Lecture 10

The linearized Einstein Equations and its gauge invariance.

Suppose the space-time metric g differs "only slightly" from the flat Minkowski metric η

$$g = \eta + h \tag{10.1}$$

where $h \in ST^0_2(M)$ is "small". By this smallness we shall understand that $h_{\mu\nu}(x)$, $\partial_\lambda h_{\mu\nu}(x)$, and $\partial^2_{\lambda\sigma} h_{\mu\nu}(x)$ are all small in modulus compared to 1 at all points $x \in M$ in a chart where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

The leading order corrections to the Riemann-, Ricci-, and Einstein tensor are then linear in h and ∂h ($\partial^2 h$ anyway). Since in a chart where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ the $\eta_{\mu\nu}$ are constant, the Christoffel symbols $\Gamma = \frac{1}{2} g^{-1}(-\partial g + \partial g + \partial g)$ $= \frac{1}{2}\eta^{-1}(-\partial h + \partial h + \partial h)$ is already linear in h and the quadratic $\Gamma\Gamma$-terms in Riem do not contribute

$$\overset{(1)}{R}{}^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \partial_\nu \Gamma_\mu{}^\alpha{}_\beta \qquad (10.2)$$

$$= \frac{1}{2}\eta^{\alpha\lambda}\Big(-\partial^2_{\mu\lambda} h_{\nu\beta} + \partial^2_{\mu\beta} h_{\lambda\nu} + \underline{\partial^2_{\mu\nu} h_{\beta\lambda}}$$
$$+ \partial^2_{\nu\lambda} h_{\mu\beta} - \partial^2_{\nu\beta} h_{\lambda\mu} - \underline{\partial^2_{\nu\mu} h_{\beta\lambda}}\Big)$$

Where $\partial^2_{\mu\lambda} = \partial^2 / \partial x^\mu \partial x^\lambda$ etc.

The underlined terms cancel and we get

$$\overset{(1)}{R}{}^{\alpha}{}_{\beta\mu\nu} = -\frac{1}{2}\eta^{\alpha\lambda}\Big(\partial^2_{\lambda\mu} h_{\beta\nu} + \partial^2_{\beta\nu} h_{\lambda\mu}$$
$$- \partial^2_{\lambda\nu} h_{\beta\mu} - \partial^2_{\beta\mu} h_{\lambda\nu}\Big) \qquad (10.3)$$

or, in short:

$$\overset{(1)}{R}{}_{\alpha\beta\mu\nu} = -\frac{1}{2}\left(\partial^2 \owedge h\right)_{\alpha\beta\mu\nu} \qquad (10.4)$$

Where $\owedge$ is the Kulkarni-Nomizu product, here formally applied to ∂^2 and h:

$$\left.\begin{aligned}(\partial^2 \owedge h)_{\alpha\beta\mu\nu} = {} & \partial^2_{\alpha\mu} h_{\beta\nu} + \partial^2_{\beta\nu} h_{\alpha\mu} \\ & - \partial^2_{\alpha\nu} h_{\beta\mu} - \partial^2_{\beta\mu} h_{\alpha\nu}\end{aligned}\right\} \qquad (10.5)$$

From (10.4) the Ricci-Tensor follows by contracting α and μ; renaming indices we get

$$\overset{(1)}{R}_{\alpha\beta} = \overset{(1)}{R}{}^{\lambda}{}_{\alpha\lambda\beta} = \eta^{\lambda\sigma} \overset{(1)}{R}_{\lambda\alpha\sigma\beta}$$

$$= -\frac{1}{2}\Big(\Box h_{\alpha\beta} + \partial^2_{\alpha\beta} h - \partial_\alpha \partial^\lambda h_{\lambda\beta} - \partial_\beta \partial^\lambda h_{\lambda\alpha} \Big) \qquad (10.6)$$

where

$$\left.\begin{aligned} \Box &:= \eta^{\mu\nu} \partial^2_{\mu\nu} \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial z^2} \\ &= \frac{1}{c^2}\frac{\partial}{\partial t^2} - \Delta \end{aligned}\right\} \qquad (10.7)$$

is the d'Alembert- or wave-operator in the flat Minkowski metric η, and

$$h := \eta^{\mu\nu} h_{\mu\nu} \qquad (10.8)$$

is the trace of h with respect to η. Also, indices on ∂_μ are raised and lowered with η:

$$\partial^\lambda := \eta^{\lambda\mu} \partial_\mu \qquad (10.9)$$

and also on h (for later use)

$$h^{\lambda}{}_{\beta} = \eta^{\lambda\alpha} h_{\alpha\beta} \quad (10.10)$$

$$h^{\lambda\sigma} = \eta^{\lambda\alpha} \eta^{\sigma\beta} h_{\alpha\beta} \quad (10.11)$$

Note that then

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + O(h^2), \quad (10.12)$$

because

$$\begin{aligned} g^{\alpha\sigma} g_{\beta\sigma} &= (\eta^{\alpha\sigma} - h^{\alpha\sigma})(\eta_{\beta\sigma} + h_{\beta\sigma}) \\ &= \delta^{\alpha}{}_{\beta} - h^{\alpha}{}_{\beta} + h_{\beta}{}^{\alpha} - h^{\alpha\sigma} h_{\beta\sigma} \\ &= \delta^{\alpha}{}_{\beta} + O(h^2). \end{aligned} \quad (10.13)$$

It will turn out later to be convenient to introduce another field $\bar{h}_{\alpha\beta}$ next to $h_{\alpha\beta}$ which is the "trace reversed" version of $h_{\alpha\beta}$:

$$\bar{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h \quad (10.14)$$

with inverse relation

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} \bar{h} \quad (10.15)$$

where $\bar{h} := \eta^{\alpha\beta} \bar{h}_{\alpha\beta} = -h \quad (10.16)$

In terms of $\bar{h}$ the last 3 terms on the right-hand side of (10.6) combine to

$$\partial^2_{\alpha\beta} h - \partial_\alpha \partial^\lambda h_{\lambda\beta} - \partial_\beta \partial^\lambda h_{\lambda\alpha}$$

$$= - \partial^2_{\alpha\beta} \bar{h} + \tfrac{1}{2} \partial_\alpha \partial_\beta \bar{h} + \tfrac{1}{2} \partial_\beta \partial_\alpha \bar{h}$$

$$- \partial_\alpha \partial^\lambda \bar{h}_{\lambda\beta} - \partial_\beta \partial^\lambda \bar{h}_{\lambda\alpha}$$

$$= - \partial_\alpha \partial^\lambda \bar{h}_{\lambda\beta} - \partial_\beta \partial^\lambda \bar{h}_{\lambda\alpha} \qquad (10.17)$$

so that

$$\overset{(1)}{R}_{\alpha\beta} = -\tfrac{1}{2} \Big\{ \Box \big(\bar{h}_{\alpha\beta} - \tfrac{1}{2} \eta_{\alpha\beta} \bar{h} \big)$$

$$- \partial_\alpha \partial^\lambda \bar{h}_{\lambda\beta} - \partial_\beta \partial^\lambda \bar{h}_{\lambda\alpha} \Big\} \qquad (10.18)$$

The 1st-order Ricci-Scalar is

$$\overset{(1)}{R} = -\tfrac{1}{2} \Big\{ \Box (\bar{h} - 2\bar{h}) - 2 \partial^{\alpha\beta} \bar{h}_{\alpha\beta} \Big\}$$

$$= \partial^{\alpha\beta} \bar{h}_{\alpha\beta} + \tfrac{1}{2} \Box \bar{h} \qquad (10.19)$$

The Einstein-Tensor is the "trace-reversed" Ricci-Tensor

$$\overset{(1)}{G}_{\alpha\beta} = \overset{(1)}{R}_{\alpha\beta} - \tfrac{1}{2} \eta_{\alpha\beta} \overset{(1)}{R} \qquad (10.20)$$

which, using (10.18) gives (note that $\bar{h}_{\alpha\beta}$ is the "trace-reversed" of $(\bar{h}_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\bar{h})$):

$$\overset{(1)}{G}_{\alpha\beta} = -\frac{1}{2}\Big\{ \Box \bar{h}_{\alpha\beta} - \partial_\alpha \partial^\lambda \bar{h}_{\lambda\beta} - \partial_\beta \partial^\lambda \bar{h}_{\lambda\alpha} + \eta_{\alpha\beta}\, \partial^\lambda \partial^\sigma \bar{h}_{\lambda\sigma} \Big\} \qquad (10.21)$$

Note that Ricci- and Einstein-Tensor are 2nd-order linear differential Operators applied to $h_{\alpha\beta}$ or, alternatively $\bar{h}$. We may write

$$\overset{(1)}{R}_{\alpha\beta} = D^{\mu\nu}_{\alpha\beta}\, h_{\mu\nu} \qquad (10.22)$$

$$\overset{(1)}{G}_{\alpha\beta} = \bar{D}^{\mu\nu}_{\alpha\beta}\, \bar{h}_{\mu\nu} \qquad (10.23)$$

where, from (10.6) and (10.21), we read off:

$$D^{\mu\nu}_{\alpha\beta} = -\frac{1}{2}\Big[\delta^\mu_\alpha \delta^\nu_\beta \Box + \eta^{\mu\nu} \partial^2_{\alpha\beta} - \delta^\nu_\beta \partial_\alpha \partial^\mu - \delta^\nu_\alpha \partial_\beta \partial^\mu \Big], \qquad (10.24)$$

$$\bar{D}^{\mu\nu}_{\alpha\beta} = -\frac{1}{2}\Big[\delta^\mu_\alpha \delta^\nu_\beta \Box + \eta_{\alpha\beta} \partial^\mu \partial^\nu - \delta^\nu_\beta \partial_\alpha \partial^\mu - \delta^\nu_\alpha \partial_\beta \partial^\mu \Big]. \qquad (10.25)$$

The linearised field equations make sense if the background metric around which we linearise is itself a solution to the field equation for some r.h.s. $T_{\alpha\beta}$. Minkowski space is a solution for $T_{\alpha\beta} = 0$ and $\Lambda = 0$. So the linearised field eq. make sense for small $T_{\alpha\beta}$ and $\Lambda = 0$. Its two forms corresponding to (6.19) and (6.21) are, respectively,

$$\boxed{\bar{D}^{\mu\nu}_{\alpha\beta}\,\bar{h}_{\mu\nu} = \kappa\,\overset{(1)}{T}_{\alpha\beta}}\,, \tag{10.26}$$

$$\boxed{D^{\mu\nu}_{\alpha\beta}\,h_{\mu\nu} = \kappa\left(\overset{(1)}{T}_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\overset{(1)}{T}\right)}\,. \tag{10.27}$$

An important observation is that the 2nd-order linear differential operators $\bar{D}$ and D on the left-hand sides have huge kernels. Suppose $h_{\mu\nu}$ is of the form

$$\overset{(\Lambda)}{h}_{\mu\nu} := \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \tag{10.28}$$

for some $\Lambda \in ST^0_1(M)$, and correspondingly,

$$\overset{(\Lambda)}{\bar{h}}_{\mu\nu} := \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu - \eta_{\mu\nu}\,\partial^\lambda \Lambda_\lambda\,, \tag{10.29}$$

then one immediately checks that

$$\mathcal{D}^{\mu\nu}_{\alpha\beta}\, h^{(\Lambda)}_{\mu\nu} \equiv 0 \qquad (10.30)$$

$$\bar{\mathcal{D}}^{\mu\nu}_{\alpha\beta}\, \bar{h}^{(\Lambda)}_{\mu\nu} \equiv 0 \qquad (10.31)$$

and moreover that

$$\partial^{\alpha} \circ \bar{\mathcal{D}}^{\mu\nu}_{\alpha\beta} = 0 \qquad (10.32)$$

(10.30) means: Whenever $h_{\mu\nu}$ is a solution to our field equations for given source $T_{\alpha\beta}$, then

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \qquad (10.33)$$

is also a solution of the same equation with same source $T_{\alpha\beta}$.

(10.32) means: Equation (10.26) has no solution unless the source $T_{\alpha\beta}$ satisfies the integrability condition

$$\partial_\alpha \overset{(1)}{T}{}^{\alpha\beta} = 0. \qquad (10.34)$$

(10.34) is the linear version of the general integrability condition (6.18) that follows from (8.20) [twice contracted 2nd. Bianchi Id.]

The interpretation of (10.33) is that of a "first-order diffeomorphism". To explain this, let $s \mapsto \phi_s$ be a one-parameter family of diffeomorphisms acting on the metric g, where

$$\phi_{s=0} = \mathrm{id}_M, \quad \left.\frac{d}{ds}\right|_{s=0} \phi_s = X \qquad (10.35)$$

Then

$$g \mapsto g_s := \phi_s^* g = \phi_s^*(\eta + h).$$

For small s, g_s will again be close to η:

$$g_s = \eta + h_s \qquad (10.36)$$

hence

$$\begin{aligned} h_s &= g_s - \eta = \phi_s^*(\eta + h) - \eta \\ &= \mathrm{id}_M^*(\eta + h) - \eta + s \left.\frac{d}{ds}\right|_{s=0} \phi_s^*(\eta + h) + O(s^2) \\ &= h + s\, L_X(\eta + h) + O(s^2) \\ &\overset{(1)}{=} h + s\, L_X \eta = h + L_\Lambda \eta \end{aligned} \qquad (10.37)$$

where $\Lambda := sX$ and s and h are both taken to be of first order, so $s L_X h = O(2)$.

$$\begin{aligned} \Rightarrow (h_s)_{\mu\nu} &\overset{(1)}{=} h_{\mu\nu} + s (L_X \eta)_{\mu\nu} \\ &= h_{\mu\nu} + s\left[X^\lambda \partial_\lambda \eta_{\mu\nu} + \partial_\mu X^\lambda \eta_{\lambda\nu} + \partial_\nu X^\lambda \eta_{\mu\lambda} \right] \\ &= h_{\mu\nu} + s(\partial_\mu X_\nu + \partial_\nu X_\mu) \end{aligned}$$

here we used (7.44) for $(L_X \eta)$ d.f.

or

$$(h_S)_{\mu\nu} = h_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu \qquad (10.38)$$

This is the first-order change in h due to a diffeomorphism generated by X.

The corresponding statement for the exact Einstein equation is as follows:

Let the Lorentz manifold (M, g) solve Einstein equation with energy-momentum tensor T, and let $\phi : M \to M$ be any diffeomorphism of type C^2. Then

$$g' = \phi^* g \qquad (10.39)$$

solves Einsteins equations for energy-momentum tensor $\phi^* T$. This is called diffeomorphism covariance. If T is a function of g and matter variables φ (fields), and if $T[g, \varphi]$ is equivariant in the sense of

$$T[\phi^* g, \phi^* \varphi] = \phi^* (T[g, \varphi]) \qquad (10.40)$$

then the coupled system of Einstein matter equations is said to be diffeomorphism invariant if for any solution (g, φ) of the combined Einstein-matter equations, $(\phi^* g, \phi^* \varphi)$ is again a solution of the very same equations for any $\phi \in \mathrm{Diff}(M)$.

In the case of linearised Einstein eqns the $T_{\alpha\beta}$ on the r.h.s is considered small of first order (like $h_{\alpha\beta}$). But so is the difference $\phi - id_M$ of the diffeomorphism ϕ from the identity. For that reason we have $\phi^* g = \phi^*(\eta + h) \overset{(1)}{=} h + \phi^* \eta$ and $\phi^* T \overset{(1)}{=} T$, so that diffeo ~~diffeo~~-covariance implies diffe-invariance at linearized level, that is: If h solves linearised eqns. for T, then

$$h' = \phi^* g - \eta = h + (\phi^* \eta - \eta) \qquad (10.41)$$

also solves the linearized equations for the very same T. If $\phi = \phi_s$ is a one parameter family of diffeos with $\phi_{s=0} = id$, we have for small s

$$\begin{aligned} h'_s &= h + \phi^*_s \eta - \eta \\ &= h + \phi^*_0 \eta - \eta + s \left.\frac{d}{ds}\right|_{s=0} \phi^*_s \eta + O(s^2) \\ &= h + s L_X \eta + O(s^2) \qquad (10.42) \end{aligned}$$

If $X = \left.\frac{d}{ds}\right|_{s=0}$. Setting $\Lambda = sX$ we just get the statement from L 10.9.

So we see that the linearized Einstein equations determine the field $h_{\alpha\beta}$, or $\bar{h}_{\alpha\beta}$, only up to redefinitions of the form

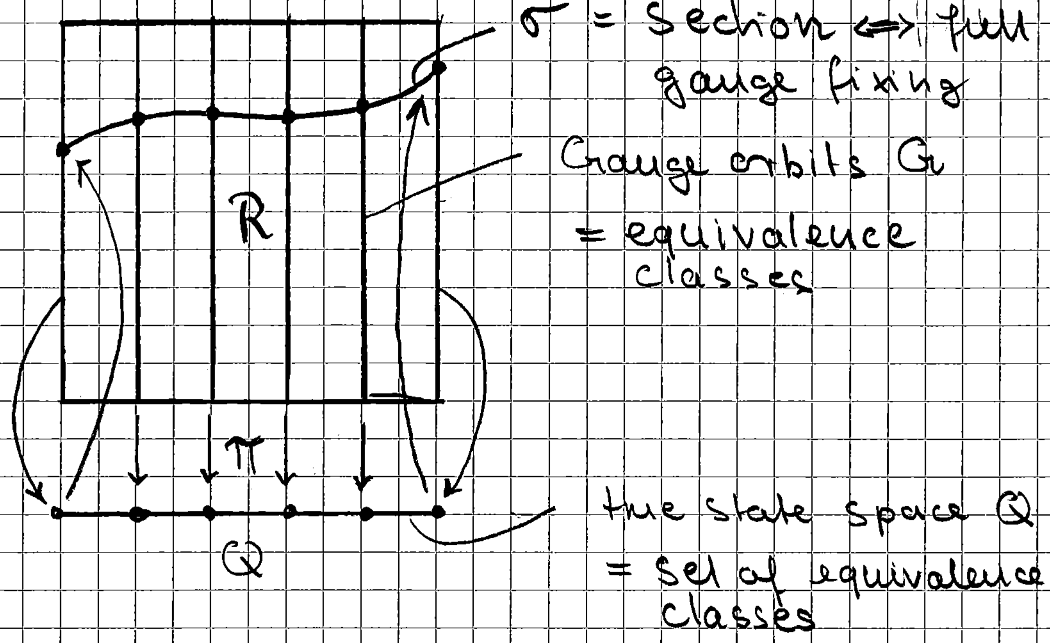
$$\left.\begin{aligned} h_{\alpha\beta} &\mapsto h'_{\alpha\beta} := h_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha \\ \bar{h}_{\alpha\beta} &\mapsto \bar{h}'_{\alpha\beta} := \bar{h}_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha - \eta_{\alpha\beta}\partial^\lambda \Lambda_\lambda \end{aligned}\right\} \quad 10.13$$

which correspond to (active interpretation) diffeomorphism-related fields on the same manifold and in the same coordinates. Alternatively one may say (passive interpretation) that $h'_{\alpha\beta}$ is the same field expressed in a different coord. system. We clearly prefer the active interpretation, because it properly refers to the geometric objects and can be applied to fields selectively: Given two fields, F_1 and F_2, and a diffeomorphism $\phi: M \to M$, we can transform F_1 but not F_2, i.e. consider $\phi^* F_1 = F_1'$ and F_2 and form combinations like $F_2 - \phi^* F_1$. Passively this would not make any sense: Taking sums or differences of tensor fields whose components refer to different charts makes no sense.

The transformations (10.43) map solutions to the field equations (linearised) to solutions to the very same equations and same source $T_{\alpha\beta}$. They are interpreted as

Gauge transformations: Are symmetries of the equations of motion, i.e. map solutions to solutions, whose interpretation is that of a redundancy transformation, in the following sense: Any two set of fields that differ by a gauge transformation represent the same physical state; they are indistinguishable by physical observables. This should be contrasted with proper physical symmetries, which map solutions to new, physically distinguishable solutions. Being related by a gauge transformation is an equivalence relation, the equivalence classes of which faithfully label physical states / solutions.

The procedure of gauge-fixing is meant to remove the redundancy in the mathematical description by selecting precisely one member per equivalence class. If more than one member is selected, one speaks of "partial gauge fixing".



The typical situation in "gauge theory":

R = redundant space of states

G = gauge orbits defining equivalence relation $\sim$ on R

$Q = R/\sim$ (also denoted by R/G)
= set of equivalence classes

$\pi: R \to Q$, $p \longmapsto [p]$ <u>natural projection</u>

σ: Section of "bundle" with <u>total space</u> R, <u>fibre</u> G and <u>base</u> Q, i.e.,

$$\sigma: Q \to R, \quad \pi \circ \sigma = id|_Q \tag{10.44}$$

= <u>gauge fixing</u>

Note: A global σ may not exist ($\to$ QCD "Gribov")

In our case, any two linearised gravitational fields $h_{\alpha\beta}$ (or $\bar{h}_{\alpha\beta}$) are gauge equivalent if and only if (10.43) holds. For $\bar{h}$ this means

$$\bar{h}'_{\alpha\beta} \sim \bar{h}_{\alpha\beta} \quad \text{iff}$$

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha - \eta_{\alpha\beta} \partial^\lambda \Lambda_\lambda \qquad (10.45)$$

A gauge fixing consists of finding a condition on the field $\bar{h}'$ so that Λ is fixed.

A <u>partial</u> gauge fixing is given by the <u>De Donder gauge condition</u>:

$$\partial^\alpha \bar{h}'_{\alpha\beta} = 0$$

It leads to

$$0 = \partial^\alpha \bar{h}_{\alpha\beta} + \Box \Lambda_\beta \qquad (10.46)$$

which can be solved by e.g.

$$\left.\begin{aligned} \Lambda_\beta(x) &= -\int G_{ret}(x, x')\, \partial^\alpha \bar{h}_{\alpha\beta}(x') \\ &= -\frac{1}{4\pi} \int \frac{\partial^\alpha \bar{h}_{\alpha\beta}\left(t - \frac{\|\vec{x}-\vec{x}'\|}{c}, \vec{x}'\right)}{\|\vec{x}-\vec{x}'\|}\, d^3x' \\ G_{ret}(x, x') &= \frac{1}{4\pi}\, \Theta(x^0 - x'^0)\, \delta\left((x-x')^2\right) \end{aligned}\right\} \qquad (10.47)$$

This shows: Given any $\bar{h}_{\alpha\beta}$, there exists a $\bar{h}'_{\alpha\beta}$ in the same gauge orbit as $\bar{h}_{\alpha\beta}$ that satisfies the De Donder condition.

<u>But</u>: There exist more than one $\bar{h}'_{\alpha\beta}$ in the same orbit. Any Λ_β that satisfies

$$\Box \Lambda_\beta = 0 \qquad (10.48)$$

defines a residual gauge freedom preserving the De Donder condition. We will later see how to use that in order to prove that gravitational waves have exactly two polarization degrees of freedom.

Now, given that we may impose the De Donder condition next to the field equation, the linearized Einstein eq. (10.26) turns into

$$\left.\boxed{\begin{aligned} \Box \bar{h}_{\alpha\beta} &= -2\kappa T_{\alpha\beta} \\ \partial^\alpha \bar{h}_{\alpha\beta} &= 0 \end{aligned}} \quad \begin{matrix}\text{linearized} \\ \text{Einstein Eq.} \\ \text{in De Donder} \\ \text{gauge}\end{matrix} \right\} \qquad (10.49)$$

Look what happend: De Donder's condition decoupled the 10 equations for $h_{\alpha\beta}$, which where formerly coupled by term $\sim \partial^\alpha \bar{h}_{\alpha\beta}$. (compare (10.21)).

This should be seen in analogy to the Lorenz-gauge in electrodynamics

$$\partial^\mu A_\mu = 0 \quad \text{(Lorenz gauge)} \qquad (10.50)$$

There one has

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu \qquad (10.51)$$

with

$$A^\mu = \left(\tfrac{1}{c}\phi, \vec{A}\right) \qquad (10.52)$$

$$j^\mu = \left(c\rho, \vec{j}\right) \qquad (10.53)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \qquad (10.54)$$

$$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu \qquad (10.55)$$

homogeneous Maxwell eq.

$$\left.\begin{array}{c} dF = 0 \iff F = dA \\ \Updownarrow \\ \vec{\nabla}\cdot\vec{B} = 0 \text{ and } \dot{\vec{B}} + \vec{\nabla}\times\vec{E} = 0 \end{array}\right\} \quad (10.56)$$

inhomogeneous Maxwell eq.

$$\left.\begin{aligned} &\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \\ \Leftrightarrow \quad &\vec{\nabla} \vec{E} = \frac{1}{\varepsilon_0} \rho \\ &\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \dot{\vec{E}} = \mu_0 \vec{j} \end{aligned}\right\} \quad (10.57)$$

Its integrability cond. is obviously

$$\partial_\nu j^\nu = 0 \Leftrightarrow \dot{\rho} + \vec{\nabla} \vec{j} = 0 \qquad (10.58)$$

(charge conservation)

Using $F = dA$ in (10.57) we get

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \Box A^\nu - \partial^\nu (\partial^\lambda A_\lambda) \\ &= D^\nu{}_\lambda A^\lambda \end{aligned} \qquad (10.59)$$

where

$$D^\nu{}_\lambda = \delta^\nu{}_\lambda \Box - \partial^\nu \partial_\lambda \qquad (10.60)$$

Observe that all A^λ of the form $\partial^\lambda \Lambda$ are in the kernel of D:

$$D^\nu{}_\lambda \partial^\lambda \Lambda = \Box \partial^\nu \Lambda - \partial^\nu \Box \Lambda \equiv 0. \qquad (10.61)$$

also $\quad \partial_\nu \circ D^\nu{}_\lambda \equiv 0 \qquad (10.62)$

in analogy to (10.31) and (10.32).

Hence have gauge-transformation

$$A_\lambda \mapsto A'_\lambda := A_\lambda + \partial_\lambda \Lambda \tag{10.63}$$

which does not change $F_{\mu\nu}$ and hence none of the observable quantities $\vec{E}$ and $\vec{B}$.
If we impose Lorenz gauge

$$\partial^\lambda A'_\lambda = \partial^\lambda A_\lambda + \Box \Lambda = 0 \tag{10.64}$$

then

$$\Lambda(x) = - \int G_{ret}(x, x') \, \partial^\lambda A_\lambda(x') \, d^4x' \tag{10.65}$$

Hence each gauge orbit contains a member satisfying the Lorenz gauge. Imposing it, Maxwells equations are equivalent to $F = dA$ and

$$\boxed{\begin{array}{l} \partial_\mu F^{\mu\nu} = \Box A^\nu = \mu_0 j^\nu \\ \text{together with } \partial_\lambda A^\lambda = 0 \end{array}} \tag{10.66}$$

The Lorenz condition is only a partial gauge fixing. Residual gauge transformations with $\Box \Lambda = 0$ are still possible while keeping Lorenz gauge. Again this leads to the possibility to remove one more component of A outside matter (where $j^\mu = 0$).