General Relativity

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Conventions.

- We refer to General Relativity as "GR"
- Signature convention is "mostly minus", i.e. the Minkowski-Metric is

$$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$$

- Greek indices $\{\alpha, \beta, \ldots, \lambda, \mu, \nu, \ldots\}$ range over $\{0, 1, 2, 3\}$; latin indices $\{a, b, \ldots, l, m, n, \ldots\}$ over $\{1, 2, 3\}$.

- Index convention: Repeated indices on different levels (one upstairs one downstairs) are summed over. This convention is also followed in $\mathbb{R}^3$ with euclidean metric δ_{ab}, δ^{ab} to lower and raise indices.

Lecture 1 : Newtonian concepts

Law of Inertia states the existence of preferred reference systems and preferred clocks, with respect to which orbits in space of "free" (i.e. force-less) mass points are <u>straight</u> and <u>uniform in time</u> (equal spatial distances in equal intervals of time). Such reference frames and clocks are called <u>inertial systems</u> and <u>inertial time scales</u>. They are characterised by that property and nothing else. Relative to them the Newtonian law of motion is valid in its usual (simplest) form:

$$\underbrace{\vec{F}}_{\text{Force}}(\vec{X}(t), t) = \underbrace{m_i}_{\text{inertial mass}} \underbrace{\ddot{\vec{X}}(t)}_{\text{acceleration}} \qquad (1.1)$$

The law of inertia may be understood as path structure on space-time: To each point $(\vec{x}, t)$ on spacetime and each tangential direction $(\vec{v} = d\vec{x}/d\lambda,\ dt/d\lambda)$ there is a unique path (= unparametrised curve) in spacetime through $(\vec{x}, t)$ tangential to the given direction. It is the path that a free particle with these initial conditions would take.

Remark 1: For $n \leq 3$ particles we can always find a spatial reference system with respect to which they move on straight lines (modulo singular exceptions). Doing this for $n = 3$ fixes the reference frame completely. The non-trivial statement of the law of inertia is, that any fourth, fifth etc. free particle will also move on a straight line with respect to that very same reference frame. Similarly for clocks: For $n = 1$ we can always find a clock s.t. a single particle moves uniform; that fixes the timescale up to affine rescalings: $t \mapsto t' := at + b,\ a \neq 0$.

But it becomes a non-trivial statement that any second, third, etc. free particle will likewise move uniformly with respect to that affine class of time scales.

Remark 2: Laws of motion can, of course, also be written down in non-inertial reference frames and with respect to non-inertial time scales. The way to do this is to <u>first</u> write down the Newtonian law $\vec{F} = m\vec{a}$ in an inertial frame and with respect to an inertial time scale and <u>then</u> rewrite this in terms of non-inertial reference parameters. This will lead to additional terms, so-called "inertial-forces" (german: Scheinkräfte), like centrifugal, Coriolis, Euler-force.

Now, in Newtonian physics, the word "force" or "true force" (in contrast to „inertial" or „apparent" force) is used for any causes of deviations from inertial motion. In particular: Gravity is a force! This will not be true in GR.

Using inertial systems and timescales we write the usual

$$\vec{F} = m_i \ddot{\vec{x}} \quad , \text{ or sometimes}$$

$$\vec{F} + (-m_i \ddot{\vec{x}}) = \vec{0} \tag{1.2}$$

external acting force

inertial reaction "force".

<u>Concepts of mass</u> : m_i, $m_g^{(p)}$, $m_g^{(a)}$

- Inertial mass : Defined as constant of proportionality between external acting force $\vec{F}$ and reactive acceleration $\ddot{\vec{x}}$ with respect to inertial frames: $\vec{F} = m_i \ddot{\vec{x}}$

- Passive gravitational mass: Defined as constant of proportionality between gravitational field strength $\vec{g}$ and the force it exerts on a body :

$$\vec{F}_g = m_g^{(p)} \cdot \vec{g} \tag{1.3}$$

- Active gravitational mass: Defined as ability to generate gravitational field:

$$\vec{g}(\vec{x}) = -G \cdot m_g^{(a)} \cdot \frac{(\vec{x} - \vec{x}')}{\|\vec{x} - \vec{x}'\|^3} \tag{1.4}$$

is the gravitational field of a point active gravitational mass $m_g^{(a)}$ located at $\vec{x}'$.

There is a general assumption according to which none of these masses can be negative (compare problem sheet 1).

What is the relation between these three masses?

Regarding the relation between the two gravitational masses, we can make the following observation: Consider two point masses at positions $\vec{x}_1$ and $\vec{x}_2$. Their passive / active gravitational masses are

$$(m_g^{(p)})_1 \,/\, (m_g^{(a)})_1 \quad , \quad (m_g^{(p)})_2 \,/\, (m_g^{(a)})_2$$

respectively.

The gravitational field produced by 1 at the position 2 is

$$\vec{g}_{12} = -G \left(m_g^{(a)}\right)_1 \frac{\vec{X}_2 - \vec{X}_1}{\|\vec{X}_2 - \vec{X}_1\|^3} , \qquad (1.5)$$

acting on 2 with the force

$$\vec{F}_{12} = \left(m_g^{(p)}\right)_2 \cdot \vec{g}_{12}$$

$$= -G \left(m_g^{(a)}\right)_1 \left(m_g^{(p)}\right)_2 \frac{\vec{X}_2 - \vec{X}_1}{\|\vec{X}_2 - \vec{X}_1\|^3} \qquad (1.6)$$

Similarly, the gravitational field produced by 2 at 1 leads to a force on 1:

$$\vec{F}_{21} = -G \left(m_g^{(a)}\right)_2 \left(m_g^{(p)}\right)_1 \frac{\vec{X}_1 - \vec{X}_2}{\|\vec{X}_1 - \vec{X}_2\|^3} \qquad (1.7)$$

$$1 \; \bullet\!\!\longrightarrow \vec{F}_{21} \qquad \vec{F}_{12} \longleftarrow\!\!\bullet \; 2$$

If we invoke Newtons <u>third law</u> also known as "actio = re-actio", we infer

$$\vec{F}_{21} = -\vec{F}_{12} \qquad (1.8)$$

Given the expressions above this is equivalent to

$$(m_g^{(a)})_1 \, (m_g^{(p)})_2 = (m_g^{(a)})_2 \, (m_g^{(p)})_1$$

or $$\left(\frac{m_g^{(a)}}{m_g^{(p)}} \right)_1 = \left(\frac{m_g^{(a)}}{m_g^{(p)}} \right)_2 .$$

Since this must hold for all pairs of masses we must have

$$\frac{m_g^{(a)}}{m_g^{(p)}} = \text{universal constant}$$

By choice of units we can then arrange

$$m_g^{(a)} = m_g^{(p)} =: m_g$$

for all masses.

The question remains: How is m_i related to m_g? No fundamental theoretical reason seems to exist that implies $m_g = m_i$ — except

<u>Experience</u> !

Consider a mass-point in an external gravitational field $\vec{g}(\vec{x})$. Its equation of motion is

$$m_g \, \vec{g}(\vec{x}(t)) = m_i \, \ddot{\vec{x}}(t) \qquad (1.9)$$

or

$$\ddot{\vec{x}}(t) = \frac{m_g}{m_i} \, \vec{g}(\vec{x}(t)) \, . \qquad (1.10)$$

Now, <u>experience</u> supports the following <u>principle of UFF</u>

UFF (Universality of Free Fall):

In a given gravitational field $\vec{g}$ the orbit $\vec{x}(t)$ of a <u>test mass</u> only depends on the initial data. Given the initial data, it is the <u>same</u> for <u>all</u> test masses.

Given (1.10) this principle implies

$$\frac{m_g}{m_i} = \text{universal constant.}$$

Again by choosing units appropriately we have

$$m_g = m_i$$

Note that UFF makes a statement that

1) is not a logical consequence of Newtonian theory but merely refers to experience. It is open to falsification within Newtonian theory.

2) it makes a statement about so-called <u>test-masses</u> only. This is a difficult concept! These are masses which are not too large ($\vec{g}$ must vary only slightly over its extent) and not too small (the gravitational field produced by the mass itself at its location should be small compared to the gravitational field $\vec{g}$ it is supposed to "test"). Also, it should not possess any structural features, like higher multipole moments, charge, spin etc, that could account for deviations from $\ddot{\vec{x}} = \vec{g}$. Whether a given object qualifies as "test mass" depends on the context! This makes the UFF principle somewhat tricky!

Newtonian gravity as field theory

Gravitational field

$$\vec{g} : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(t, \vec{x}) \longmapsto \vec{g}(t, \vec{x}) \qquad (1.11)$$

Sources of $\vec{g}$ are (gravitational) mass densities ϱ

$$\varrho : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0} \qquad (1.12)$$

so that

$$\vec{\nabla} \vec{g} = - 4 \pi G \varrho . \qquad (1.13)$$

Here the constant $4\pi G$ is conventional and the minus-sign indicates that a positive ϱ acts such that gravity fied-lines point towards the sourcing mass, not away from it.

The $\vec{g}$-field is "conserving", i.e curl-free :

$$\vec{\nabla} \times \vec{g} = \vec{0} \qquad (1.14)$$

Hence, by Poincaré's lemma, there exists

$$\left.\begin{aligned} &\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R} \\ \text{s.t.} \quad &\vec{g} = -\nabla\phi \end{aligned}\right\} \quad (1.15)$$

(the minus sign is conventional).

In total we have

Field-equation:

$$\Delta\phi = 4\pi G \rho \qquad (1.16)$$

Equation of motion for test mass:

$$\ddot{\vec{x}}(t) = -\vec{\nabla}\phi(t, \vec{x}(t)) \qquad (1.17)$$

We will generally assume that

$$\int_{\mathbb{R}^3} \rho \, d^3x = M < \infty \qquad (1.18)$$

and often, in addition, that ρ has compact support:

closure

$$\mathrm{supp}(\rho) = \overline{\{\vec{x}\in\mathbb{R}^3 : \rho(\vec{x}) \neq 0\}} \qquad (1.19)$$

If we impose boundary condition

$$\phi(t, \|\vec{x}\| \to \infty) \to 0$$

solution to (1.16) is uniquely given by

$$\phi(t,\vec{x}) = -G \int_{\mathbb{R}^3} d^3x' \, \frac{\rho(t,\vec{x}')}{\|\vec{x}-\vec{x}'\|} \, . \qquad (1.20)$$

Note that this integral exists by hypothesis. In fact, convergence would already be implied if there exist three positive real numbers C, ε, and R s.t.

$$\left.\begin{aligned} |\rho(t,\vec{x})| &< \frac{C}{r^{2+\varepsilon}} \\ \forall \vec{x} &: \|\vec{x}\| > R \\ \forall t & \end{aligned}\right\} \qquad (1.21)$$

That is to say: outside some fixed distance R ρ falls of faster than $\sim r^{-2}$ at all times.

Equation (1.20) displays the instantaneous action of gravity in Newton's theory: changing ϱ at time t is felt instantaneously in a change of ϕ <u>anywhere</u> in space. This has to do with the elliptic character of the differential equation (1.16).

Recall:

Δ is an elliptic operator

$\left(\frac{\partial}{\partial t} - \kappa \Delta\right)$ is a parabolic operator

$\Box := \left(\frac{\partial^2}{\partial t^2} - \kappa \Delta\right)$ is a hyperbolic operator

Parabolic operators occur e.g. in the heat- or diffusion equation, hyperbolic operators in wave equations and Maxwell's theory.

In Newtonian theory the matter distribution <u>determines</u> the grav. field entirely, unlike in Electromagnetism, where the charge- and current distribution do not determine $(\vec{E}, \vec{B})$, since you can always add solutions to homogeneous Maxwell eqns.

In Maxwell's theory there are plenty of solutions to the homogeneous equs., in particular electromagnetic waves. This fact indicates that the electro-magnetic field itself has physical degrees of freedom. In Newtonian gravity, on the other hand, there is no (regular) solution to the homogeneous equation $\Delta\phi = 0$, except $\phi \equiv 0$. The gravitational field has no degrees of freedom by itself; it is a pure attribute of matter!