THE WEAKLY-INTERACTING BOSE GAS. UNIFORM CASE

Let's consider a dilute Bose gas in which the average interparticle distance $d = 1/n^{1/3} >> r_0 \equiv$ range of the interaction. Hence we can limit ourselves to binary collisions (since 3 body processes are safely negligible), as we mentioned already in p. 86.

The distance d is large enough to use the asymptotic expressions for the scattering. Since we will deal with very low energies $kr_0 \ll 1$ this means that only the s-wave scattering will be of relevance. In the following we will always consider the diluteness condition

$$n|a|^3 \ll 1$$

The Hamiltonian describing a weakly-interacting Boson gas in free space is given by

$$\hat{H} = \int d^3r \frac{\hbar^2}{2m} \vec{\nabla}\hat{\psi}^\dagger(\vec{r}) \cdot \vec{\nabla}\hat{\psi}(\vec{r}) + \frac{1}{2}\int d\vec{r} \int d^3r' \, \hat{\psi}^\dagger(\vec{r})\hat{\psi}^\dagger(\vec{r}\,') V(\vec{r}\,'-\vec{r}) \hat{\psi}(\vec{r}\,')\hat{\psi}(\vec{r})$$

For a uniform gas occupying a volume V the field operators are

$$\hat{\psi}(\vec{r}) = \sum_{\vec{p}} \hat{a}_{\vec{p}} \frac{e^{i\vec{p}\cdot\vec{r}/\hbar}}{\sqrt{V}}$$

$\hat{a}_{\vec{p}}$ = annihilates a particle with momentum $\vec{p}$.

Then:

$$\hat{H} = \sum_p \frac{p^2}{2m} \hat{a}_p^\dagger \hat{a}_p + \frac{1}{2V} \sum_{p_1,p_2,q} V_{\vec{q}} \, \hat{a}^\dagger_{\vec{p}_1+\vec{q}} \hat{a}^\dagger_{\vec{p}_2-\vec{q}} \hat{a}_{\vec{p}_2} \hat{a}_{\vec{p}_1}$$

where $V_{\vec{q}} = \int V(\vec{r}) e^{-i\vec{q}\cdot\vec{r}/\hbar} d^3r$

Since only small momenta contribute, then we just take $V_{\vec{q}} \simeq V_0$

$$V_0 = \int V_{eff}(\vec{r}) \, d\vec{r}$$

Hence:

$$\hat{H} = \sum_p \frac{p^2}{2m} \hat{a}_p^+ \hat{a}_p + \frac{1}{2V} V_0 \sum_{p_1 p_2 q} \hat{a}^+_{\bar{p}_1+\bar{q}} \hat{a}_{\bar{p}_2-\bar{q}} \hat{a}_{\bar{p}_2} \hat{a}_{\bar{p}_1}$$

- For $T \ll T_c$, the lowest eigenstate ($\bar{p}=0$) is macroscopically populated. Hence we will perform the so-called Bogoliubov approximation $\hat{a}_0, \hat{a}_0^+ \simeq \sqrt{N_0}$

Note: This approximation demands BEC in a given state. Remember our caveat of p. 76-81 about systems without BEC!

(Note II: More about the Bog. approx. later in p. 107)

In a first approximation we can neglect all $\hat{a}_{\bar{p}\neq 0}$, and we can assume $N_0 \simeq N$, and we can also assume $V_0 \simeq \frac{4\pi\hbar^2 a}{m} \equiv g$ in first approximation.

Then $E_0 = \frac{1}{2V} N^2 g$ ← ground-state energy

Hence $E_0 = \frac{1}{2} N g n$

- The chemical potential $\mu \equiv \frac{\partial E_0}{\partial N} = gn$

Note that for the non-interacting gas μ was zero for $T<T_c$.

- Up to now we have just considered $\hat{a}_0$ and $\hat{a}_0^+$. Terms with only one particle operator with $\bar{p}\neq 0$ don't enter due to momentum conservation. One may retain all quadratic terms in the particle operators with $p\neq 0$ to obtain (I skip some details of the derivation here):

$$\hat{H} = \text{Constant} + \sum_p \left\{ \frac{p^2}{2m} \hat{a}_p^+ \hat{a}_p + \frac{1}{2} gn \left(2\hat{a}_p^+ \hat{a}_p + \hat{a}_p^+ \hat{a}_{-p}^+ + \hat{a}_p \hat{a}_{-p} \right) \right\}$$

Note that now in addition to the more familiar terms $\hat{a}_p^+ \hat{a}_p$ we have some "strange" terms $\hat{a}_p \hat{a}_{-p}$ and $\hat{a}_p^+ \hat{a}_{-p}^+$.

* This Hamiltonian can be diagonalized by means of a so-called

Bogoliubov transformation

$$\hat{a}_p = u_p \hat{b}_p + v^*_{-p} \hat{b}^+_{-p}$$

$$\hat{a}^+_p = u^*_p \hat{b}^+_p + v_{-p} \hat{b}_{-p}$$

(Note: This is a general procedure appearing in many physical problems. It's just a diagonalization.)

This transformation introduces a new set of operators $\hat{b}_p$ and $\hat{b}^+_p$ to which we impose bosonic commutation rules

$$[\hat{b}_p, \hat{b}^+_{p'}] = \delta_{pp'} \longrightarrow$$ since $[a_p, a^+_{p'}] = \delta_{pp'}$ this imposes the condition $|u_p|^2 - |v_{-p}|^2 = 1$

Just few words about the new operators $\hat{b}_p$ and $\hat{b}^+_p$. They are annihilation and creation operators, but clearly they don't annihilate or create particles. They are excitations, and we will see in a moment what is the energy of these excitations but we cannot associate these excitations to individual excited particles, but rather to "quasiparticles".

* OK, let's come back to $|u_p|^2 - |v_{-p}|^2 = 1$. Remember that for the hyperbolic functions $\cosh^2\alpha - sh^2\alpha = 1$. Hence we can assign:

$$u_p = \cosh \alpha_p$$

$$v_{-p} = \sinh \alpha_p$$

If now we substitute the Bogoliubov transformation into $\hat{H}$ we find something like:

$$\hat{H} = \text{constant} + \sum_p \left\{ A(p)\, b^+_p b_p + B(p) \left[b^+_p b^+_{-p} + b_p b_{-p} \right] \right\}$$

where $B(p) = \frac{gn}{2} \left[|u_p|^2 + |v_{-p}|^2 \right] + \left(\frac{p^2}{2m} + gn \right) u_p v_{-p}$

$$= \frac{gn}{2} \cosh 2\alpha_p + \frac{1}{2} \left(\frac{p^2}{2m} + gn \right) \sinh 2\alpha_p$$

· We impose $B_p = 0$, and this gives us

$$\coth 2\alpha_p = -\left[\frac{p^2/2m + gn}{gn}\right]$$

$$u_p, v_{-p} = \pm\left[\left(\frac{p^2/2m + gn}{2\epsilon(p)}\right) \pm \frac{1}{2}\right]^{1/2}$$

and finally

$$\hat{H} = \text{CONSTANT} + \sum_p \epsilon(p)\, \hat{b}_p^+ \hat{b}_p$$

where $\boxed{\epsilon(p) = \left[\frac{gn}{m} p^2 + \left(\frac{p^2}{2m}\right)^2\right]^{1/2}} \longrightarrow$ Bogoliubov spectrum

Note that the original system of independent particles can be described by independent quasiparticles of energy $\epsilon(p)$.
This is a very important result with far-reaching consequences as we will see very soon.

· The CONSTANT in $\hat{H}$ is the ground-state energy (i.e. in absence of quasiparticles carrying excitations) and it's of the form

$$E_0 = \frac{N}{2} gn \left[1 + \underbrace{\frac{128}{15\sqrt{\pi}} (na^3)^{1/2}}_{\text{correction due to interactions}}\right]$$

: remember that $n|a|^3 \ll 1$

and hence

$$\mu = gn\left[1 + \frac{32}{3\sqrt{\pi}} (na^3)^{1/2}\right]$$

· In the following we will try to understand the important physics behind the Bogoliubov spectrum.

* THE BOGOLIUBOV SPECTRUM

* We have just seen that the excited states of an interacting Bose gas can be described in terms of a gas of non-interacting quasiparticles with a dispersion law

$$\epsilon(p) = \left[\left(\frac{p^2}{2m}\right)^2 + p^2 c_s^2 \right]^{1/2} \qquad \text{where } c_s = \left(\frac{gn}{m}\right)^{1/2}$$

* For small momenta $p \ll m c_s$, we can approximate

$$\epsilon(p) \simeq c_s p$$

This is the same linear dispersion law that you have for example for photons (remember that for a photon $\omega = kc \to \hbar\omega = \hbar k c \Rightarrow E = pc$). It's also the dispersion law that one has for phonons (remember that a phonon is the collective excitation of a crystalline structure).

For phonons $\epsilon = c_s p$ where c_s is the sound velocity.

Phonons are nothing else than sound waves.

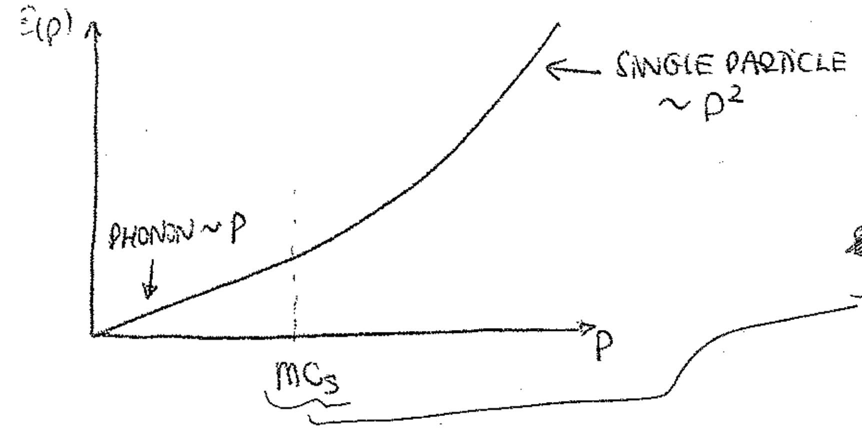
Then small momenta (long wave length) excitations are sound waves.

* For large momenta $p \gg m c_s$

$$\epsilon(p) \simeq \frac{p^2}{2m} + m c_s^2$$

and we recover the typical dispersion law of a massive particle, i.e. the typical kinetic energy we all now. This is the so-called single-particle part of the spectrum.

The transition between the phonon and the single-particle regime occurs for $p \sim m c_s$.



For $\varepsilon \sim \mu$ we have the rough border between the 2 behaviors.

• By setting $p^2/2m = gn$ with $p = \hbar/\xi$ we can define a characteristic length

$$\xi = \sqrt{\frac{\hbar^2}{2mgn}} = \frac{1}{\sqrt{2}}\frac{\hbar}{mc_s}$$

$\longrightarrow$ This is the so-called <u>healing-length</u>. This is an important length scale as we will see in the future.

• Just a final remark concerning the excitations we just discussed. Remember that we are dealing here with quasiparticles and not with particles. This is particularly clear from the following brief discussion. Since the quasiparticles are bosons of energy $\varepsilon(p)$ we can easily write the occupation number

$$N_p \equiv \langle \hat{b}_p^+ \hat{b}_p \rangle = \frac{1}{e^{\beta\varepsilon(p)} - 1}$$

(we set $\mu = 0$ because we deal here with an ideal gas of quasiparticles)

By means of the Bogoliubov transform we can calculate $n_p = \langle \hat{a}_p^+ \hat{a}_p \rangle$ i.e. the <u>particle</u> occupation number:

$$n_p = \langle \hat{a}_p^+ \hat{a}_p \rangle = |v_{-p}|^2 + |u_p|^2 \langle \hat{b}_p^+ \hat{b}_p \rangle + |v_{-p}|^2 \langle \hat{b}_{-p}^+ \hat{b}_{-p} \rangle \quad \left(\begin{array}{c}\text{holding}\\ \text{for } p \neq 0\end{array}\right)$$

The number of condensed atoms is hence

$$N_0 = N - \sum_{p\neq 0} n_p = \left[N - \sum_{p\neq 0} |v_{-p}|^2\right] - \sum_{p\neq 0}\left(|u_p|^2 \langle \hat{b}_p^+ \hat{b}_p \rangle + |v_{-p}|^2 \langle \hat{b}_{-p}^+ \hat{b}_{-p} \rangle\right)$$

At absolute zero, $\langle b_p^+ b_p \rangle = 0$ for $p \neq 0$. However the interactions lead to a depletion of the condensate:

$$N_0 \xrightarrow[T\to 0]{} N - \sum_{p\neq 0} |v_{-p}|^2$$

→ Even at $T=0$ not all particles are in the condensate!

This is the so-called quantum depletion (Note: This is different for the ideal gas, where we had $N_0/N = 1$ at $T=0$)

A proper calculation yields for a uniform gas at $T=0$:

$$n_0 \equiv \frac{N_0}{V} = n\left[1 - \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}\right]$$

Note that for $na^3 \ll 1 \rightarrow n_0 \underset{T=0}{\simeq} n$ as we had before.

But when the interaction increases (i.e. at a Feshbach resonance) the depletion of the condensate is larger and larger.

For sufficiently large interactions we cannot talk any more of a condensate and we enter into the strongly-interacting regime where we have to employ a rather different (and more involved) theory.

The Bogoliubov spectrum will play a crucial role in our discussion of superfluidity, as we will see later on.

NON-UNIFORM BOSE GASES AT ZERO TEMPERATURE

THE ORDER PARAMETER. BOGOLIUBOV APPROXIMATIONS

Let's recall our discussion about the off-diagonal long-range order (p. 80)

Remember that we defined the density matrix as

$\rho(\vec{r},\vec{r}') = \langle \hat{\psi}^{+}(\vec{r})\, \hat{\psi}^{+}(\vec{r}') \rangle$

We can always diagonalize this matrix (also for interacting systems) to get $\rho(\vec{r},\vec{r}') = \sum_i n_i\, \varphi_i^{*}(\vec{r})\, \varphi_i(\vec{r}')$

where n_i are the eigenvalues and $\varphi_i(\vec{r})$ the eigenfunctions.

We can use these eigenfunctions to write the field operator $\hat{\psi}(\vec{r})$ in the form $\hat{\psi}(\vec{r}) = \sum_i \varphi_i \hat{a}_i$

where $\hat{a}_i$ ($\hat{a}_i^{+}$) are the annihilation (creation) operators of a particle in the state φ_i, and which obey the usual bosonic commutation relations $[\hat{a}_i, \hat{a}_j^{+}] = \delta_{ij}$, $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^{+}, \hat{a}_j^{+}] = 0$

Hence

$\rho(\vec{r},\vec{r}') = \sum_{ij} \varphi_i^{*}(\vec{r})\, \varphi_j(\vec{r}') \langle \hat{a}_i^{+} \hat{a}_j \rangle \Rightarrow \langle \hat{a}_i^{+} \hat{a}_j \rangle = \delta_{ij}\, n_i$

Remember that for $T<T_c$, there is a macroscopically large eigenvalue $n_0 = N_0$ (which remember was the off-diagonal long-range order). Hence the corresponding wave function $\varphi_0(\vec{r})$ plays a crucial role in the BEC theory, being the so-called condensate wave function.

- Hence: $\hat{\psi}(\vec{r}) = \varphi_0(\vec{r})\hat{a}_0 + \sum_{i\neq 0} \varphi_i(\vec{r})\hat{a}_i$ (actually we have seen it already in a previous discussion in p. 100)

- Now we will perform a rather crucial approximation, known as the Bogoliubov approximation, which consists on replacing

$$\hat{a}_0,\ \hat{a}_0^{+} \longrightarrow \sqrt{N_0} \quad \text{which is a c-number.}$$

Note that by doing this we are forgetting the non-commutativity of the operators. Why? Well, because $[\hat{a}_0, \hat{a}_0^{+}] = 1$, but since $\langle \hat{a}_0^{+}\hat{a}_0 \rangle = N_0 \gg 1$, this means that $\hat{a}_0$ and $\hat{a}_0^{+}$ are of the order of $\sqrt{N_0} \gg 1$. Hence, we can neglect the operator character.

As a consequence, we can re-write the field operator in the form:

$$\hat{\psi}(\vec{r}) \cong \psi_0(\vec{r}) + \delta\hat{\psi}(\vec{r})$$

where $\psi_0(\vec{r}) = \sqrt{N_0}\,\varphi_0(\vec{r}) \longleftarrow$ condensed part (c-number)

$\delta\hat{\psi}(\vec{r}) = \sum_{i\neq 0} \varphi_i(\vec{r})\hat{a}_i \longleftarrow$ non-condensed parte (operator)

At very low temperatures we can forget the non-condensed part and hence $\hat{\psi}(\vec{r}) \simeq \psi_0(\vec{r})$

<u>Note</u>: this is equivalent to what one does in quantum optics when substituting the quantum fields by the classical electromagnetic field, for fields with large photon occupation).

The function $\psi_0(\vec{r})$ is called the <u>condensate wavefunction</u> and clearly plays the role of an order parameter characterizing the BEC phase, since it vanishes for $T > T_c$.

* The function $\psi_0(\vec{r})$ is a complex quantity

$$\psi_0(\vec{r}) = |\psi_0(\vec{r})| e^{iS(\vec{r})} \rightarrow \text{phase}$$

$|\psi_0(\vec{r})|^2 \rightarrow$ density distribution of the condensate

The phase $S(\vec{r})$ is quite important as we will see later in these lectures

(Note: ψ_0 is defined up to a phase factor $e^{i\alpha}$ without changing the physics. This (so-called gauge) symmetry is broken when choosing a particular phase of the BEC, i.e. there's a spontaneous symmetry breaking here.)

* Just a final point before leaving our discussion on the order parameter

one can interpret

$\psi_0(\vec{r}) = \langle \hat{\psi}(\vec{r}) \rangle$ because $\hat{\psi}(\vec{r}) \cong \varphi_0(\vec{r}) \hat{a}_0$

but $\langle \hat{\psi}(\vec{r}) \rangle$ means here $\langle N-1 | \hat{\psi} | N \rangle$ (i.e. it destroys one particle). But the states $|N\rangle$ and $|N-1\rangle$ are physically equivalent up to corrections of the order $1/N_0 \ll 1$

Let $|N\rangle$ be ~~the~~ stationary state for N particles

then $|N\rangle(t) = |N\rangle(0) e^{-iE(N)t/\hbar}$

Hence $\psi_0(\vec{r},t) = e^{-i[E(N)-E(N-1)]t/\hbar} \psi_0(\vec{r},0)$

The chemical potential is defined as the change of the energy of the system when adding a particle $\rightarrow \mu = \frac{\partial E}{\partial N} \simeq E(N) - E(N-1)$

Hence $\boxed{\psi_0(\vec{r},t) = e^{-i\mu t/\hbar} \psi_0(\vec{r})}$ The chemical potential μ is, as we will see, a crucial parameter here.

(Note: the ~~energy~~ evolution of the ψ_0 is not given by the energy but by the chemical potential)

• THE GROSS-PITAEVSKII EQUATION (GPE)

· Remember that the Hamiltonian describing a weakly-interacting Bose gas is given by:

$$\hat{H} = \int d^3r \; \psi^\dagger(\vec{r}) \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) \right] \psi(\vec{r})$$

$$+ \frac{1}{2} \int d^3r \int d^3r' \; \hat{\psi}^\dagger(\vec{r}) \hat{\psi}^\dagger(\vec{r}\,') \hat{\psi}(\vec{r}\,') \hat{\psi}(\vec{r}) V(\vec{r}-\vec{r}\,')$$

(Note: the only difference with the Hamiltonian discussed in page 99 is that now we consider an external potential $V_{ext}(\vec{r})$. This is e.g. the trapping potential)

In the Heisenberg representation the operator $\hat{\psi}(\vec{r},t)$ fulfills the equation:

using the commutation rules $[\hat{\psi}(\vec{r}), \hat{\psi}^\dagger(\vec{r}\,')] = \delta(\vec{r}-\vec{r}\,')$

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\vec{r},t) = [\hat{\psi}(\vec{r},t), \hat{H}] =$$

$$= \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) + \int d^3r' \; \hat{\psi}^\dagger(\vec{r}\,',t) V(\vec{r}\,'-\vec{r}) \hat{\psi}(\vec{r}\,',t) \right] \hat{\psi}(\vec{r},t)$$

· Doing the Bogoliubov approximation: $\hat{\psi}(\vec{r},t) \longrightarrow \psi_0(\vec{r},t)$ we arrive to the Gross-Pitaevskii equation [assuming as in a previous discussion) that $V(\vec{r}\,'-\vec{r}) \simeq g\, \delta(\vec{r}\,'-\vec{r})$]:

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi_0(\vec{r},t) = \left[\frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(\vec{r},t) + g |\psi_0(\vec{r},t)|^2 \right] \psi_0(\vec{r},t)}$$

This equation is the main theoretical tool for investigating non-uniform dilute Bose gases at low temperatures. This equation is a particular example of a nonlinear Schrödinger equation (similar to that appearing in nonlinear optics in Kerr media) and hence the BEC physics is inherently nonlinear. We will have a look to nonlinear phenomena later.

* Note, that since the stationary solution fulfills

$$\Psi_0(\vec{r},t) = e^{-i\mu t/\hbar}\,\Psi_0(\vec{r})$$

then the time-independent Schrödinger equation is of the form:

$$\mu\,\Psi_0(\vec{r}) = \left[\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V_{ext}(\vec{r}) + g|\Psi_0(\vec{r})|^2\right]\Psi_0(\vec{r})$$

the value of the chemical potential μ is fixed by the normalization condition

$$\int |\Psi_0(\vec{r})|^2\, d^3r = N$$

: For a uniform system $V_{ext}(\vec{r})=0$, $|\Psi_0(\vec{r})|^2 = n$ = uniform density

and hence $\mu = gn$ as we already knew (p. 100)

BEC IN A BOX POTENTIAL

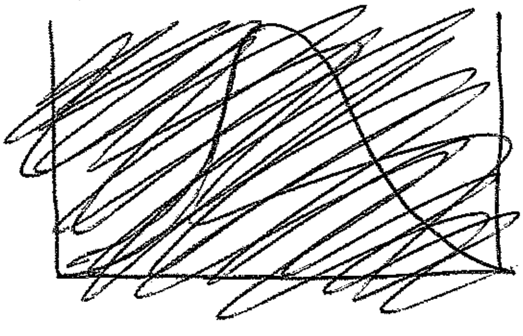
: In the following we consider a very simple but instructive problem, namely that of N interacting bosons (with $g>0 \rightarrow$ repulsive gas) confined in a box potential of the form

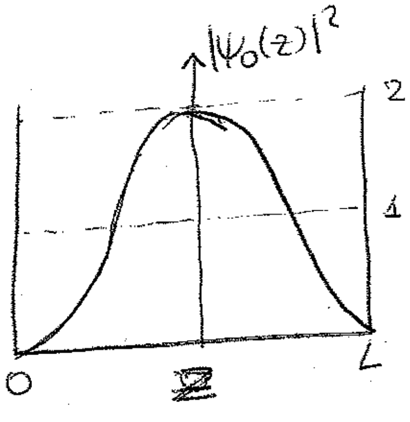
$$V_{ext}(z) = \begin{cases} 0 & 0<z<L \\ \infty & \text{otherwise} \end{cases} \longrightarrow \Psi_0(0) = \Psi_0(L) = 0$$

Hence, in absence of any interactions ($g=0$), the GPE becomes an usual Schrödinger equation, and hence

$$\Psi_0(\vec{r}) = \sqrt{2\bar{n}}\,\sin\frac{\pi}{L}z \qquad \text{with } \bar{n} = \frac{N}{V}$$

V = volume of the box (we assume periodic boundary condition in x and y)





: Let's see what happens if the interactions are present.
Now the system is controlled by the GPE

$$\mu \psi_0(z) = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + g|\psi_0(z)|^2\right]\psi_0(z)$$

Since $g>0$, the system minimizes the energy by making the density uniform in the interior of the box. Clearly the maximal density is minimized (and with it the interaction energy $g|\psi_0|^2$) if we spread $|\psi_0|$ as uniform as possible in the box. Only at the walls of the box something will happen, because obviously at the walls we have to fulfill $|\psi_0|^2 = 0$.
So except close to the walls we can consider a uniform density n.
Remember the definition of healing length associated with the density n: $\xi = \sqrt{\frac{\hbar}{2mgn}}$

Let $\tilde{\psi}_0(z) = \psi_0(z)/\sqrt{n}$, hence (introducing $\tilde{z} = z/\xi$)

$$\mu \tilde{\psi}_0(z) = \left[\frac{-\hbar^2}{2m\xi^2}\frac{d^2}{d\tilde{z}^2} + gn|\tilde{\psi}_0(z)|^2\right]\tilde{\psi}_0(z)$$

Assuming that the box size L is much larger than the border region at which the density is different than n, we can the forget the border and approximate

$$\mu \approx gn = \frac{\hbar^2}{2m\xi^2}$$

Hence

$$\tilde{\psi}_0(\tilde{z}) = -\frac{d^2}{d\tilde{z}^2}\tilde{\psi}_0(\tilde{z}) + \tilde{\psi}_0(\tilde{z})^3$$

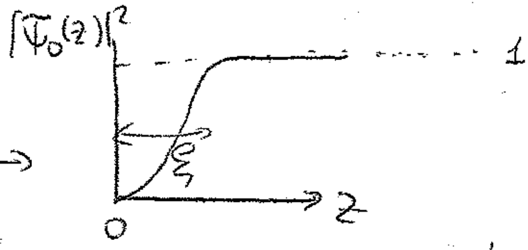
* let's see what happens close to the border $z=0$.

At $z=0 \longrightarrow \tilde{\Psi}_0(z)=0$

Out of the border region $\longrightarrow \tilde{\Psi}_0(z) \xrightarrow[z \gg \text{border}]{} 1$

This equation has an analytic solution, namely:

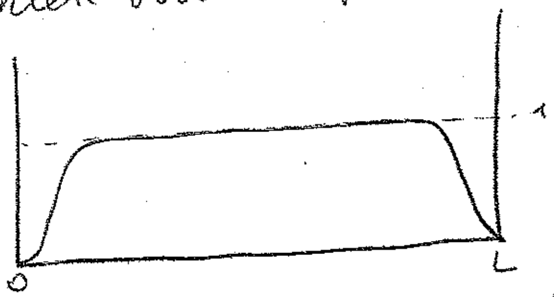
$$\tilde{\Psi}_0(\tilde{z}) = \tanh\left(\tilde{z}/\sqrt{2}\right)$$



Hence $\Psi_0(z) = \sqrt{n}\,\tanh\left(\frac{z}{\sqrt{2}\,\xi}\right)$

Then we can now identify the border region as an interval of the order of the healing length ξ (hence this calculation is ok for $\xi \ll L$)

· The complete solution for the interacting BEC in the box looks



The <u>healing length</u> it is therefore a crucial length scale in the problem.

The physical meaning of the healing length becomes evident. It's the length scale at which a perturbation of the density (i.e. the border of the box wall) will die out into a uniform density (hence the name healing length)

Basically the healing length is the result of comparing the typical kinetic energy associated with a variation of the wavefunction in a length scale $\xi \longrightarrow \frac{\hbar^2}{2m\xi^2}$ and the chemical potential $\mu = gn$ (the interaction energy).

- Clearly when the interaction increase (g increases) then ξ decreases, and the border region becomes narrower. On the contrary when g decreases ξ becomes larger.

We have thus seen that the 2-body interactions can significantly modify the ground state profile of a BEC. We will see later what happens in an harmonic trap.

HYDRODYNAMIC EQUATIONS. THE THOMAS-FERMI LIMIT

- Recall the GPE

$$i\hbar \frac{\partial}{\partial t}\psi_0(\vec{r},t) = \left[\frac{-\hbar^2\nabla^2}{2m} + V_{ext}(\vec{r}) + g|\psi_0(\vec{r},t)|^2\right]\psi_0(\vec{r},t)$$

remember that (p. 108) $\psi_0(\vec{r},t) = \sqrt{n(\vec{r},t)}\, e^{iS(\vec{r},t)}$

- let's multiply by ψ_0^*

$$i\hbar\,\psi_0^* \frac{\partial}{\partial t}\psi_0 = \psi_0^*\left[-\frac{\hbar^2\nabla^2}{2m}\right]\psi_0 + V_{ext}(\vec{r})|\psi_0|^2 + g|\psi_0|^4$$

let's take the complex conjugate of this equation:

$$-i\hbar\,\psi_0 \frac{\partial}{\partial t}\psi_0^* = \psi_0\left[\frac{-\hbar^2}{2m}\nabla^2\right]\psi_0^* + V_{ext}(\vec{r})|\psi_0|^2 + g|\psi_0|^4$$

let's substract both equations

$$i\hbar\frac{\partial}{\partial t}n = \frac{-\hbar^2}{2m}\left[\psi_0^*\nabla^2\psi_0 - \psi_0\nabla^2\psi_0^*\right] = \frac{-\hbar^2}{2m}\nabla\left[\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right]$$

then $\frac{\partial}{\partial t}n = \frac{i\hbar}{2m}\nabla(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*)$

Since $\psi_0 = n^{1/2}e^{iS} \rightarrow \nabla\psi_0 = \left[\frac{1}{2}\nabla n + i\nabla S\right]n e^{iS} = \left(\frac{\nabla n}{2} + i\nabla S\right)\psi_0$

Then $\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^* = n\left[\frac{\nabla n}{2} + i\nabla S - \frac{\nabla n}{2} + i\nabla S\right] = 2in\nabla S$

Hence: $$\frac{\partial n}{\partial t} = \frac{i\hbar}{2m}\nabla[2in\nabla S] = -\nabla\left[\frac{\hbar}{m}n\nabla S\right]$$

$$= -\nabla\vec{J}$$

- Then $\boxed{\dfrac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$

this has the form of a continuity equation, where $\vec{J}$ plays the role of a density current (recall the notion of current in quantum mechanics)

The physical meaning of this equation is that the number of particles are conserved by the equation (by the GPE).

- The current is the number of atoms per surface area per second hence $\boxed{\vec{v}_S = \dfrac{\hbar}{m}\vec{\nabla}S}$ has units of velocity.

It's indeed the velocity of the condensate flow. Note that $\vec{\nabla}\times\vec{v}_S = \dfrac{\hbar}{m}\vec{\nabla}\times(\vec{\nabla}S) = 0$.

$\vec{v}_S$ is an example of a so-called irrotational fluid, something crucial for discussing the properties of superfluids as we will see later.

- Inserting $\psi_0 = \sqrt{n}\, e^{iS}$ into the GPE we obtain (in addition to the continuity equation) a second equation providing the evolution of the phase

$$\boxed{\hbar\frac{\partial}{\partial t}S + \left(\frac{m v_S^2}{2} + V_{ext}(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right) = 0}$$

This equation and the continuity equation provide a closed set of coupled equations equivalent to the GPE.

· Hence:

$$m\frac{\partial}{\partial t}\vec{v}_s = -\vec{\nabla}\left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}\right]$$

If you have a close look to this equation and the continuity equation you will see that $\hbar$ just appears in the last term at the r.h.s. of the last equation. Hence the quantum character is pretty much reduced to that term, which receives the name "quantum pressure"

The quantum pressure becomes small if the density changes slowly in space. Let's see first when this happens.

Let's call R the typical length characterizing density variations, typically the condensate size for the ground state configuration

Hence the quantum pressure term scales as:

$$\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \sim \frac{1}{R^2}$$

Remember that $gn = \dfrac{\hbar^2}{2m\xi^2}$

Hence if $R \gg \xi$, then $\dfrac{\hbar^2}{2m\xi^2} \gg \dfrac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}$, and we can neglect the quantum pressure. This is the so-called Thomas-Fermi limit.

In this limit

$$\boxed{m\frac{\partial}{\partial t}\vec{v}_s + \vec{\nabla}\left[\frac{m v_s^2}{2} + V_{ext}(\vec{r}) + gn\right] = 0}$$

* This equation is well known in hydrodynamics, being the Euler equation for a nonviscuous gas with pressure $P = gn^2/2$

- Note: the sound velocity in a fluid is defined as $mc_s^2 = \frac{\partial P}{\partial n} = gn \longrightarrow c_s = \sqrt{\frac{gn}{m}}$ which is exactly the sound-velocity we got from the Bogoliubov spectrum (so all matches, fortunately!).

- In the ground state configuration $\vec{v}_S = 0$, and hence $\vec{\nabla}\left[V_{ext}(\vec{r}) + gn \right] = 0 \longrightarrow V_{ext}(\vec{r}) + gn = \text{constant}$

This constant is the chemical potential (as it becomes clear by removing the kinetic energy in the time-independent GPE).

Hence $\boxed{\mu = V_{ext}(\vec{r}) + gn(\vec{r})}$

as always μ is fixed by setting $\int n(\vec{r})\, d^3r = N$.

- In the following we will employ this general expression for the case of a BEC in an harmonic trap.

* BEC in a harmonic trap

* Let's consider the harmonic confinement:

$$V_{ext}(\vec{r}) = \frac{m\omega_x^2}{2}x^2 + \frac{m\omega_y^2}{2}y^2 + \frac{m\omega_z^2}{2}z^2$$

We will consider the case $g>0$ (repulsive BEC), and leave the attractive case ($g<0$) for a later discussion.

Let's consider the Thomas-Fermi limit:

$$\mu = V_{ext}(\vec{r}) + gn(\vec{r})$$

Hence

$$n(\vec{r}) = \frac{1}{g}(\mu - V_{ext}(\vec{r}))$$

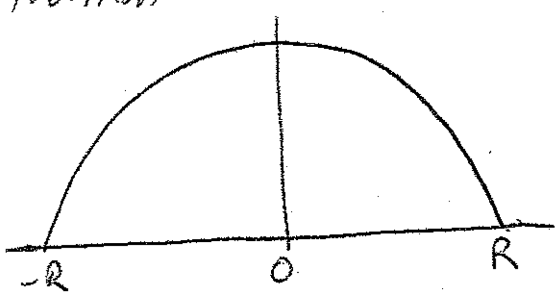
$$= \frac{\mu}{g}\left[1 - \frac{m\omega_x^2}{2\mu}x^2 - \frac{m\omega_y^2}{2\mu}y^2 - \frac{m\omega_z^2}{2\mu}z^2\right]$$

$$= n_0\left[1 - (x/R_x)^2 - (y/R_y)^2 - (z/R_z)^2\right]$$

where $n_0 \equiv$ central density

$R_{k=x,y,z} \Rrightarrow$ are the so-called Thomas-Fermi Radii.

Therefore the density profile of a BEC in a trap in the Thomas-Fermi limit acquires the form of an inverted parabola



The Thomas-Fermi radius is provided by the classical turning point

$$\mu = \frac{m\omega^2}{2}R^2$$

* It's interesting to have a look to the actual form of μ and R_k as a function of the system parameters.

- We know that

$$N = \int d^3r\, n(\vec{r}) = n_0 \int d^3r \left[1 - \left(\frac{x}{R_x}\right)^2 - \left(\frac{y}{R_y}\right)^2 - \left(\frac{z}{R_z}\right)^2\right]$$

$$= n_0 R_x R_y R_z \int_0^1 r^2 dr\,(1-r^2)\, 4\pi$$

$$= \frac{8\pi}{15} n_0 R_x R_y R_z = \frac{8\pi}{15} \frac{\mu}{g} \left(\frac{2\mu}{m\omega_x^2}\right)^{1/2} \left(\frac{2\mu}{m\omega_y^2}\right)^{1/2} \left(\frac{2\mu}{m\omega_z^2}\right)^{1/2} \underset{\bar{\omega} = (\omega_x \omega_y \omega_z)^{1/3}}{=}$$

$$= \frac{8\pi}{15 g} \left(\frac{2}{m\bar{\omega}^2}\right)^{3/2} \mu^{5/2} = \frac{8\pi}{15 \cdot \frac{4\pi\hbar^2}{m} a} \left(\frac{2}{m\bar{\omega}^2}\right)^{3/2} \mu^{5/2} \underset{l_{HO} = \sqrt{\frac{\hbar}{m\bar{\omega}}}}{=}$$

$$= \left(\frac{2\mu}{\hbar\bar{\omega}}\right)^{5/2} \frac{l_{HO}}{15a} \longrightarrow \boxed{\mu = \frac{\hbar\bar{\omega}}{2}\left(\frac{15aN}{l_{HO}}\right)^{2/5}}$$

As a consequence:

$$\boxed{R = \left(\frac{2\mu}{m\bar{\omega}^2}\right)^{1/2} = l_{HO}\left(\frac{15aN}{l_{HO}}\right)^{1/5}} \longrightarrow R_k = \frac{\bar{\omega}}{\omega_k} R$$

As a consequence note that the size of the system increases as $N^{1/5}$. For the ideal gas the size of the condensate is fixed by the oscillator length. In the case of a repulsive gas, the size of the system grows when we increase N, because the particles repel each other. (Note: However a growth $N^{1/5}$ may be considered as relatively quite mild.)

- If we compare R and the healing length:

$$\xi = \left(\frac{\hbar^2}{2m\mu}\right)^{1/2} \longrightarrow \frac{\xi}{R} = \left(\frac{\hbar\bar{\omega}}{2\mu}\right)^2 = \left(\frac{15Na}{l_{HO}}\right)^{-2/5}$$

Remember that the Thomas-Fermi limit is justified if $\xi \ll R$, which is true if $\mu \gg \hbar\bar{\omega}$, which in turn is true for $Na/l_{HO} \gg 1$.

* Typical values of $a/a_{HO} \sim 10^{-3}$, hence $N >> 10^3$ to get the Thomas-Fermi limit. Typically $N \sim 10^4, 10^5$ so the Thomas-Fermi limit is typically OK (but not allways).

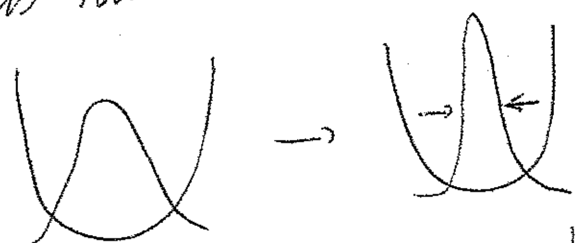
BEC WITH ATTRACTIVE ~~INT~~ERACTIONS

If $g<0$, the interactions are attractive, and the behaviour of the condensates is very different.

Let's write the GPE:

$$\mu \Psi_0 = \left[\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r}) - |g|\,|\Psi|^2\right]\Psi$$

It's clear that the system minimizes its energy by contracting and increasing its ~~energy~~ density in the trap center. In a classical gas this would be unstoppable. However, in quantum mechanics we must take into account the zero-point oscillation ~ the correspon-ding to the trap. The zero-point kinetic energy tends to compensate the contraction.



$$\underbrace{\left(\frac{-\hbar^2}{2m}\nabla^2 + V_{ext}(\vec{r})\right)\Psi}_{\substack{\text{zero point energy: tends}\\ \text{to prevent excessive contraction}}} \quad \underbrace{-|g|\Psi^3}_{\substack{\text{tends to contract}\\ \text{the gas towards}\\ \text{the trap center}}}$$

zero point energy: tends to prevent excessive contraction → This is a single particle effect. It's N independent

tends to contract the gas towards the trap center → This effect grows with N

* As a consequence, for a sufficiently large number of particles the kinetic energy can't compensate the non-linear ~~attract~~ compression (non linear focusing in the language of nonlinear optics) and as a consequence the condensate collapses!

Let's see this in more detail.
Let's consider a spherical trap with frequency ω.
Let's use the following Gaussian ansatz for the condensate

$$\phi(\vec{r}) = \left(\frac{N}{\lambda^3 a_{HO}^3 \pi^{3/2}}\right)^{1/2} e^{-r^2/2\lambda^2 a_{HO}^2}$$

λ fixes the width of the condensate and works as our variational parameter.
. Now that we have set a Gaussian "bubble" of width λa_{HO}, let's calculate the energy as a function of λ.
Remember that

$$E = \int d^3r \left[\frac{\hbar^2}{2m}|\nabla\psi|^2 + V_{ext}(\vec{r})|\psi|^2 + |g|\frac{|\psi|^4}{2}\right]$$

After plugging the Ansatz one gets:

$$\frac{E(\lambda)}{N\hbar\omega} = \frac{3}{4}\left(\frac{1}{\lambda^2} + \lambda^2\right) - \frac{N|a|}{\sqrt{2\pi}\, l_{HO}}\lambda^{-3}$$

kinetic energy

trap ~~[illegible]~~

interaction energy.

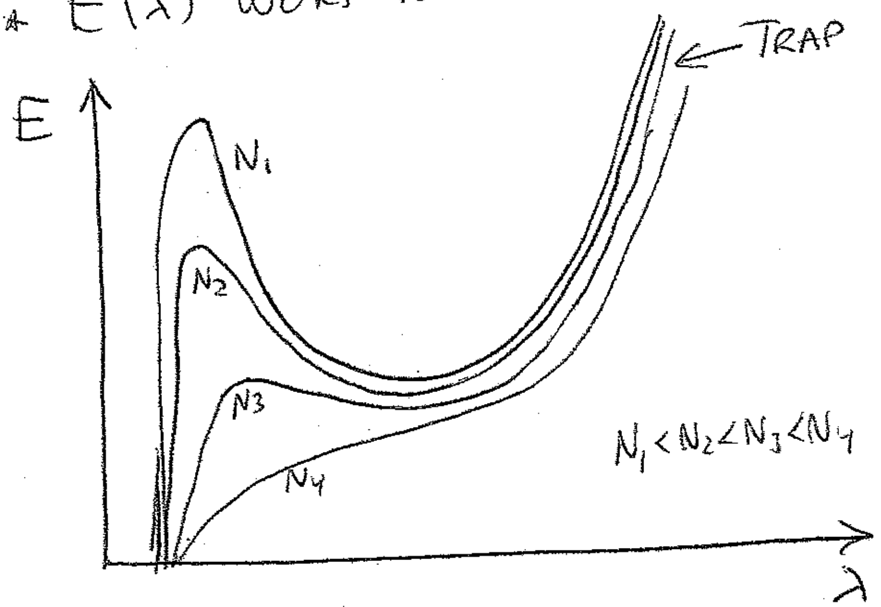
. Here it is quite crucial how the different terms scale with λ.

- The kinetic energy $\sim 1/\lambda^2$ (this was expected because the kinetic energy is a 2nd derivative, and hence scales as $\sim 1/\text{length}^2$)

The trap energy $\sim \lambda^2$ (also expected because we consider an harmonic potential $\sim x^2$)

The interaction energy $\sim \frac{1}{\lambda^3}$ (also expected because the interaction energy is proportional to the density and the density $= \frac{1}{\text{volumen}} \sim \frac{1}{\text{length}^3}$)

- $E(\lambda)$ looks like this:

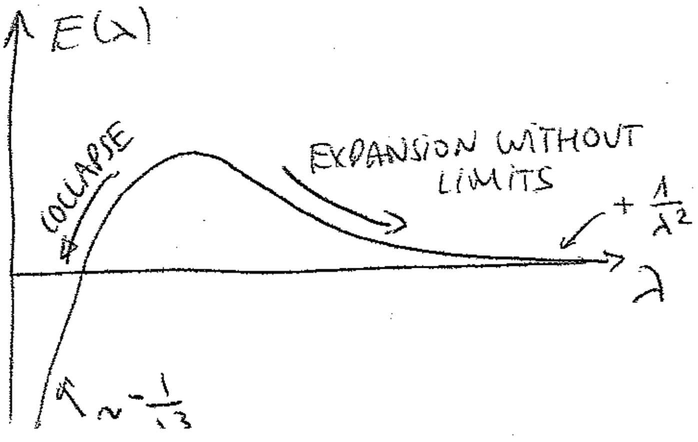


- For small values of N $E(\lambda)$ shows a potential barrier, which prevents the system to collapse.
- For large N, the barrier disappears and the collapse occurs.

- The numerical value at which collapse occurs is

$$\frac{N_{cr}|a|}{a_{ho}} = 0.575$$

- Note that in absence of a trap and the one just has kinetic energy and interactions, and the energy curves like this:



* Note that the system either collapses or expands without limits. As a consequence there cannot be a self-bound ("bubble-like") solution for the GPE in a 3D environment.

The situation is radically different in 1D systems, where self-bound solutions are possible, as we will discuss when talking about solitons