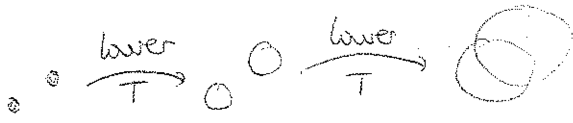
• IDEAL DEGENERATE QUANTUM GASES

* In the previous lectures we have learned how to cool atoms to extremely low temperatures by combining laser and evaporative cooling.
* Remember that the delocalization of an atomic wavepacket is proportional to $\sqrt{T}$ $\longrightarrow$ $\Delta x \propto \sqrt{T}$

The cooler the atoms the more spread they are, and, as already mentioned when talking about atom optics, the more matter-wave-like they are.

The width of the atomic wavepackets is given by the thermal de Broglie wavelength

$$\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T} \right)^{1/2}$$



* It's clear that something remarkable should happen when λ_T becomes comparable to the interparticle distance R, which is proportional to $1/\text{density}^3 = 1/n^3$.

Remember (p. 43) that we define the phase-space density as $n\lambda_T^3$. So something remarkable occurs when $n\lambda_T^3 > 1$. The wavepackets overlap, the particles can't be distinguished any more, and hence quantum statistics becomes very important, i.e. it becomes crucial whether the atoms are bosons or fermions.

* In order to understand what happens in this regime, called the regime of quantum degeneracy, we should recall some basic ideas of quantum statistical mechanics.

* Remember that bosons are integer-spin particles which may occupy in any number a given energy state, whereas fermions are half-integer-spin particles such that each energy state may be occupied by a single fermion at most (this is the famous Pauli exclusion principle)

We shall perform our discussion in the grand canonical ensemble. The grand-canonical partition function is of the form: (Note: for a derivation see e.g. K. Huang, Statistical Mechanics)

• Bosons: $\mathcal{Z}(z, V, T) = \prod_p (1 - z e^{-\beta \varepsilon_p})^{-1}$

• Fermions: $\mathcal{Z}(z, V, T) = \prod_p (1 + z e^{-\beta \varepsilon_p})$

where $z = e^{\beta\mu}$ → $\beta = 1/k_B T$, $\mu \equiv$ chemical potential

it's called fugacity

and ε_p are the possible energy values.

p are the possible states (e.g. in free space, momentum states).

The thermodynamics of these systems may be obtain from the equation of state $\frac{PV}{k_B T} = \ln \mathcal{Z}(z, V, T)$ (P = Pressure, V = Volume) and the normalization $N = z \frac{\partial}{\partial z} \ln \mathcal{Z}(z, V, T)$

* The last expression may be written as

$$N = \sum_p \langle n_p \rangle$$

where $\langle n_p \rangle$ is the mean occupation of the p-level

$$\boxed{\langle n_p \rangle = \frac{1}{e^{\beta(\varepsilon_p - \mu)} \mp 1}}$$

BOSONS

FERMIONS

* In this lecture we shall consider that the atoms are in 3D free space (we will see later what happens in the presence of a trap). We shall also consider that the gas is ideal, i.e. we neglect for the moment interatomic interactions. In future lectures we will incorporate interactions to the problem, and show that they actually play a major role in the physics of ultra cold gases.

* We will now split our discussion in Bosons and fermions, since, as we will see, the physics is crucially different.

* Ideal Bose Gas : The Bose-Einstein Condensation

* In free space the ground single-particle state is given by $\vec{p}=0$, $\epsilon_{\vec{p}} = p^2/2m \longrightarrow \epsilon_0 = 0$. This is particularly important for the next discussion.

* Remember from the previous discussion that for bosons

$$\frac{PV}{k_BT} = -\sum_{\vec{p}} \ln\left[1 - z\, e^{-\beta\epsilon_{\vec{p}}}\right]$$

$$N = \sum_{\vec{p}} \frac{1}{e^{\beta[\epsilon_{\vec{p}} - \mu]} - 1}$$

Note that something quite remarkable occurs when μ tends to $\epsilon_0 = 0$ (i.e. when $z \to 1$)

(Note: It should be clear that $\mu \leqslant 0$, because otherwise there would be states with negative population, and this obviously can't be!)

* When $\mu \to 0$ the term with $\vec{p}=0$ diverges, i.e. it may become as large as the rest of the sum (or much larger actually). This is actually the key point in the idea of Bose-Einstein condensation as we will see in a while.

* Since $\vec{p}=0$ is rather special, we will split the previous sums into two parts, namely $\vec{p}=0$ and the rest. For the rest we may convert sums into integrals: $\sum_{\vec{p}} \longrightarrow \frac{V}{h^3}\int d^3\vec{p}$. Then:

$$\frac{P}{k_BT} = -\frac{1}{V}\ln(1-z) - \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \ln\left[1 - z\, e^{-\beta p^2/2m}\right]$$

$$\frac{N}{V} = \frac{1}{V}\frac{z}{1-z} + \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \frac{1}{e^{\beta\left(\frac{p^2}{2m}\right)} z^{-1} - 1}$$

* We may perform the integrals to obtain

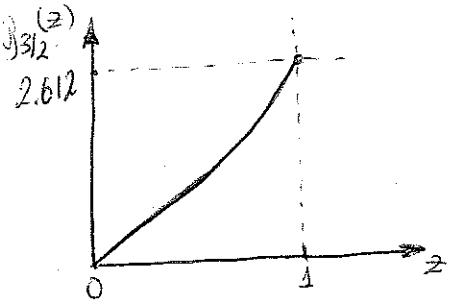
$$\left.\begin{aligned} \frac{P}{k_B T} &= -\frac{1}{V}\ln(1-z) + \frac{1}{\lambda_T^3} g_{5/2}(z) \\ \frac{N}{V} &= \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z) \end{aligned}\right\}$$

where $\lambda_T = \left(\frac{2\pi\hbar^2}{m k_B T}\right)^{1/2}$ is the thermal de Broglie wavelength which we have already met before.

The functions $g_s(z)$ are defined as

$$g_s(z) = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^s}$$

* The behaviour of $g_{3/2}(z)$ is particularly important for our discussion.



The function $g_{3/2}(z)$ is in the interval $0 \le z < 1$ (which is the relevant one for us) a positive and monotonously-growing function of z, with

$$g_{3/2}(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^{3/2}} = \zeta(3/2) = 2.612..$$

↑ Riemann zeta function.

The fact that $g_{3/2}$ reaches a maximum for $z=1$ is crucial.

* Let's call $N_0 = \frac{z}{1-z}$ ⟶ population of the lowest single-particle state

Then $\frac{N}{V} = \frac{N_0}{V} + \frac{1}{\lambda_T^3} g_{3/2}(z)$ ⟶ $\frac{N_0}{V} = \frac{N}{V} - \frac{1}{\lambda_T^3} g_{3/2}(z)$

$$\frac{N_0}{V}\lambda_T^3 = n\lambda_T^3 - g_{3/2}(z)$$

where we define the density $n = N/V$. We can recognize the phase-space density $n\lambda_T^3$.

* Recall that for $0 \le z < 1$ ⟶ $g_{3/2}(z) \le g_{3/2}(1) \simeq 2.612$

For low-enough T or/and large-enough densities one reaches that

$$n\lambda_T^3 > 2.612$$

* When this occurs something very remarkable occurs.
When $n\lambda_T^3$ reaches 2.612, we can't put more particles in states different than $\bar{p}=0$. This is clear because

$$N = N_0 + \underbrace{\frac{V}{\lambda_T^3} g_{3/2}(z)}_{\text{number of particles in } \bar{p}\neq 0}$$

$$\sum_{\bar{p}\neq 0} N_{\bar{p}} < \frac{V}{\lambda_T^3} g_{3/2}(1)$$

* This means that all the other extra particles must necessarily go to the $\bar{p}=0$ state. This is the <u>Bose-Einstein Condensation</u> (BEC)
* The critical temperature for onset of the Bose-Einstein condensation is that in which the maximal possible number of $\bar{p}\neq 0$ particles equals the total number of particles.

$$N = \frac{V}{\lambda_T(T_c)^3} g_{3/2}(1) \rightarrow n\lambda_T(T_c)^3 = 2.612$$

critical phase space density

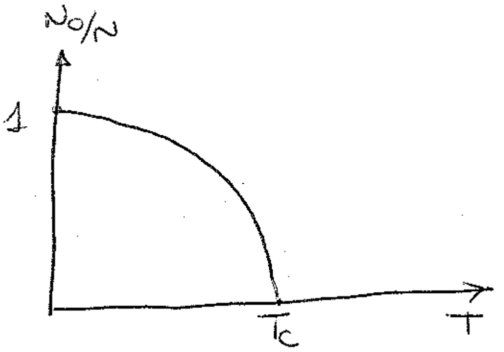
$$\boxed{k_B T_c = \frac{2\pi\hbar^2}{m}\left(\frac{n}{2.612}\right)^{2/3}}$$

critical temperature

* For $T<T_c$, $z=1$ and the number of non-condensed particles is $N_T(T) = \frac{V}{\lambda_T(T)^3} g_{3/2}(1) = \frac{N\lambda_T(T_c)^3}{\lambda_T(T)^3} = N\left(\frac{T}{T_c}\right)^{3/2}$

Hence the condensate fraction is of the form (for $T<T_c$)

$$\boxed{\frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$



At $T=T_c$ one has then a phase transition into a condensed state. This phase transition is very remarkable, since it may occur without interactions. It is of purely quantum origin!

* Note that the BEC is something quite remarkable.
Let's consider a cubic box of length $L \rightarrow V = L^3$. Then

$$k_B T_C = \frac{2\pi\hbar^2}{m}\left[\frac{N/L^3}{2.612}\right]^{2/3} \longrightarrow \left[\frac{2\pi^2\hbar^2}{mL^2}\right]\frac{1}{\pi(2.612)^{2/3}} N^{2/3}$$

Note that $\epsilon_1 = \dfrac{2\pi^2\hbar^2}{mL^2}$ is the energy of the 1st excited state of the box.

Then $\dfrac{k_B T_c}{\epsilon_1} \propto N^{2/3}$.

This means that the condensation temperature is much larger than the separation between the ground-state of the box and the 1st excited state (Note: if this were not the case the effect would not be so remarkable)

. As you may see from our previous discussion a crucial point in the theory is that the number of non-condensed particles (I will call them also thermal particles) ~~at~~ a given temperature is limited

$$N_T \leq \frac{V}{\lambda_T^3(T)} g_{3/2}(1)$$

This is however not always the case and this is particularly important to understand.

. In our previous calculation we considered a 3D Bose gas in a box of volume V. Let's consider now a 2D gas. We will get a similar integral as that at the end of p. 73:

$$\frac{N}{S} = \frac{N_0}{S} + \frac{2\pi}{h^2}\int_0^\infty dp\, p\, \frac{1}{z^{-1}e^{\beta p^2/2m}-1} = \frac{N_0}{S} + \frac{1}{\lambda_T^2}\ln(1-z)$$

surface

Note that now $\ln(1-z)$ plays the role of $g_{3/2}(z)$. However $\ln(1-z)$ is NOT limited between $0 \leq z \leq 1$, and hence the number of $\vec{p} \neq 0$ particles is NOT limited. As a consequence there's NO condensation in 2D free space (in thermodynamic limit $\begin{smallmatrix} N\to\infty \\ S\to\infty \end{smallmatrix}$ but N/S finite)

* In 1D is even worse:

$$\frac{N}{L} = \frac{N_0}{L} + \frac{1}{h}\int_{-\infty}^{\infty} dp \, \frac{1}{z^{-1}e^{\beta p^2/2m}-1} = \frac{N_0}{L} + \frac{1}{\lambda_1(T)} g_{1/2}(z)$$

and again $g_{1/2}(z)$ isn't limited. So there's no condensation in 1D either.

* We may understand the absence of condensation in 1D or 2D in a free gas by having a look into <u>the density of states</u>.

let's remember this important idea.

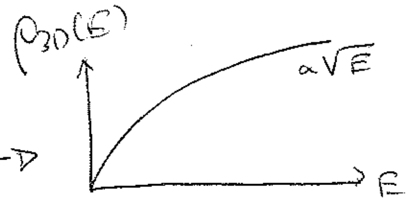
$$\text{In 3D} \to N_T = \int_0^{\infty} \frac{V}{h^3} 4\pi p^2 dp \, \frac{1}{z^{-1}e^{\beta p^2/2m}-1} \overset{E=p^2/2m}{=}$$

(box of $V=L^3$)

$$= \int_0^{\infty} dE \left[\frac{V \cdot 4\pi m\sqrt{2m}}{h^3}\sqrt{E}\right] \frac{1}{z^{-1}e^{\beta E}-1} = \int_0^{\infty} \frac{\rho_{3D}(E)\,dE}{z^{-1}e^{\beta E}-1}$$

where we have introduced the density of states

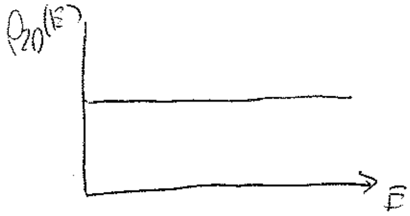
$$\rho_{3D}(E) = 2\pi\left(\frac{mL^2}{2\pi^2\hbar^3}\right)^{3/2}\sqrt{E}$$



So in 3D $\rho_{3D}(E) \propto \sqrt{E}$ $\longrightarrow$

Clearly $\rho(E)$ tends to zero at $E \to 0$, and hence the role of the excited states isn't important

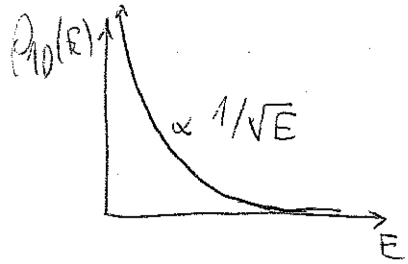
* In 2D we may make a similar calculation and get



$$\rho_{2D}(E) = \frac{mL^2}{2\pi\hbar^2} \longrightarrow \text{constant}$$

So now the excited states are clearly more important. This leads to the logarithmic divergence we found before.

* Finally in 1D



* $$\rho_{1D}(E) = \left(\frac{mL^2}{2\pi\hbar^2}\right)^{1/2} \frac{1}{\sqrt{E}}$$

* The density of states diverges at $E \to 0$ (!!)
* Hence excitations at low energies are terribly important in 1D gases.

* This makes the properties of 1D gases very remarkable

(Note: if we have time I will tell you something about 1D gases at the end of these lectures)

· Bose-Einstein condensation in external traps

* In the previous discussion we ~~[illegible]~~ considered the case of a free gas. Let's consider now a Bose gas in a harmonic trap, which is actually an important case experimentally.
* Let's denote the eigenstates of the harmonic oscillator $\vec{n} = (n_x, n_y, n_z)$ with eigenenergies (I limit myself here for simplicity to the isotropic oscillator):

$$\varepsilon(\vec{n}) = \hbar\omega(n_x + n_y + n_z + 3/2) = \hbar\omega(n + 3/2)$$

(Note: since $\frac{3}{2}\hbar\omega$ is a constant I'll re-absorb it in the definition of chemical potential. Hence $\varepsilon(\vec{n}) \to \hbar\omega(n_x+n_y+n_z) = \hbar\omega n$)

* For p. 72 we have then:

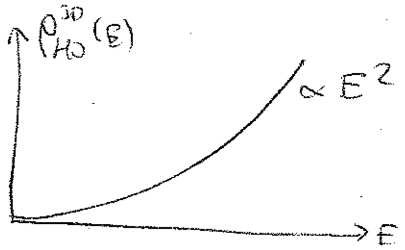
$$N = \sum_{\vec{n}} \frac{1}{z^{-1}e^{\beta\varepsilon(\vec{n})} - 1}$$

We remove the $n=0$ case as we did in p. 73

$$= \sum_{n=0}^{\infty} \underbrace{\sum_{n_x=0}^{n} \sum_{n_y=0}^{n-n_x}}_{(n+1)(n+2)/2} \frac{1}{z^{-1}e^{\beta\varepsilon(n)} - 1} = \frac{z}{1-z} + \sum_{n\neq 0}^{\infty} \frac{(n+1)(n+2)}{2} \frac{1}{z^{-1}e^{\beta\varepsilon(n)} - 1}$$

$$\underset{\frac{k_BT}{\hbar\omega} \gg 1}{\simeq} \frac{z}{1-z} + \int_0^{\infty} dn \, \frac{n^2}{2} \frac{z e^{-\beta\varepsilon(n)}}{1 - z e^{-\beta\varepsilon(n)}} \overset{E = \hbar\omega n}{=} \frac{z}{1-z} + \int_0^{\infty} dE \left[\frac{E^2}{2(\hbar\omega)^3}\right] \frac{1}{-1 + z^{-1}e^{\beta E}}$$

* Note that $\rho^{3D}_{HO}(E) = \frac{E^2}{2(\hbar\omega)^3}$ is the corresponding density of states $\longrightarrow$

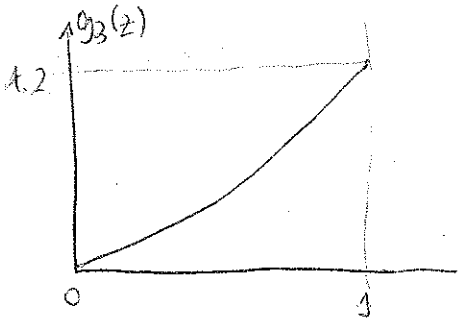


Note that $\rho^{3D}_{HO}(E)$ goes to zero for $E \to 0$.
So we would expect a nice condensation here, and this is exactly what happens.

* Let's see that

$$N_T(T) = \frac{1}{2(\hbar\omega)^3}\int_0^\infty dE\, \frac{E^2}{-1+z^{-1}e^{+\beta E}} = \left(\frac{k_B T}{\hbar\omega}\right)^3 g_3(z)$$

The function $g_3(z)$ looks like this:



So like $g_{3/2}(z)$ for the 3D free gas, $g_3(z)$ is also positive and monotonously growing, being limited by

$$g_3(z) \le g_3(1) = \zeta(3) \simeq 1.2$$

* Proceeding as for the free gas we may obtain the critical temperature for condensation:

$$N = \left(\frac{k_B T_c}{\hbar\omega}\right)^3 g_3(1) \longrightarrow \boxed{k_B T_c = \hbar\omega\left(\frac{N}{1.2}\right)^{1/3}} \quad (\simeq 0.94\,\hbar\omega\, N^{1/3})$$

(Note: if the oscillator isn't isotropic the result is very similar but one must substitute ω by $\bar{\omega} = \sqrt[3]{\omega_x\omega_y\omega_z}$.)

(Note II: Remember that in free gas $T_c \propto N^{2/3}$).

* As for the free gas, for $T<T_c$, $z=1$ and hence the condensate fraction is

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{k_B T}{\hbar\omega}\right)^3 \frac{g_3(1)}{N} = 1 - \left(\frac{T}{T_c}\right)^3}$$

(Recall that for free gas it was $1-(T/T_c)^{3/2}$)

* Off-diagonal long-range order

* To finish our discussion about ideal Bose gases, let's have a look to a very important idea related to the BEC idea (but actually much more general than that). This is the idea of off-diagonal long-range order.

* let's recall the idea of one-body density matrix

$$\rho(\vec r, \vec r') = \langle \hat\psi^+(\vec r)\,\hat\psi(\vec r')\rangle$$

where $\hat\psi^+(\vec r)$, $\psi(\vec r)$ are respective the field operator creating/annihilating a particle at $\vec r$.

* For a non-interacting system, the eigenstates of the density matrix are exactly the eigenstates of the single particle Hamiltonian. $(\varphi_i(\vec r))$

Hence:
$$\rho(\vec r, \vec r') = \sum_i \underbrace{\bar n_i}_{\text{mean occupation}} \varphi_i^*(\vec r)\,\varphi_i(\vec r')$$

For a free gas $\rho(\vec r, \vec r') = \rho(\vec r - \vec r')$ (due to translational invariance).
The eigenstates for a free case are given by plane waves

$$\varphi_{\vec p}(\vec r) = \frac{1}{\sqrt V} e^{i\vec p\,\vec r/\hbar}$$

* Hence for

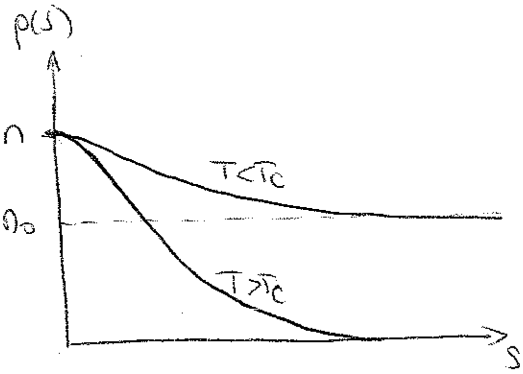
$$T > T_c \longrightarrow \rho(s) = \frac{1}{h^3}\int d^3p\, \frac{e^{i\vec p\cdot\vec s/\hbar}}{z^{-1}e^{\beta p^2/2m}-1}$$

$$T < T_c \longrightarrow \rho(s) = n_0 + \frac{1}{h^3}\int d^3p\, \frac{e^{i\vec p\cdot\vec s/\hbar}}{e^{\beta p^2/2m}-1}$$ with $n_0 = \frac{N_0}{V}$ ← condensate density

* It's quite interesting to discuss the behaviour of $\rho(s)$ for large s:

• $T > T_c \longrightarrow \rho(s) \simeq n\, e^{-\pi s^2/\lambda_T^2}$

• $T < T_c \longrightarrow \rho(s) \simeq n_0 + \dfrac{1}{(2\pi)^3 \lambda_T^3} \dfrac{1}{s}$



Hence, there's a dramatic difference.

• For $T > T_c$, $\rho(s)$ decays exponentially down to zero (actually like a gaussian)

• For $T < T_c$, $\rho(s)$ decays a $1/s$ but it doesn't go to zero, but to a finite value. This finite value is the so-called <u>off-diagonal long-range order (ODLRO)</u> which coincides with the condensate density n_0.

• As I mentioned above this is a really very important idea, with applications to other systems, and which holds also in the presence of interactions.

In the presence of interaction $\rho(\vec{r},\vec{r}\,') = \sum_i n_i \chi_i^*(\vec{r}) \chi_i(\vec{r}\,')$ where the eigenfunction of the density matrix aren't those of the single-particle Hamiltonian, but the so-called <u>natural states</u>. BEC occurs when one of the natural states $(i=0)$ acquire a population $n_0 \simeq n$.

* We will discuss later in these lectures the role of interactions.

* IDEAL FERMI GAS

* let us discuss now the case of an ideal (spinless) Fermi gas in a 3D free space. Remember that

$$\frac{PV}{K_BT} = \sum_{\bar{p}} \ln\left[1+ze^{-\beta\epsilon_{\bar{p}}}\right] \xrightarrow[\sum_{\bar{p}} \to \frac{V}{h^3}\int d^3\bar{p}]{\text{passing to the continuum}} \frac{P}{K_BT} = \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \ln\left[1+ze^{-\beta p^2/2m}\right]$$

$$N = \sum_{\bar{p}} \frac{1}{z^{-1}e^{\beta\epsilon_{\bar{p}}}+1} \longrightarrow \frac{N}{V} = \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \frac{1}{z^{-1}e^{\beta p^2/2m}+1}$$

Note that for Fermi gases there's no singular behaviour at $\bar{p}=0$. Actually contrary to the case of bosons z isn't limited in the interval $0 \le z \le 1$, but on the contrary it may attain all positive values.

* We may perform the previous integrals to obtain

$$\left.\begin{aligned} \frac{P}{K_BT} &= \frac{1}{\lambda_T^3(T)} f_{5/2}(z) \\ \frac{N}{V} &= \frac{1}{\lambda_T^3(T)} f_{3/2}(z) \end{aligned}\right\}$$

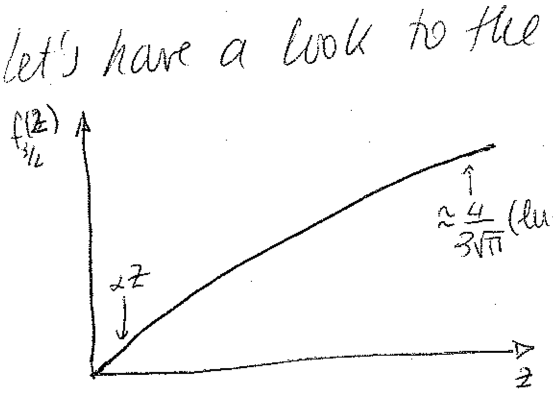
where $f_s(z) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} z^\ell}{\ell^s}$

Note again the appearance of the thermal de Broglie-wavelength.

* The second equation may be expressed in terms of the phase-space density

$$n\lambda_T^3(T) = f_{3/2}(z)$$

let's have a look to the function $f_{3/2}(z)$:



* The function $f_{3/2}(z)$ is monotonously growing with z.
* Since $f_{3/2}(z) = n\lambda_T^3(T)$, then for $n\lambda_T^3(T)$ sufficiently large, it's the large z behaviour the one that dominates.

* Without entering into the details, one can show that for large z:

$$n\lambda_T^3(T) = f_{3/2}(z) \simeq \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2} \longrightarrow \ln z \simeq \left[\frac{3\sqrt{\pi}\, n\lambda_T^3(T)}{4}\right]^{2/3}$$

* And thus: $z = e^{\left[\frac{3\sqrt{\pi}}{4} n \lambda_T^3(T)\right]^{2/3}}$

let's have a closer look to the exponent

$$\left[\frac{3\sqrt{\pi}}{4} n \lambda_T^3(T)\right]^{2/3} = \left[\frac{3\sqrt{\pi}}{4} n \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}\right]^{2/3} = \beta \frac{\hbar^2}{2m}\left(6\pi^2 n\right)^{2/3}$$

where as always $\beta = 1/k_BT$

Hence $z \cong e^{\beta\epsilon_F}$, where $\epsilon_F = \frac{\hbar^2}{2m}(6\pi^2 n)^{2/3}$ is the Fermi energy

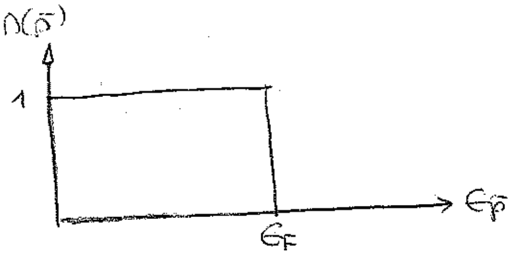
* Since $z = e^{\beta\mu}$, this means that for a sufficiently large $n\lambda_T^3$, then $\mu \simeq \epsilon_F$

* Hence $\langle n_{\vec{p}}\rangle \cong \dfrac{1}{e^{\beta(\epsilon_p - \epsilon_F)}+1}$

The distribution is quite remarkable when $T \to 0$ (i.e. $\beta \to \infty$).

In that case

* For $\epsilon_p < \epsilon_F \longrightarrow \langle n_{\vec{p}}\rangle = 1$
* For $\epsilon_p > \epsilon_F \longrightarrow \langle n_{\vec{p}}\rangle = 0$



* The physical meaning of ϵ_F is then quite clear. Due to the Pauli exclusion principle one cannot place 2 particles at the same level (we consider for the moment spinless particles). As a consequence at $T=0$ the particles fill the lowest modes completely until reaching ϵ_F.

This allow us to calculate ϵ_F in an alternative way.

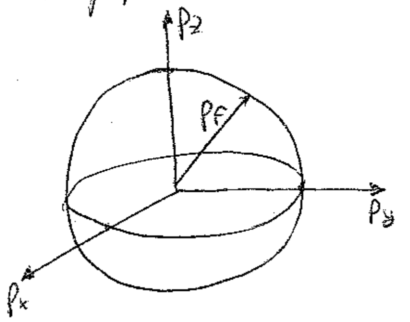
At $T=0$:

$$\frac{N}{V} = \frac{1}{h^3}\int_{\epsilon_p<\epsilon_F} d^3p \overset{\epsilon_F = p_F^2/2m}{=} \frac{4\pi}{h^3}\int_0^{p_F} dp\, p^2 = \frac{1}{h^3}\frac{4\pi}{3}p_F^3 \rightarrow \frac{p_F}{h} = \left(\frac{3}{4\pi}n\right)^{1/3}$$

$$\longrightarrow \epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m}\left(6\pi^2 n\right)^{2/3}$$ as we had before.

p_F is the so-called Fermi momentum

* Hence, in momentum representation, the particles fill a sphere of radius p_F



This is the so-called Fermi-sphere, and it's surface it's (as you can imagine) the <u>Fermi surface</u>.

* The ideas of Fermi energy and Fermi surface are extremely important in physics!

* <u>Note</u>: If one has particles with a spin s, then one must take into account that each $\bar{p}$ state may be occupied by at most $(2s+1)$ particles (i.e. from $m=-s$ to $m=s$). This must be taken into account in the corresponding partition function. Basically it just amounts for substituting n by $n/(2s+1)$. Hence

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{2s+1}n\right)^{2/3} \xrightarrow[\text{for } s=1/2]{} \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$

Note that n is the total number density in all spin states)

* Let's see what happens at finite temperature. Without entering into all details one may obtain that for low temperatures

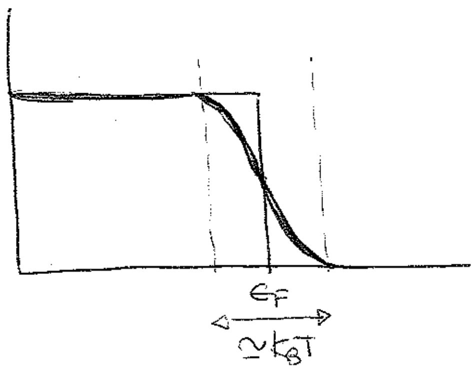
$$\mu \simeq \epsilon_F \left\{1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right\}$$

where we have introduced the important idea of <u>Fermi Temperature</u>

$$T_F = \epsilon_F / k_B$$

* This temperature is very important, because it actually sets the scale for what we call low temperatures. For $T < T_F$ one enters into the so-called <u>Fermi degeneracy</u>. This regime is characterized by a large z, and hence it's the regime that we have discussed above.

* Since for finite T, μ isn't exactly ϵ_F, then the borders of the Fermi distribution



$$\langle n(\vec{p}) \rangle_{T \neq 0} = \frac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1}$$

are not any more sharp but blurred in an energy scale $\simeq k_B T$.

* Note that for $T < T_F$ only the border at the Fermi surface reacts against temperature.

Actually for a Fermi degenerate gas the physics concentrates basically at the Fermi surface. Deep inside the Fermi sphere (the so-called Fermi sea) the gas is basically "dead", stalled into $n_p = 1$.

* As a final ~~comment~~ point I would like to comment on how pressure behaves in a degenerate Fermi gas.

One may show that

$$P \simeq \frac{2}{5} n \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{T}{T_F} \right)^2 \right]$$

Hence for zero temperature we have a zero-point pressure $\frac{2}{5} n \epsilon_F$.

This is because only one particle (or at most 2s+1) have $\bar{p} = 0$. The most the others have $\bar{p} \neq 0$ and hence exert a pressure. This must be compare with a pure BEC, with all atoms at $\bar{p} = 0$. In that case pressure dissapears at $T = 0$.

* This concept is particularly important in trapped gases. Typically a Fermi gas in a trap can contract beyond a given volume due to this zero-point pressure. Actually this was one of the ways of probing Fermi degenerancy in experiments.