}{4}E_F$.

. This is very different to the classical gas where $E/N = 3k_BT$ (for a trapped Bose gas, the energy at $T=0$ tends to a value $\ll E_F$)

This deviation of the released energy was measured in 1999 as one of the 1st signatures of quantum degeneracy in a trapped Fermi gas (Jin's group, JILA)



• INTERACTING FERMI GASES

• The ideal gas model discussed above provides a good description of a cold spin-polarized Fermi gas, since, as we have mentioned several times, identical fermions can't interact via s-wave scattering.

• In the presence of a mixture (of species or spin components) two-body interactions may become important, especially if the interactions are attractive as we shall discuss in a moment.

• In the following we consider the simplest case of two spin states $\{\uparrow, \downarrow\}$. with mass m. The two components interact via s-wave scattering with an interaction coupling constant $g = 4\pi\hbar^2 a/m$, where a is, as always, the s-wave scattering length.

The relevance of the interaction term is easily estimated by comparing the expectation value of the interaction energy in the ground-state of the ideal case

$$E_{int} = g \int d^3r \, n_\uparrow(\vec{r}) \, n_\downarrow(\vec{r})$$

and the corresponding values of the oscillator energy

$$E_{HO} = \int d^3r \, \frac{1}{2} m\omega^2 r^2 \left(n_\uparrow(\vec{r}) + n_\downarrow(\vec{r}) \right)$$

Let $n_\uparrow(\vec{r}) = n_\downarrow(\vec{r}) = n(\vec{r}) = \dfrac{8N}{\pi^2 R^3} \left(1 - r^2/R^2\right)^{3/2}$

we consider isotropic trap of frequency ω, and

hence $\epsilon_F = (6N)^{1/3} \hbar\omega = m\omega^2/2 \; R^2 \rightarrow R^2 = 2 (6N)^{1/3} \ell_{HO}^2$

with $\ell_{HO} = \sqrt{\hbar/m\omega}$)

• Then $E_{ho} = \left(\frac{8N}{\pi^2 R^3}\right) 2 \; \frac{m\omega^2}{2} \int d^3r \, r^2 \left(1 - r^2/R^2\right)^{3/2}$

$$= \frac{3}{8} m\omega^2 N R^2$$

and

$$E_{int} = g \left(\frac{8N}{\pi^2 R^3}\right)^2 \int d^3r \; \left(1 - r^2/R^2\right)^3$$

$$= g \left(\frac{8N}{\pi^2}\right)^2 \frac{4\pi}{R^3} \int_0^1 r^2 dr \, (1-r^2)^3$$

$$= g \left(\frac{8N}{\pi^2}\right)^2 \frac{1}{R^3} \left(\frac{64\pi}{315}\right)$$

$K_F = p_F/\hbar = 1.91 \, N^{1/6}/l_{ho}$

Then $\frac{E_{int}}{E_{ho}} \cong 0.56 \, N^{1/6} \frac{a}{l_{ho}} = 0.29 \, K_F a$

• Hence the relevance of the interaction term is fixed by the combination $K_F a$, which is usually a small quantity (although at Feshbach resonances this may change as we will see).

Typical values of $K_F^{-1} \sim$ average interatomic distance $\sim n^{-1/3}$

Hence $K_F a \sim (n a^3)^{1/3} \ll 1$ under the diluteness condition $n a^3 \ll 1$.

Hence the effects of the interactions on ground state properties are typically very small. For example for $a > 0$, at $T = 0$ one obtains for an uniform gas (Huang and Yang, 1957)

$$\mu \simeq \epsilon_F \left[1 + \frac{4}{3\pi}(K_F a) + \dots\right]$$

WEAKLY ATTRACTIVE INTERACTIONS (BCS transition)

* In the following we shall constraint our discussion to the case of attractive interactions ($a<0$)

Note: repulsive interactions may lead to relevant deviations of e.g. collective modes, but we won't discuss it here. For more details see the book of Pitaevskii and Stringari.)

· The case of attractive interactions is particularly relevant because at low enough temperatures a Fermi gas interacting attractively undergoes a phase transition into the superfluid state. This is the analogue of the transition to superconductivity in electrons, the so-called BCS transition (BCS ≡ Bardeen-Cooper-Schrieffer)

· According to BCS theory, fermions with opposite momenta and spin at the Fermi surface form a bound state (so-called Cooper pair). These pairs behave as bosons and exhibit BEC in the zero-momentum state.

* In the following we shall introduce the BCS theory for the uniform case. The grand-canonical Hamiltonian is of the form

$$\hat H - \mu \hat N = \int d^3r \left\{ \hat\psi_\uparrow^\dagger(\vec r)\left(-\frac{\hbar^2\nabla^2}{2m}-\mu\right)\hat\psi_\uparrow(\vec r) + \hat\psi_\downarrow^\dagger(\vec r)\left(-\frac{\hbar^2\nabla^2}{2m}-\mu\right)\hat\psi_\downarrow(\vec r) + g\,\hat\psi_\uparrow^\dagger(\vec r)\hat\psi_\downarrow^\dagger(\vec r)\psi_\downarrow(\vec r)\psi_\uparrow(\vec r)\right\}$$

· We perform now a mean-field analysis, where we employ the so-called Wick's theorem:

$$\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}^\dagger_\downarrow(\vec{r})\,\hat{\psi}_\downarrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r}) \simeq$$

$$\simeq \langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}^\dagger_\downarrow(\vec{r})\rangle\,\hat{\psi}_\downarrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r}) + \langle\hat{\psi}_\downarrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r})\rangle\,\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}^\dagger_\downarrow(\vec{r})$$

$$- \langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}^\dagger_\downarrow(\vec{r})\rangle\langle\hat{\psi}_\downarrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r})\rangle$$

$$+ \langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r})\rangle\,\hat{\psi}^\dagger_\downarrow(\vec{r})\,\hat{\psi}_\downarrow(\vec{r}) + \langle\hat{\psi}^\dagger_\downarrow(\vec{r})\,\hat{\psi}_\downarrow(\vec{r})\rangle\,\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r})$$

$$- \langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r})\rangle\langle\hat{\psi}^\dagger_\downarrow(\vec{r})\,\hat{\psi}_\downarrow(\vec{r})\rangle$$

(where we assume $\langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}_\downarrow(\vec{r})\rangle = 0$. this would mean pair-hole pairing).

- let's consider the uniform case

$$\Lambda = \langle\hat{\psi}^\dagger_\uparrow(\vec{r})\,\hat{\psi}^\dagger_\downarrow(\vec{r})\rangle = \langle\psi_\downarrow(\vec{r})\,\psi_\uparrow(\vec{r})\rangle$$

$$n = \langle\psi^\dagger_\uparrow(\vec{r})\,\psi_\uparrow(\vec{r})\rangle = \langle\psi^\dagger_\downarrow(\vec{r})\,\psi_\downarrow(\vec{r})\rangle$$

- the term n just slightly changes μ, but we will neglect that.

We will concentrate on the pairing mean field Λ.

Apart from a constant

$$\hat{H}-\mu\hat{N} \simeq \int d^3r \left\{ \hat{\psi}^\dagger_\uparrow(\vec{r})\left(\frac{-\hbar^2\nabla^2}{2m}-\mu\right)\hat{\psi}_\uparrow(\vec{r}) + \hat{\psi}^\dagger_\downarrow(\vec{r})\left(\frac{-\hbar^2\nabla^2}{2m}-\mu\right)\hat{\psi}_\downarrow(\vec{r}) \right.$$

$$\left. + g\Lambda\left[\hat{\psi}_\downarrow(\vec{r})\,\hat{\psi}_\uparrow(\vec{r}) + \hat{\psi}^\dagger_\uparrow(\vec{r})\,\psi^\dagger_\downarrow(\vec{r})\right]\right\}$$

Let $\boxed{g\Lambda \equiv \Delta}$

* Let's Fourier transform

$$\hat{H}-\hat{\mu}\hat{N} \simeq V\int \frac{d^3p}{(2\pi\hbar)^3} \begin{Bmatrix} \eta(\bar{p}) \left[\hat{\psi}^\dagger_\uparrow(\vec{p})\hat{\psi}_\uparrow(\bar{p}) + \hat{\psi}^\dagger_\downarrow(\bar{p})\hat{\psi}_\downarrow(\bar{p}) \right] \\ + \Delta\left[\psi^\dagger_\downarrow(-\bar{p})\hat{\psi}_\uparrow(\bar{p}) + \hat{\psi}^\dagger_\uparrow(\bar{p})\psi^\dagger_\downarrow(-\bar{p}) \right] \end{Bmatrix}$$

Let
$$\begin{cases} \hat{\psi}_\uparrow(\bar{p}) = u(\bar{p})\hat{\Phi}_\uparrow(\bar{p}) + v(-\bar{p})\hat{\Phi}^\dagger_\downarrow(-\bar{p}) \\ \hat{\psi}_\downarrow(\bar{p}) = u(\bar{p})\hat{\Phi}_\downarrow(\bar{p}) - v(-\bar{p})\hat{\Phi}^\dagger_\uparrow(-\bar{p}) \end{cases}$$
Bogoliubov transformation

Due to the isotropy of the gas $v(\bar{p}) = v(-\bar{p})$, $u(\bar{p}) = u(-\bar{p})$

The new quasi-particle operator $\hat{\Phi}_n$ should satisfy the anticommutation rules. This demands $u(\bar{p})^2 + v(\bar{p})^2 = 1$,

i.e. $u(\bar{p}) = \cos\varphi(\bar{p})$
$v(\bar{p}) = \sin\varphi(\bar{p})$

* If we insert the Bogoliubov transformation into the mean-field Hamiltonian we get:

$$\hat{H}-\mu\hat{N} = \text{constant} + \frac{V}{(2\pi\hbar)^3}\int d^3p \begin{Bmatrix} \left[\eta(\bar{p})\left[u^2(\bar{p}) - v^2(\bar{p})\right] - 2\Delta u(\bar{p})v(\bar{p})\right]\left[\phi^\dagger_\uparrow(\bar{p})\phi_\uparrow(\bar{p}) + \phi^\dagger_\downarrow(\bar{p})\phi_\downarrow(\bar{p})\right] \\ + \left[2\eta(\bar{p})u(\bar{p})v(\bar{p}) + \Delta\left[u^2(\bar{p}) - v^2(\bar{p})\right]\right]\left[\phi^\dagger_\uparrow(\bar{p})\phi^\dagger_\downarrow(-\bar{p}) + \phi_\downarrow(\bar{p})\phi_\uparrow(\bar{p})\right] \end{Bmatrix}$$

We impose $2\eta uv + \Delta(u^2 - v^2) = 0$

$$\longrightarrow \tan 2\varphi(\vec{p}) = \frac{-g\Lambda}{\eta(\bar{p})}$$

- Hence the prefactor of $\phi_\uparrow^\dagger \phi_\uparrow + \phi_\downarrow^\dagger \phi_\downarrow$ is:

$$\eta(\bar{p}) \cos 2\theta(\bar{p}) - \Delta \sin \varphi(\bar{p}) = \sqrt{\eta^2_{(\bar{p})} + \Delta^2} = \xi(\bar{p})$$

- Hence:

$$\hat{H} - \mu \hat{N} = \text{constant} + \frac{V}{(2\pi\hbar)^3} \int d^3p \; \xi(\bar{p}) \left[\hat{\Phi}_\uparrow^\dagger(\bar{p}) \hat{\Phi}_\uparrow(\bar{p}) + \hat{\Phi}_\downarrow^\dagger(\bar{p}) \hat{\Phi}_\downarrow(\bar{p}) \right]$$

- Hence the quasiparticles present an spectrum:

$$\xi(\bar{p}) = \left[\eta(\bar{p})^2 + \Delta^2 \right]^{1/2}$$

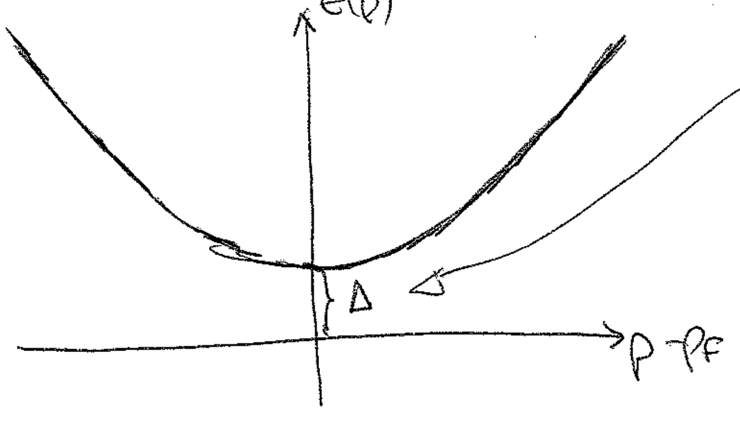
$$\eta(\bar{p}) = \frac{p^2}{2m} - \mu \simeq \frac{p^2 - p_F^2}{2m} = \frac{(p_F + (p - p_F))^2 - p_F^2}{2m} = \frac{p_F}{m}(p - p_F) + \frac{(p - p_F)^2}{2m} \overset{|p-p_F| \ll p_F}{\simeq} v_F (p - p_F)$$

v_F: Fermi velocity.

- Hence for $|\bar{p} - \bar{p}_F| \ll p_F$:

$$\boxed{\xi(\bar{p}) \simeq \sqrt{v_F^2 (p - p_F)^2 + \Delta^2}}$$

- This expression shows that the pairing mechanism is responsible for the appearance of an <u>energy gap</u> Δ in the excitation spectrum.



The gap has important consequences as we will see in a moment.

* The gap may be easily calculated

$$V\cdot\Lambda = \int d^3r \, \langle \hat{\psi}_\downarrow^{\dagger}(\bar{r})\, \psi_\uparrow(\bar{r}) \rangle$$

$$= \left[\frac{V}{(2\pi\hbar)^3}\right]^2 \int d^3p \int d^3p' \, \psi_\downarrow(\bar{p})\, \psi_\uparrow(\bar{p})\, e^{i(\bar{p}+\bar{p}')\bar{r}/\hbar}$$

$$\Lambda = \frac{1}{(2\pi\hbar)^3} \int d^3p \, \langle \psi_\downarrow(-\bar{p})\, \psi_\uparrow(\bar{p}) \rangle$$ statistical average

$$= \frac{1}{(2\pi\hbar)^3} \int d^3p \, \langle \left[u(\bar{p})\phi_\downarrow(-\bar{p}) - v(\bar{p})\phi_\uparrow^{\dagger}(+\bar{p})\right] \cdot \left[u(\bar{p})\,\phi_\uparrow(\bar{p}) + v(-\bar{p})\,\phi_\downarrow^{\dagger}(-\bar{p})\right] \rangle$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3p \; uv \left[\langle \phi_\downarrow(\bar{p})\phi_\downarrow^{\dagger}(\bar{p})\rangle - \langle \phi_\uparrow^{\dagger}(\bar{p})\,\phi_\uparrow(\bar{p})\rangle\right]$$

$$= \frac{1}{(2\pi\hbar)^3} \int d^3p \, \frac{1}{2} \left[\frac{-g\Lambda}{\sqrt{\eta(p)^2 + g^2\Lambda^2}}\right] \left\{1 - 2\,n(\bar{p},T)\right\}$$

$n(\bar{p},T) \equiv \langle \phi_\downarrow^{\dagger}\phi_\downarrow\rangle = \langle \phi_\uparrow^{\dagger}\phi_\uparrow\rangle$

Then

GAP EQUATION

$$\boxed{1 = \frac{-g}{2(2\pi\hbar)^3} \int d^3p \, \frac{(1 - 2n(\bar{p},T))}{\sqrt{\eta(p)^2 + \Delta^2}}}$$

· At $T=0 \rightarrow n(\bar{p},T)=0$ and

$$1 = \frac{-g}{2(2\pi\hbar)^3}\int \frac{d^3p}{\sqrt{\Delta_0^2+\eta(\bar{p})^2}}$$

(<u>Note</u>: It's clear that this equation just has a solution for $g<0$.)

· The physical contribution to this integral comes from momenta $\Delta \ll v_F|p-p_F| \ll v_F p_F \sim \mu$. In this interval we may approximate

$$\int \frac{p^2dp}{\sqrt{\Delta_0^2+\eta(\bar{p})^2}} \simeq \quad \eta(\bar{p}) \simeq v_F(p-p_F) \rightarrow d\eta = v_F\,dp \quad \leftarrow p^2dp \simeq \frac{p_F^2}{v_F}d\eta$$

$$\simeq \frac{p_F^2}{v_F}\int \frac{d\eta}{\sqrt{\Delta_0^2+\eta^2}} \simeq 2\,\frac{p_F^2}{v_F}\ln\frac{\tilde{\varepsilon}}{\Delta_0}$$

where we have introduced an upper energy cut-off $\tilde{\varepsilon} = \tilde{p}^2/2m$ in the integral (note that this cut-off is necessary to avoid an ultraviolet divergence at large momenta, which was motivated by our p-independent form of the interaction.)

Then $$1 = \frac{-g}{(2\pi\hbar)^3}\cdot 4\pi\,\frac{p_F^2}{v_F}\ln\frac{\tilde{\varepsilon}}{\Delta_0}$$

Hence $$\boxed{\Delta_0 = \tilde{\varepsilon}\, e^{-\pi/2k_F|a|}} \longrightarrow \text{energy gap at } T=0$$

· <u>Note</u>: I will comment on $\tilde{\varepsilon}$ below.

* The occurrence of the gap has very important consequences. According to Landau's criterion of superfluidity (p. 125) the excitation spectrum of p. 219 leads to a superfluid critical velocity:

$$\min_p \frac{\mathcal{E}(p)}{p} \cong \min_p \frac{1}{p}\sqrt{v_F(p-p_F)^2+\Delta^2} = \frac{|\Delta|}{p_F} = v_c$$

Since $|\Delta| \ll \varepsilon_F \rightarrow v_c \ll v_F$

* <u>At sufficiently low T the gas is hence superfluid!</u>

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* Connected to the ~~gap~~ energy Δ, we may define a corresponding <u>healing length</u>

$$\xi = \hbar v_F/|\Delta|$$

Superfluid phenomena involve space variations $\gg \xi$.

Quantized vortices have a core size $\sim \xi$.

. Note however that $|\Delta|$ is typically very small, and hence ξ is typically much larger than for BECs.

. Let's see now what happens <u>at finite T</u>. The occupations are

$$n(\bar{p},T) = \frac{1}{e^{\xi(\bar{p})/k_BT}+1} \longrightarrow 1-2n(\bar{p},T) = \tanh\left[\frac{\xi(\bar{p})}{2k_BT}\right]$$

The gap equation becomes

$$1 = -\frac{g}{2(2\pi\hbar)^3}\int d^3p \frac{\tanh\left(\frac{\xi(\bar{p})}{2k_BT}\right)}{\xi(\bar{p})}$$

* The transition from the superfluid to the normal phase takes place at $T = T_C$, such that $\Delta = 0$:

$$1 = \frac{-g}{2(2\pi\hbar)^3}\int d^3p \, \frac{\tanh[\eta(\bar{p})/k_B T_C]}{\eta(\bar{p})}$$

We introduce again an upper cut-off $\tilde{\varepsilon}$. In the limit $T_C \ll \tilde{\varepsilon}$ the integral approaches $\simeq \ln\left[\frac{2\gamma\tilde{\varepsilon}}{k_B T_C \pi}\right]$, with $\gamma \simeq 1.781$ = Euler's constant

Then:

$$T_C = \frac{\gamma}{\pi}\tilde{\varepsilon}\, e^{-\pi/2k_F|a|} = 0.57\Delta_0$$

The <u>critical temperature T_C</u> is hence directly related to the value of the gap at $T=0$.

- Note that $\tilde{\varepsilon}$ was introduced phenomenologically. The prefactor $\tilde{\varepsilon}$ may be evaluated by properly considering the re-normalization of the scattering length due to screening effects in the medium (Gorkov and Melik-Barkhudarov, 1961) yielding the result $\tilde{\varepsilon} = \left(\frac{2}{e}\right)^{7/3} E_F \simeq 0.49\, E_F$

Hence $\boxed{T_C \simeq 0.28\, T_F\, e^{-\pi/2k_F|a|}}$

Note that due to the exponential factor $T_C \ll T_F$, being typically a regime of much lower T as that needed for BEC.