SUPERFLUIDITY

LANDAU'S CRITERION

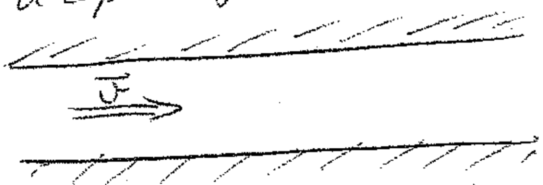
Before starting with Landau's theory of superfluidity, let's recall the transformation laws of energy and momentum under <u>Galilean transformations</u>.
Let E and $\vec{p}$ the energy and momentum of a fluid in a reference system K.
Let's consider a second reference system K' moving with velocity $\vec{V}$ relative to K. For the system K'

$$E' = E - \vec{p}\cdot\vec{V} + \frac{MV^2}{2} \quad \text{and} \quad \vec{p}\,' = \vec{p} - M\vec{V}$$

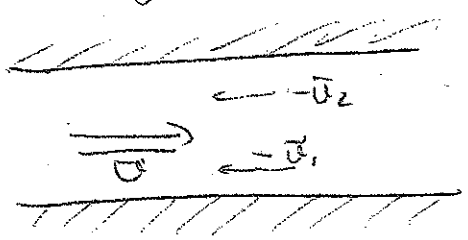
like the Doppler effect

where M is the total mass of the fluid.
We will use these results in a second.

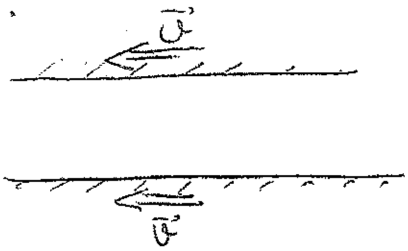
Let us now consider an uniform fluid at zero temperature flowing along a capillary at constant velocity $\vec{v}$



If the fluid is viscous then the motion will produce energy dissipation with the consequent heating and decrease of kinetic energy.
One can understand this process as taking place through the creation of elementary excitations on top of the fluid moving against it



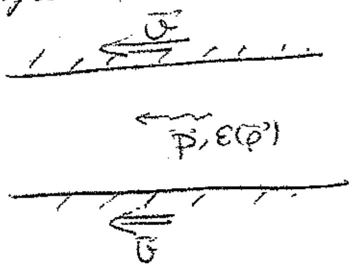
* Let's now describe the process in the reference system moving with the fluid.



Now the fluid is at rest, and the walls are moving

Let $E_0 \equiv$ energy of the ground state (with no excitation)
In absence of any excitation the fluid is at rest (in this reference frame) and hence $\vec{P}_0 = 0$

Let's consider now that a single excitation with momentum $\vec{p}$ appears. The momentum of the system becomes $\vec{P} = \vec{p}$, and the energy is $E = E_0 + \epsilon(\vec{p})$, where $\epsilon(\vec{p})$ is the dispersion.



(Remember that for a Bose-Einstein condensate $\epsilon(\vec{p})$ is the Bogoliubov spectrum)

Let's go back now to the reference frame where the capillary is at rest. This system moves with a velocity $-\vec{v}$ when compared to the system at which the fluid is at rest. Hence, by applying the galilean transformation, we obtain:

$$E' = E_0 + \frac{1}{2}Mv^2 + \epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$$

$$\vec{P}' = \vec{P} + M\vec{v}.$$

Hence by creating one excitation on top of the fluid we have changed the momentum by $\vec{p}$ and the energy by an amount $\epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$.

· The crucial point is that the spontaneous creation of excitations (linked with the flow dissipation) is just possible if this change of energy is negative;

$$\epsilon(\vec{p}) + \vec{p}\cdot\vec{v} < 0$$

i.e. if the energy is reduced by creating the excitation.

· Note that the most favourable situation for creating the excitation occurs when $\vec{p}$ and $\vec{v}$ are counterpropagating, i.e. when $\vec{p}\cdot\vec{v} = -pv$.

Hence we arrive to $\epsilon(\vec{p}) - pv < 0 \longrightarrow v > \frac{\epsilon(\vec{p})}{p}$

If this is the case the flow is unstable and its kinetic energy will be transformed into heat.

But: if the velocity v is smaller than

$$v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{p}$$

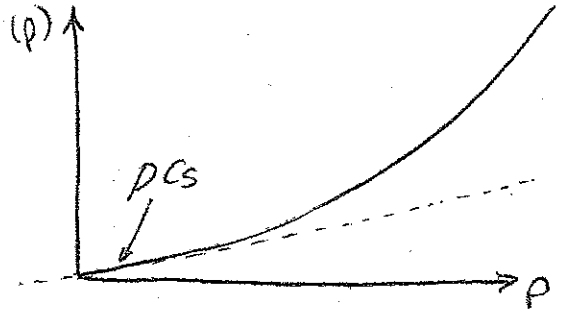
Then the condition for the creation of excitations is never fulfilled and no excitation will spontaneously grow in the fluid. the flow presents no dissipation. As the consequence of that, i.e. the fluid becomes <u>SUPERFLUID</u>

<u>Landau's criterion for superfluidity</u> can then be written in the form

$$\boxed{v < v_c = \min_{\vec{p}} \frac{\epsilon(\vec{p})}{p}}$$

* let's see now that the weakly-interacting Bose gas fulfills the Landau criterion for superfluidity.

. Remember that for a weakly interacting Bose gas we found that the dispersion law $\epsilon(\vec{p})$ was provided by the Bogoliubov spectrum



$$\epsilon(\vec{p}) = \left[\left(\frac{p^2}{2m}\right)^2 + (p c_s)^2\right]^2$$

where $c_s \equiv$ sound velocity

. let's apply now the Landau criterion.

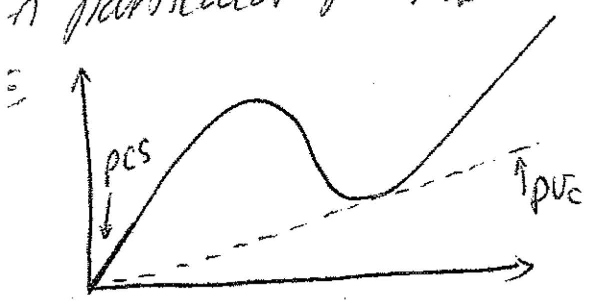
It's clear that $\min_p \frac{\epsilon(p)}{p} = c_s$ (see the dashed line in the figure)

Hence the critical velocity is the sound velocity $\boxed{v_c = c_s}$

For other superfluids the dispersion law is different.

For other superfluids the dispersion law looks like this

In particular for ~~helium~~ ^4He, the dispersion law looks like this



where the minimum is the so called roton minimum

If we apply Landau's criterion we observe that now $v_c < c_s$ because it's the roton minimum which determines the critical superfluid velocity.

Note that according to the Landau criterion an ideal Bose gas is NOT superfluid, since $\epsilon(\vec{p}) = p^2/2m$, and hence $v_c = 0$.

* Two-fluid model

* The previous discussion is basically at zero Temperature. We will consider now an uniform fluid at finite small temperature. The thermodynamics will be given by that of a non-interacting gas of quasiparticles.
* The additional mass flow associated to thermally-excited quasiparticles is not superfluid. These excitations can collide with the walls, and hence the corresponding motion shows dissipation.
* Thus we have like 2 different fluids at $T \neq 0$:
  * A normal viscous fluid
  * A superfluid without viscosity

Note: They are not actually 2 physically distinguishable fluids, as we would have in a mixture. The model is more a way of conceptually approach the problem.)

* The gas of excitations is in thermodynamic equilibrium, with a velocity $\vec{v}_n$ equal to the velocity of the frame in which the capillary is at rest. The superfluid velocity $\vec{v}_s$ and $\vec{v}_n$ are not the same.

Note: we will relate $\vec{v}_s$ and the condensate phase in a moment).

Remember that the dispersion law $\epsilon(\vec{p})$ (i.e. the Bogoliubov spectrum for the weakly-interacting Bose gas) is in the frame where the superfluid is at rest. Hence in the original frame (for $v_n = 0$), we have $\epsilon(\vec{p}) \longrightarrow \epsilon(\vec{p}) + \vec{p} \cdot \vec{v}_s$. Since now the gas of excitations has a velocity $\vec{v}_n$ equal to the velocity of the capillary frame, then in the capillary frame the energy of elementary excitations is given by

$$\epsilon(\vec{p}) + \vec{p} \cdot (\vec{v}_s - \vec{v}_n).$$

This is the energy that comes in the expression for the equilibrium distribution function $N_{\vec{p}}$ of elementary excitations.

$$N_{\vec{p}} = \frac{1}{e^{\left[\frac{\epsilon(\vec{p}) + \vec{p}\cdot(\vec{v}_s - \vec{v}_n)}{k_B T}\right]} - 1}$$

(note that only $|\vec{v}_s - \vec{v}_n| < v_c$ allows $N_{\vec{p}} > 0$ for all $\vec{p}$. That's once more Landau's criterion)

* According to the two-fluid model the total mass density

$$\rho = \underbrace{\rho_s}_{\text{superfluid mass density}} + \underbrace{\rho_n}_{\text{normal mass density}}$$

(Note: we introduce the mass density ρ as $\rho = m\,n$ ← particle density; m ← particle mass)

* The mass current is hence

$$m\vec{J} = \rho_s \vec{v}_s + \rho_n \vec{v}_n$$

* Remember from p. 123 that the total momentum carried by the fluid may be written as

$$\vec{P} = M\vec{v}_s + \underbrace{\vec{P}'}_{\sum_i \vec{p}_i}$$

← sum over excitations.

If we re-write this equation per unit volume:

$$m\vec{J} = \rho\vec{v}_s + \int \frac{d^3p}{(2\pi\hbar)^3}\, \vec{p}\, N_{\vec{p}}$$

Comparing the two expressions for $m\vec{J}$ we get that

$$\rho_n(\vec{v}_n - \vec{v}_s) = \int \frac{d^3p}{(2\pi\hbar)^3}\, \vec{p}\, N_{\vec{p}}$$

Let's multiply by $\vec{v} = \vec{v}_n - \vec{v}_s$

$$\rho_n(v) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\vec{p}\cdot\vec{v}}{v^2} \frac{1}{e^{\left[\frac{\epsilon(\vec{p}) - \vec{p}\cdot\vec{v}}{k_B T}\right]} - 1}$$

- Let $N_0(\vec{p}) = \dfrac{1}{e^{\varepsilon(p)/k_BT}-1}$

~~Let~~'s assume tha v is small, then

$$N(\vec{p}) \cong N_0(\vec{p}) - (\vec{p}\cdot\vec{v})\frac{dN_0}{d\varepsilon(p)}$$

Then:

$$\rho_n(v) \simeq \underbrace{\int \frac{d^3p}{(2\pi\hbar)^3}\frac{\vec{p}\cdot\vec{v}}{v^2}N_0(p)}_{\text{This is zero due to parity of } N_0(p)} - \int\frac{d^3p}{(2\pi\hbar)^3}\underbrace{\frac{(\vec{p}\cdot\vec{v})^2}{v^2}}_{p^2\cos^2\theta}\underbrace{\frac{dN_0}{d\varepsilon(p)}}$$

This is zero due to parity of $N_0(p)$ (Note that $\vec{p}\cdot\vec{v}$ is odd whereas $N_0(p)$ is even)

This just depends on $\vec{p}$

$\vec{p}^2\cos^2\theta$

Note that $\int_0^\pi d\theta \sin\theta\cos^2\theta = \dfrac{2}{3}$
$= \dfrac{1}{3}\int_0^\pi d\theta\sin\theta$

Hence

$$\boxed{\rho_n(T) = -\frac{1}{3}\int\frac{d^3p}{(2\pi\hbar)^3}\,p^2\,\frac{dN_0}{d\varepsilon(p)}}$$

This is the central equation of Landau's theory of superfluidity. It allows us to calculate the normal component in terms of the energy spectrum of elementary excitations. Note that it doesn't depend any more on v. It's a universal function of the temperature T. (and c_s)

- Note that $\dfrac{dN_0}{d\varepsilon} = \dfrac{-1}{k_BT}\dfrac{e^{\varepsilon/k_BT}}{(e^{\varepsilon/k_BT}-1)^2}$

In the integral of $\rho_n(v)$ only the energies $\varepsilon \sim k_BT$ contribute. If $k_BT \ll \mu$, then only energies $\varepsilon \ll \mu$ contribute, this means that only phonon-like excitations $\varepsilon(p) = c_s p$ contribute.

Hence $dp\,p^4 = \dfrac{1}{c_s^5}d\varepsilon\,\varepsilon^4$

$$\rho_n(\omega) \simeq +\frac{1}{3}\,\frac{4\pi}{C_s^5}\int_0^\infty \frac{dE\, E^4}{(2\pi\hbar)^3 K_B T}\,\frac{e^{\epsilon/K_BT}}{(e^{\epsilon/K_BT}-1)^2}$$

$$= \frac{(K_BT)^4}{6\pi^2\hbar^3 C_s^5}\int_0^\infty dx\, x^4\,\frac{e^x}{(e^x-1)^2} = \frac{2\pi^2}{45}\,\frac{(K_BT)^4}{\hbar^3 C_s^5}$$

Therefore at low T, the normal component dissappears as $\sim T^4$.

* The previous expression tells us something interesting about the typical wavenumber of thermally excited phonons. Note that the typical wavenumber of a thermal phonon is $K_T \sim \frac{K_BT}{\hbar C_s}$ (this is because $\epsilon = \hbar K C_s = K_BT$). The volume of a sphere in wavenumber space having a radius K_T tells us the density of excitations (note: volume in K-space is density in position space).

$$\text{Then } \underbrace{\rho_n(T)}_{\text{normal mass density}} \sim \underbrace{\left(\frac{K_BT}{C_s^2}\right)} \underbrace{\left(\frac{K_BT}{\hbar C_s}\right)^3}_{\text{density of excitations}}$$

This plays hence the role of the mass of the thermal phonon!!

Since $mC_s^2 = \mu$ and $K_BT \ll \mu$ $\Rightarrow$ $\frac{K_BT}{C_s} \ll m$

The "mass" of the thermal excitation is much less than the atomic mass!!

* Clearly when K_BT approaches μ, the mass of the excitation becomes of the order of m.

- For $K_BT \gg \mu$ $\to$ the single particle limit dominates and hence $\epsilon(p) \simeq p^2/2m$.

then:

$$\rho_n(T) = -\frac{4\pi}{3}\int \frac{dp\, p^4}{(2\pi\hbar)^3}\left(\frac{dN_0}{d\varepsilon}\right) = $$

$\varepsilon = p^2/2m$
$p\,dp = m\,d\varepsilon$
$p = (2m\varepsilon)^{1/2}$

integration by parts

$$= \frac{-4\pi}{3(2\pi\hbar)^3}\, m \int_0^\infty d\varepsilon\, (2m\varepsilon)^{3/2}\left(\frac{dN_0}{d\varepsilon}\right) =$$

$$= \frac{4\pi}{3(2\pi\hbar)^3}\, m \int_0^\infty d\varepsilon\, N_0(\varepsilon)\, \frac{3}{2}(2m)^{3/2}\varepsilon^{1/2} d\varepsilon$$

$$= \frac{4\pi}{\cancel{3}(2\pi\hbar)^3}\, \cancel{3} m \int_0^\infty N_0(p)\, p^2 dp$$

$$= m \underbrace{\int \frac{d^3p}{(2\pi\hbar)^3}\, N_0(p)}_{\text{This is simply the number of thermal particles (i.e. the depletion of the condensate)}}$$

Hence

$$\boxed{\rho_n(T) = m\, n_T}$$

and, in this case, normal part and thermal depletion of the condensate coincide.

(Note: in this calculation we neglect interactions between the excitations and hence it's just valid sufficiently below T_c).

* The relation between superfluidity and BEC is however not one-to-one. In particular, as we have already seen, at $T=0$ the gas is purely superfluid $\rho = \rho_s$, whereas we know already (p. 105) that interactions lead always to a quantum depletion of the condensate. In the following we will explore this particularly important point.

• THE RELATION BETWEEN BEC AND SUPERFLUIDITY

• To understand this important connection it is useful to consider the properties of the condensate wavefunction Ψ_0 under galilean transformations.

• Let's consider the uniform case $V_{ext} = 0$. In the coordinate system where the sample is in equilibrium

$$\Psi_0 = \sqrt{n_0}\, e^{-i\mu t/\hbar}$$ (remember p. 108)

In the frame where the fluid moves with velocity $\vec{v}$

$$\Psi_0 = \sqrt{n_0}\, \exp\left\{-\frac{it}{\hbar}\left(\mu + \frac{mv^2}{2}\right) + \frac{im}{\hbar}\vec{v}\cdot\vec{r}\right\}$$

(remember the energy and momentum transformation on p. 127)

$$= \sqrt{n_0}\, e^{iS(\vec{r},t)}$$

Hence the velocity is proportional to $\vec{\nabla} S$. We can then identify the superfluid velocity

$$\boxed{\vec{v}_s = \frac{\hbar}{m}\vec{\nabla} S}$$

Remember from p. 114 that this is the flow velocity we got in our hydrodynamic calculations.

* In deriving this result we have just assumed that there's a classical field Ψ_0 (i.e. the condensate wavefunction) associated with the macroscopic component of the field operator $\hat{\Psi}$. (this is irrespective of whether the gas is dilute or not or at finite temperature).

* The identification of the superfluid velocity with the gradient of the phase of the order parameter represents a key relationship between BEC and superfluidity!

* However, this relationship doesn't involve the modulus of Ψ_0. It's certainly wrong to identify the condensate (mass) density $m|\Psi_0|^2$ and the superfluid (mass) density ρ_s. They are two different things!!

As we mentioned above, at $T=0$ $\rho=\rho_s$ but there's quantum depletion.

This is particularly dramatic in ^4He, where at $T\to 0$ the whole gas is superfluid, but only 10% is condensed (in ^4He the interactions are much stronger and the depletion much larger).

* In the following we will have a look to one of the most significant consequences of superfluidity: the appearance of quantized vortices.

(Note: there are many other striking consequences of superfluidity like e.g. the reduction of the moment of inertia)

Quantized vortices / Quantized circulation

• We have mentioned that the superfluid velocity is irrotational. Hence a superfluid cannot rotate as an ordinary liquid. If we start to stirr the condensate at low angular velocities the superfluid will remain at rest. However for a sufficiently large angular velocity such at state at rest becomes energetically unfavourable. Let's see why (although it seems intuitively almost obvious).

• Let's move to the reference in which the stirr is at rest (rotating frame). The energy becomes

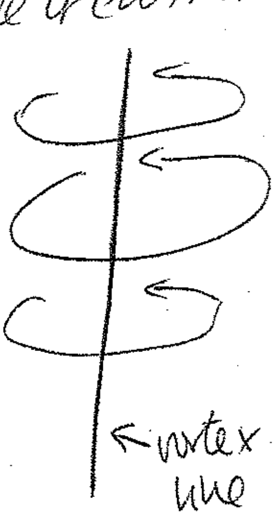
$$E \longrightarrow \underset{\uparrow \text{lab. frame}}{E} - \vec{\Omega}\cdot\vec{L}$$

$\vec{\Omega} \to$ angular velocity of the stirr
$\vec{L} \to$ angular momentum (in the laboratory frame)

We have to minimize this energy.

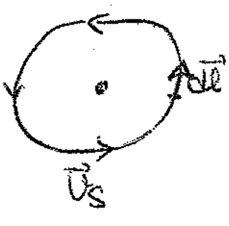
It's clear that for sufficiently large Ω, with $\vec{\Omega}\cdot\vec{L} > 0$, it's better to have angular momentum $\neq 0$ than to have $L=0$ (which is what you would have at rest)

• But, as commented already, since a superfluid cannot rotate in a rigid way, the rotation will eventually be realized through the creation of quantized vortex lines.



• Let's see first why I call these lines quantized.

• Let's see the vortex line from the top.



← length element-line of the contour

* The circulation of the superfluid velocity around a contour around the vortex line is defined as:

$$\Gamma = \oint_{\text{CONTOUR}} \vec{v}_S \cdot d\vec{l} = \oint \frac{\hbar}{m} \vec{\nabla} S \cdot d\vec{l}$$

$\oint \vec{\nabla} S \cdot d\vec{l} \longrightarrow$ tells us how much does the condensate phase when moving around the vortex and coming back to the initial point.

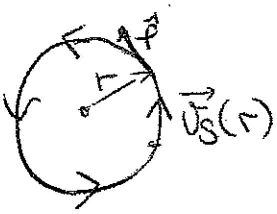
But since the condensate wavefunction $\psi_0(\vec{r},t)$ is univalued then the variation of the phase must be a multiple of 2π: $\Delta S = 2\pi s$

Hence $\Gamma = \left(\frac{2\pi\hbar}{m}\right) s$

So this means that the circulation is quantized in multiples of $\left(\frac{2\pi\hbar}{m}\right) \rightarrow$ This is what we mean by quantized vortices.

* let's consider a straight vortex line. The velocity $\vec{v}_S$ is given by circles lying on the plane perpendicular to the line. So from the top

Since $\oint_{\substack{\text{circle}\\\text{of radius}\\ r}} \vec{v}_S \cdot d\vec{l} = v_S^{(r)} 2\pi r = 2\pi \frac{\hbar}{m} s$

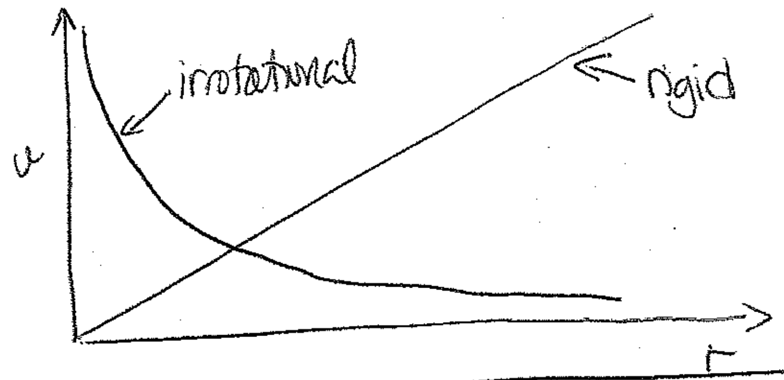


then $\boxed{v_S = s\frac{\hbar}{mr}} \longrightarrow \vec{v}_S = v_S\,\hat{\varphi}$

Note: $\hat{\varphi} \longrightarrow$ tangential vector (see figure)

* Compare this with the rigid body expression.

For a rigid body $\vec{v} = \vec{\Omega} \times \vec{r} \longrightarrow v = \Omega r$



The difference is very significant indeed!!

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* THE VORTEX CORE. CRITICAL ANGULAR VELOCITY

* Remember that $\vec{v}_S = \frac{\hbar}{m} s \frac{1}{r} \hat{\varphi} = \vec{\nabla} S = \frac{1}{r}\frac{dS}{d\varphi}\hat{\varphi}$

Hence the condensate phase is $S = s\varphi$

Therefore the condensate wavefunction is

$$\psi_0(\vec{r}) = e^{is\varphi} |\psi_0(\vec{r})|$$

* Let's put this into the stationary GPE. We get

$$\mu|\psi_0| = -\frac{\hbar^2}{2m}\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]|\psi_0| + \underbrace{\frac{\hbar^2 s^2}{2mr^2}|\psi_0|}_{\text{centrifugal barrier}} + g|\psi_0|^3$$

Far away from the vortex ($r \to \infty$) it's clear that we recover the uniform density $|\psi_0| \longrightarrow \sqrt{n}$.

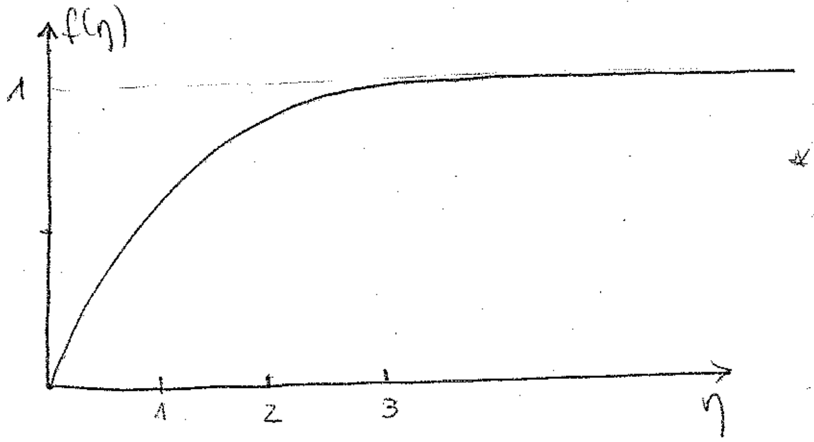
Let's proceed like in our discussion of the boundary effects in the box potential.

We substitute (dimensionless form) $r \longrightarrow r/\xi \equiv \eta$, $|\psi_0| \longrightarrow \sqrt{n}\, f(\eta)$ } $\xi = \sqrt{\frac{\hbar^2}{2mgn}}$ is the healing length (p. 104)

· With this substitution we obtain a dimensionless equation for the vortex core:

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\eta\frac{df}{d\eta}\right]+\left(1-\frac{s^2}{\eta^2}\right)f-f^3=0 \qquad \begin{pmatrix}\text{with } f(\infty)=1\\ f(0)=0\end{pmatrix}$$

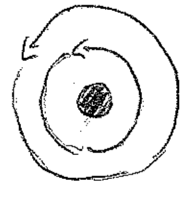
the solution of this equation looks like this (for $s=1$)
singly-quantized vortex



* You can see that the typical size of the vortex core is provided by the healing length ξ

(Note: again we have done a perturbation, like in our discussion of p. 112, this time the vortex singularity, and this perturbation heals after some distance $\sim\xi$).

* So, if you see from the top you see a hole in the density



* We may calculate the energy of the vortex by inserting $\Psi_0(\vec{r})=e^{is\varphi}|\Psi_0(\vec{r})|$ into the energy functional:

$$E=\int\left[\frac{\hbar^2}{2m}|\vec{\nabla}\Psi_0|^2+V_{ext}(\vec{r})|\Psi_0|^2+\frac{g}{2}|\Psi_0|^4\right]d^3r$$

and substracting from this expression the ground state energy of the uniform gas occupying a cylinder of volume $V=\pi R^2 L$

- $E_g = \frac{V}{2} g n^2$
- the corrections to the chemical potential due to the core are negligible $\rightarrow \mu \simeq gn$.

Using the variable η and the function $f(\eta)$ we get

$$E_v = \frac{L\pi \hbar^2 n}{m} \int_0^{R/\xi} d\eta \, \eta \left[\left(\frac{df}{d\eta}\right)^2 + \frac{s^2}{\eta^2} f^2 + \frac{1}{2}(f^2-1)^2 \right]$$

For singly quantized vortices ($s=1$)

$$\boxed{E_v = L\pi n \frac{\hbar^2}{m} \ln\left[1.46 \frac{R}{\xi}\right]}$$

Note that the vortex energy is macroscopically large $\sim L \ln R$ but E_v/E_g is vanishingly small when $R \to \infty$. (in accordance with our assumption that $\mu \simeq gn$).

* In a frame rotating with angular velocity $\Omega \vec{e}_z$ one should consider $H = H_0 - \Omega L_z$

H_0: in lab frame; L_z: angular momentum in lab frame.

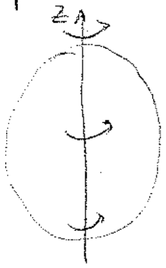
In the rotating frame, the vortex solution (which carries $L_z = N\hbar$ for $s=1$) becomes energetically favorable with respect to the ground state solution of H_0 if $\Omega > \Omega_c$ with

$$\boxed{\Omega_c = \frac{E_v}{N\hbar} = \frac{\hbar}{mR^2} \ln\left(\frac{1.46 R}{\xi}\right)}$$

* Due to the quantization of the circulation one needs a finite angular velocity to induce a vortex. This is contrary to a usual fluid where $\Omega_c \to 0$. Note that $\Omega_c \sim \hbar$ (this tells you that here is Quantum mechanics at work!)

• VORTICES IN HARMONIC TRAPS

* Up to now our discussion about vortices was reduced to the homogeneous case (cylinder). If one has a trap the situation is rather similar. Let's consider the simplest case of an axially-symmetric trap. We look for a singly quantized vortex at the trap center and stretched along z:



$$\psi_0(\vec{r}) = \psi_v(r_\perp, z)\, e^{i\varphi}$$

The G.P.E equation for ψ_v becomes:

$$\left[-\frac{\hbar^2\nabla^2}{2m} + \frac{\hbar^2}{2m r_\perp^2} + \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2) + g\,\psi_v^2(r_\perp,z)\right]\psi_v(r_\perp,z) = \mu\,\psi_v(r_\perp,z)$$

As for the uniform case the centrifugal term leads to zero density along the z axis. The size of the vortex core can be shown by using the central value of the density $n(r_\perp=0,z)$ in the absence of vortex.

* In the absence of vortex in the Thomas-Fermi regime (p. 117):

$$n_0(r_\perp,z) = n_0\left(1 - \frac{z^2}{Z^2} - \frac{r_\perp^2}{R_\perp^2}\right)$$

We can then define the <u>local healing length</u> (p. 109)

$$\xi_z = (8\pi a\, n_0(0,z))^{-1/2}$$

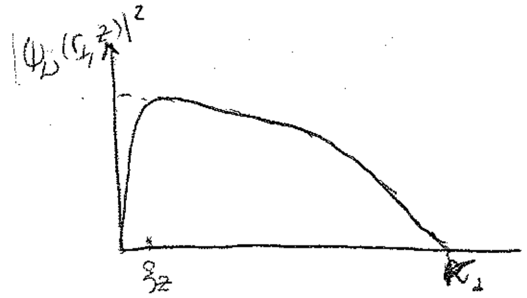
* Since the condensate size (the Thomas-Fermi radii) is much larger than the healing length, the density profile is very smooth and we can solve for each z the GPE on the xy plane.

For $r_\perp \ll R_\perp(z) = R_\perp\sqrt{1 - z^2/Z^2}$ we can treat the system as an uniform gas and use

$$\psi_v(r_\perp,z) = \sqrt{n_0(r_\perp,z)}\; f\!\left(\frac{r_\perp}{\xi_z}\right)$$

where $f(\rho)$ is the same universal wavefunction characterizing the core in homogeneous media (p. 137). Note that for $r_\perp \gg \xi_z \to \psi_0(r_\perp,z) \simeq \sqrt{n_0(r_\perp,z)}$

- Then we have now



We may then insert the expression of $\Psi_v(r_\perp,z)$ into the energy functional to obtain (after substracting the energy without vortex) the energy of the vortex:

$$E_v = \frac{4\pi}{3} n_0 Z \frac{\hbar^2}{m} \ln\left[0.67 \frac{R_\perp}{\xi}\right] \qquad \text{with } \xi = (8\pi a n_0(0,0))^{-1/2}$$

This result turns out to be in good agreement with the numerical calculations for large N.

- We may then calculate the critical rotation frequency as before:

$$\Omega_{cr} = \frac{E_v}{N\hbar}$$

Taking the Thomas-Fermi result for $n_0 = \frac{15}{8\pi} \frac{N}{Z R_\perp^3}$ (You can easily get this from p. 118)

then:

$$\Omega_{cr} = \frac{5}{2} \frac{\hbar}{m R_\perp^2} \ln\left[0.67 \frac{R_\perp}{\xi}\right]$$

Then

$$\frac{\Omega_{cr}}{\omega_\perp} = \frac{5}{4} \frac{\ln\left[1.34 (\mu/\hbar\omega_\perp)\right]}{(\mu/\hbar\omega_\perp)}$$

In the Thomas-Fermi regime $\mu/\hbar\omega_\perp >> 1$, and hence Ω_{cr} is typically a fraction of $\omega_\perp$.

* Vortices in condensates are typically created by a suitable rotating modulation of the trap to stir the condensate.
(Note: other forms of creating vortices are possible, as it was the case of the first realization of vortices at JILA using two-component BECs)

Above a critical angular velocity one observes the formation of the vortex at the trap center. Due to the small size of the cores the BECs are first expanded (the trap is switched-off and the cloud falls and expands). After the expansion the vortex core expands and it may be imaged.

* The actual critical angular frequency observed turns out to be a factor of 2 larger than that predicted from energy considerations (i.e. the one calculated before). This is because the nucleation of vortices demands that a vortex enters from the BEC surface into the center. Without entering into all details we just mention here that this occurs by the instability of surface excitations (quadrupole excitations) which occur for $\Omega \sim \omega_\perp/\sqrt{2}$ (we will talk about collective excitations in a moment)

* If one rotates faster, at sufficiently large angular velocity other vortices enter the condensate. It's now possible to create arrays containing a large number of vortices. These vortices form a typical triangular lattice (vortex lattice). This is the so-called Abrikosov lattice also found in superconductors. This lattice is formed because it's the minimal energy configuration, since vortices of the same circulation actually repel each other like 2D Coulomb charges, with a logarithmic repulsive potential $\sim \ln(r/\xi)$. For very large Ω the rotation of the system looks similar to that of a rigid body.