Lecture 11

Linearised Einstein equation: Newtonian limit and next to leading order approximation. The gravitomagnetic field.

If we impose the De Donder-condition

$$\partial^\alpha \bar{h}_{\alpha\beta} = 0 \tag{11.1}$$

the linearized field equations become

$$\Box \bar{h}_{\alpha\beta} = -2\kappa T_{\alpha\beta}. \tag{11.2}$$

Note that (11.2) is equivalent to the linearized Einstein equation in combination with (11.1). Hence, in particular, $T_{\alpha\beta}$ should satisfy

$$\partial^\alpha T_{\alpha\beta} = 0. \tag{11.3}$$

Now (11.2) is solved by, e.g., retarded Green function

$$\bar{h}_{\alpha\beta}(x) = -2\kappa \int G_{ret}(x,x')\, T_{\alpha\beta}(x')\, d^4x' \tag{11.4}$$

or

$$\bar{h}_{\alpha\beta}(t,\vec{x}) = -\frac{\kappa}{2\pi}\int d^3x' \, \frac{T_{\alpha\beta}\left(t-\frac{\|\vec{x}-\vec{x}'\|}{c}, \vec{x}'\right)}{\|\vec{x}-\vec{x}'\|} \tag{11.5}$$

For dust like matter

$$T^{\alpha\beta} = \varrho u^\alpha u^\beta \tag{11.6}$$

with $u^\alpha = \gamma c\,(1, \vec\beta)$, $\vec\beta = \frac{\vec v}{c}$, $\gamma = (1-\vec\beta^2)^{-1/2}$, this is

$$T^{\alpha\beta} = \varrho c^2 \gamma \left(\begin{array}{c|c} 1 & \vec\beta^T \\ \hline \vec\beta & \vec\beta \otimes \vec\beta^T \end{array}\right) \tag{11.7}$$

The Newtonian approximation keeps only the leading-order terms

$$T^{\alpha\beta} = \varrho c^2 \delta^\alpha_0 \delta^\beta_0 \tag{11.8a}$$

and no retardation. Note that the integrability condition implies ϱ to be static:

$$\partial_\alpha T^{\alpha\beta} = 0 \iff \partial_t \varrho = 0. \tag{11.8b}$$

Hence

$$\bar h_{00}(\vec x) = -\frac{\kappa c^2}{2\pi} \int d^3x' \, \frac{\varrho(\vec x)}{\|\vec x - \vec x'\|}$$

$$= \frac{\kappa c^2}{2\pi G}\, \phi(\vec x),$$

where

$$\phi(\vec x) := -G \int d^3x' \, \frac{\varrho(\vec x)}{\|\vec x - \vec x'\|}$$

is the "Newtonian Potential".

Hence

$$\left.\begin{aligned} h_{00} &= \bar{h}_{00} - \tfrac{1}{2}\eta_{00}\,\bar{h}_{00} = \tfrac{1}{2}\bar{h}_{00} \\ h_{aa} &= -\tfrac{1}{2}\eta_{aa}\,\bar{h}_{00} = \tfrac{1}{2}\bar{h}_{00} \end{aligned}\right\} \quad (11.11)$$

and all other h's vanish.

From the geodesic equation we have in that limit $v/c \to 0$, $d\tau = dt$ and

$$\frac{d^2 z^a}{dt^2} + \Gamma^a_{00}\, c^2 = 0$$

$$\frac{d^2 z^a}{dt^2} + \frac{1}{2}\eta^{ab}(-h_{00,b})\, c^2 = 0$$

$$\Leftrightarrow \quad \frac{d^2 z^a}{dt^2} = -\frac{c^2}{2}\,\partial_b (h_{00}) \qquad (11.12)$$

In order to get Newtonian eq.

$$\Rightarrow \quad \phi = \frac{c^2}{2} h_{00} \qquad (11.13)$$

(no constant since $\phi(\vec{x} \to \infty) = 0$ and $h_{00}(\vec{x} \to \infty) = 0$. Hence

$$h_{00} = \frac{2}{c^2}\phi = \frac{1}{2}\bar{h}_{00} = \frac{\kappa c^2}{4\pi G}\phi$$

$$\Leftrightarrow \quad \boxed{\kappa = \frac{8\pi G}{c^4}} \qquad (11.14)$$

Inserting (11.14) into (11.9) gives

$$\bar{h}_{00} = \frac{4}{c^2}\phi \qquad (11.15)$$

and this into (11.11):

$$\left.\begin{aligned} h_{00} &= \frac{2}{c^2}\phi \\ h_{0b} &= 0 \\ h_{ab} &= \delta_{ab}\frac{2}{c^2}\phi \end{aligned}\right\} \qquad (11.16)$$

The metric then becomes

$$\boxed{\begin{aligned} g &= (\eta_{\alpha\beta} + h_{\alpha\beta})\, dx^\alpha \otimes dx^\beta \\ &= \left(1 + \frac{2\phi}{c^2}\right) c^2\, dt \otimes dt \\ &\quad - \left(1 - \frac{2\phi}{c^2}\right) d\vec{x} \otimes d\vec{x} \end{aligned}} \qquad (11.17)$$

metric in Newtonian limit, wher
ϕ = Newtonian gravitational pot.

As we have seen, the geodesic equation reduces to Newtonian equation of motion if we systematically neglect terms $(V/c)^n$ for $n \geq 1$.

The next level of approximation would consist in taking into account first but not second or higher powers in (V/c). This means that we also consider mass- or energy-currents as sources of grav. fields

$$\left.\begin{aligned} T^{0a} &= \rho c^2 V^a/c \\ &= \rho c V^a \end{aligned}\right\} \quad (11.18)$$

Again we must be careful to not admit Energy-momentum tensors as sources to the linearised equations that do not satisfy the integrability condition

$$\partial_\alpha T^{\alpha\beta} = 0 \qquad (11.19)$$

$$\beta = 0: \quad \partial_0 T^{00} + \partial_a T^{a0} = 0$$

$$\Leftrightarrow \quad \partial_t \rho + \partial_a (\rho V^a) = 0 \qquad (11.20)$$

$$\beta = b: \quad \partial_0 T^{0b} + \partial_a T^{ab} = 0$$

$$\Leftrightarrow \quad \partial_t (\rho V^b) = 0. \qquad (11.21)$$

So $\rho\vec{V}$ must be time independent at each fixed spatial position and $\rho\vec{V}$ must be divergence-free (if we take $\dot{\rho} = 0$).

So essentially we get the restriction that ρ and $\vec{v}$ do not depend on t and $\vec{\nabla}(\rho\vec{v}) = 0$. This would e.g. be satisfied for rigid rotation

$$\left.\begin{aligned} \rho &= \text{const} \\ \vec{v} &= \vec{\Omega} \times \vec{x} \end{aligned}\right\} \quad (11.22)$$

since $\vec{\nabla}\cdot\vec{v} = \partial^a \varepsilon_{abc} \Omega^b x^c = \delta^{ac} \varepsilon_{abc} \Omega^b = 0$.

Given that, we have

$$\bar{h}_{0a}(\vec{x}) = \frac{\kappa c}{2\pi} \int d^3x' \, \frac{\rho(\vec{x}')\, v^a(\vec{x}')}{\|\vec{x} - \vec{x}'\|} \quad (11.23)$$

But $h_{0a} = \bar{h}_{0a} - \frac{1}{2}\eta_{0a}\bar{h} = \bar{h}_{0a}$ and h_{0a} does not contribute to the trace $h = \eta^{\alpha\beta} h_{\alpha\beta} = h_{00} - h_{11} - h_{22} - h_{33}$. Hence the other components, h_{00} and h_{ab}, are just as in the Newtonian limit and we get the only additional coefficient

$$h_{0a}(\vec{x}) = \frac{4G}{c^3} \int d^3x' \, \frac{\rho(\vec{x}')\, v^a(\vec{x}')}{\|\vec{x} - \vec{x}'\|} \quad (11.24)$$

Let us call

$$\vec{h} := (h_{01}, h_{02}, h_{03}) \quad (11.25)$$

then the metric reads

$$g = \left(1 + \frac{2\phi}{c^2}\right) c\,dt \otimes c\,dt$$

$$- \left(1 - \frac{2\phi}{c^2}\right) d\vec{x} \otimes d\vec{x}$$

$$+ \vec{h} \left(c\,dt \otimes d\vec{x} + d\vec{x} \otimes c\,dt \right) \qquad (11.26)$$

where ϕ and $\vec{h}$ are independent of time

The spatial components of the geodesic equation become

$$\ddot{z}^a + \Gamma^a_{00} c^2 + 2 \Gamma^a_{0b} c \dot{z}^b = 0 \qquad (11.27)$$

$$\Gamma^a_{00} = \frac{1}{2} \eta^{ab} \left(-h_{00,b} + 2 h_{\cancel{b0,0}}\right)$$

$$= \frac{1}{2} h_{00,a} \qquad (11.28)$$

$$\Gamma^a_{0b} = \frac{1}{2} \eta^{ac} \left(-h_{0b,c} + h_{c0,b} + h_{\cancel{bc},0}\right)$$

$$= \frac{1}{2} \left(\partial_a h_b - \partial_b h_a\right) \qquad (11.29)$$

Hence (11.27) becomes

$$\ddot{z}^a + \frac{c^2}{2} h_{00,a} + c \left(\partial_a h_b - \partial_b h_a\right) \dot{z}^b = 0$$

$$(11.30)$$

In 3-vector notation:

$$\left[\dot{\vec{Z}}\times(\vec{\nabla}\times\vec{h})\right]_a = \varepsilon_{abc}\,\dot{Z}^b\,\varepsilon^{cde}\,\partial_d h_e$$

$$= \varepsilon_{abc}\,\varepsilon^{dec}\,\dot{Z}^b\,\partial_d h_e$$

$$= (\delta_a^d\,\delta_b^e - \delta_a^e\,\delta_b^d)\,\dot{Z}^b\,\partial_d h_e$$

$$= (\partial_a h_b - \partial_b h_a)\,\dot{Z}^b \qquad (11.31)$$

Hence the spatial part of the geodesic equation is equivalent to

$$\ddot{\vec{Z}} = -\vec{\nabla}\phi + \dot{\vec{Z}}\times\vec{B} \qquad (11.32)$$

analogous to the Lorentz-force law in electrostatics (ϕ and $\vec{B}$ are time independent), where

$$\vec{B} = -c(\vec{\nabla}\times\vec{h}) \qquad (11.33)$$

and

$$\vec{h}(\vec{x}) = \frac{4G}{c^3}\int d^3x' \;\frac{\rho(\vec{x}')\,\vec{v}(\vec{x}')}{\|\vec{x}-\vec{x}'\|} \qquad (11.34)$$

$$\vec{B}(\vec{x}) = -\frac{4G}{c^2}\,\vec{\nabla}\times\int d^3x' \;\frac{\rho(\vec{x}')\,\vec{v}(\vec{x}')}{\|\vec{x}-\vec{x}'\|} \qquad (11.35)$$

This is called the "gravitomagnetic field" due to its appearence in (11.32).

Note that the "dot-derivative" $\dot{\vec{z}}$ of $\vec{z}$ in (11.32) is still with respect to proper time. The zeroth component of the geodesic equation determines its relation to Newtonian time.

$$\ddot{z}^0 + \cancel{\Gamma^0_{00}}\,\dot{z}^0\dot{z}^0 + 2\,\Gamma^0_{0b}\,\dot{z}^0\dot{z}^b + \underbrace{O(\dot{z}^2)}_{\text{neglected}} = 0 \qquad (11.36)$$

$$\ddot{z}^0 + \eta^{00}\left(-\cancel{h_{0b,0}} + h_{00,b} + \cancel{h_{b0,0}}\right)\dot{z}^0\dot{z}^b = 0$$

Writing $z^0 = ct$ this gives

$$\ddot{t} + h_{00,b}\,\dot{t}\,\dot{z}^b = 0 \qquad (11.37)$$

but $h_{00,b}\,\dot{z}^b = 2\,\dot{z}^b\,\phi_{,b}/c^2$

$$= 2\dot{\phi}/c^2$$

$$\Rightarrow \ddot{t} = -2\,\dot{t}\,\dot{\phi}/c^2$$

$$\Rightarrow \frac{\ddot{t}}{\dot{t}} = -2\,\dot{\phi}/c^2 \qquad (11.38)$$

$$\Rightarrow \dot{t}(\tau) = \exp\left(-2\phi(\vec{z}(\tau))/c^2\right) \qquad (11.39)$$

where we have chosen the integration constant such that $\dot{t} = 1$ for $\phi = 0$. This implies

$$t = \tau + \text{Terms}\left(\frac{\phi}{c^2}\right)^n \quad \text{for } n \geq 1 \qquad (11.40)$$

But since in leading order

$$\phi / c^2 \cong (v/c)^2 \qquad (11.41)$$

we also have

$$\tau = t + \text{Terms} \left(\frac{v}{c}\right)^n \quad n \geq 2 \qquad (11.42)$$

which of course we could have said straightaway. As a result, we may replace the τ derivatives in (11.32) by t-derivatives without changing anything at the order we are here considering.

As an application to (11.34-35) we calculate the gravitomagnetic field of a ball

$$B_R := \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| \leq R^3\} \qquad (11.43)$$

with constant mass density in B_R

$$\rho(\vec{x}) = \begin{cases} \rho_0 > 0 & \text{for } \vec{x} \in B_R \\ 0 & \text{for } \vec{x} \notin B_r \end{cases} \qquad (11.44)$$

This, of course, is only <u>locally constant</u> and therefore violating our conditions on the measure-zero set $S_R = \partial B_r = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| = R\}$. We shall ignore that problem here, but we emphasize that one must be careful to not violate $\partial_\alpha T^{\alpha\beta} = 0$.

We assume the mass to be in the state of constant rigid rotation (11.22), i.e.

$$\vec{v}(\vec{x}') = \vec{\Omega} \times \vec{x}' \qquad (11.45)$$

Then

$$\vec{h}(\vec{x}) = \frac{4G}{c^3} \rho_0 \vec{\Omega} \times \int_{B_R} d^3x' \frac{\vec{x}'}{\|\vec{x} - \vec{x}'\|} \qquad (11.46)$$

For the integration we use spherical polar coordinates such that

$$\vec{x} = r\,\vec{e}_z$$
$$\vec{x}' = r'\left(\sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z\right) \quad (11.47)$$

hence

$$\|\vec{x}-\vec{x}'\| = r^2 + r'^2 - 2rr'\cos\theta \quad (11.48)$$

The integration over the x- and y-comp. of $\vec{x}'$ give zero because of the φ-integration:

$$\int_0^{2\pi} d\varphi \quad f(\theta, r, r') \begin{pmatrix}\sin\varphi\\ \cos\varphi\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad (11.49)$$

which only contains φ-dependence by being linear in $\sin\varphi$ or $\cos\varphi$.

The z-component of the integral is

$$\int_{B_R} d^3x' \quad \frac{r'\cos\theta\,\vec{e}_z}{(r^2+r'^2-2rr'\cos\theta)^{1/2}}$$

$$= \vec{e}_z\ 2\pi \int r'^3\,dr' \int_{-1}^{+1} d\sigma \quad \frac{\sigma}{\left[r^2+r'^2-2rr'\sigma\right]^{1/2}} \quad (11.50)$$

where we set $\sigma := \cos\theta$

$$\int_{-1}^{+1} d\sigma \; \frac{\sigma}{\left(r^2+r'^2-2rr'\sigma\right)^{1/2}}$$

$$= \int_{-1}^{+1} d\sigma \; \sigma \; \left(\frac{-1}{rr'}\right) \frac{d}{d\sigma} \left(r^2+r'^2-2rr'\sigma\right)^{1/2}$$

$$= -\frac{\sigma}{rr'} \left(r^2+r'^2-2rr'\sigma\right)^{1/2} \Big|_{-1}^{+1}$$

$$+ \frac{1}{rr'} \int_{-1}^{+1} d\sigma \left(r^2+r'^2-2rr'\sigma\right)^{1/2}$$

$$= -\frac{1}{rr'} \left[|r-r'| + |r+r'| \right]$$

$$- \frac{1}{3}\left(\frac{1}{rr'}\right)^2 \left(r^2+r'^2-2rr'\sigma\right)^{3/2} \Big|_{-1}^{+1}$$

$$= -\frac{1}{rr'} \left[|r-r'| + |r+r'| \right]$$

$$- \frac{1}{3}\left(\frac{1}{rr'}\right)^2 \left[|r-r'|^3 - |r+r'|^3 \right] \qquad (11.51)$$

1. Case : $r > r'$

$$= -\frac{1}{rr'} \left[r - r' + r + r' \right]$$

$$- \frac{1}{3}\left(\frac{1}{rr'}\right)^2 \left[(r-r')^3 - (r+r')^3 \right]$$

$$= -\frac{2}{r'}$$

$$-\frac{1}{3}\left(\frac{1}{rr'}\right)^2 \Big[r^3 - r'^3 + 3rr'^2 - 3r^2r'$$
$$- r^3 - r'^3 - 3rr'^2 - 3r^2r'\Big]$$

$$= -\frac{2}{r'} - \frac{1}{3}\left(\frac{1}{rr'}\right)^2 \left[-2r'^3 - 6r^2r'\right]$$

$$= \frac{2}{3}\,\frac{r'}{r^2} \qquad (11.52)$$

2. Case : $r < r'$

This we do not need to do again because it just consists of interchanging the rôles of r and r'. Hence we get for the integral

$$= \frac{2}{3}\,\frac{r}{r'^2} \qquad (11.53)$$

Together

$$\int_{-1}^{+1} d\sigma \frac{\sigma}{\left[r^2 + r'^2 - 2r'r\sigma\right]^{1/2}} = \frac{2}{3}\begin{cases} r'/r^2 & \text{for } r > r' \\ r/r'^2 & \text{for } r < r' \end{cases} \qquad (11.54)$$

Since $$d^3x' = 2\pi \int dr'\, r'^2 \int_{-1}^{+1} d(\cos\theta) \qquad (11.55)$$

and $e_z = \vec{x}/r$ we obtain for the integral in (11.46) the expression

$$\int d^3x' \frac{\vec{x}'}{\|\vec{x}-\vec{x}'\|} = \int dr' \int_{S^2(r')} d\Omega' \frac{\vec{x}'}{\|\vec{x}-\vec{x}'\|}$$

$$= \int dr' \, r'^2 \, 2\pi \, r' \vec{e}_z \int_{-1}^{+1} d\sigma \frac{\sigma}{\left[r^2+r'^2-2rr'\sigma\right]^{1/2}}$$

$$= \vec{x} \cdot \frac{4\pi}{3} \int dr' \begin{cases} r'^4/r^3 & \text{for } r > r' \\ r' & \text{for } r < r' \end{cases} \qquad (11.56)$$

The most important application is to consider the gravitational field outside a mass, i.e. $r > R \geq r'$. In that case (11.46) with (11.56) for $r > r'$ gives

$$h(\vec{x}) = \frac{4G}{c^3} \frac{4\pi}{15} \varrho_0 R^5 \, \vec{\Omega} \times \frac{\vec{x}}{r^3} \qquad (11.57)$$

where R is the radius of the star.

Note that the mass and moment of inertia of our homogeneous mass ball are

$$M = \frac{4\pi}{3} R^3 \varrho_0 \qquad (11.58)$$

$$I = \int dm(r,\theta) \, r^2 \sin^2\theta$$

$$= 2\pi \underbrace{\int_0^R r^4 dr}_{\frac{1}{5}R^5} \underbrace{\int_{-1}^{+1} d\sigma \, (1-\sigma^2)}_{4/3}$$

$$I = 2\pi \rho_0 \frac{4}{15} R^5$$

$$= 2 \cdot \frac{4\pi}{15} \rho_0 R^5 \qquad (11.59)$$

Also noting that angular momentum is

$$\vec{J} = I \vec{\Omega} \qquad (11.60)$$

we can now rewrite (11.57) as

$$\boxed{\vec{h}(\vec{x}) = \frac{4}{5} \frac{G}{c^3} M R^2 \; \vec{\Omega} \times \frac{\vec{x}}{r^3}} \qquad (11.61)$$

$$\boxed{\vec{h}(\vec{x}) = 2 \frac{G}{c^3} \vec{J} \times \frac{\vec{x}}{r^3}} \qquad (11.62)$$

Note that the dimensions are right: $\vec{h}$ is dimensionless:

$$[\vec{h}] = m^3 \cdot kg^{-1} \cdot s^{-2} \cdot s^3 \cdot m^{-3} \cdot kg^1 \cdot m^2 \cdot s^{-1} \cdot m^{-2}$$

$$= [1].$$

The gravitomagnetic field $\vec{B}$ follows from this through (11.33). Hence we need to calculate

$$\vec{\nabla} \times \left(\vec{\Omega} \times \frac{\vec{x}}{r^3}\right) = \vec{\Omega} \underbrace{\left(\vec{\nabla} \cdot \frac{\vec{x}}{r^3}\right)}_{0} - (\vec{\Omega} \cdot \vec{\nabla}) \frac{\vec{x}}{r^3} \qquad (11.63)$$

$$\left.\begin{array}{l} \partial_a \left(\dfrac{x^b}{r^3} \right) = \dfrac{1}{r^3} \left(\delta^b_a - 3\, n^b n_a \right) \\ \text{where} \quad \vec{n} = \vec{x}/r \end{array}\right\} \quad (11.64)$$

Hence

$$\vec{\nabla} \times \left(\vec{\Omega} \times \frac{\vec{x}}{r^3} \right) = -\frac{1}{r^3} \left(\vec{\Omega} - 3 (\vec{\Omega} \cdot \vec{n}) \vec{n} \right) \quad (11.65)$$

$$\Rightarrow \boxed{\begin{aligned} \vec{B} &= -c \left(\vec{\nabla} \times \vec{h} \right) \\ &= \frac{4}{5} \frac{G}{c^2} M R^2 \, \frac{\vec{\Omega} - 3 (\vec{\Omega} \cdot \vec{n}) \vec{n}}{r^3} && (11.66) \\ &= 2 \frac{G}{c^2} \, \frac{\vec{J} - 3 (\vec{J} \cdot \vec{n}) \vec{n}}{r^3} && (11.67) \end{aligned}}$$

Note that the physical dimension is that of a frequency:

$$[\vec{B}] = s^{-1} = Hz \quad (11.68)$$

Note also that the dependence on $\vec{x}$ on the right-hand sides of (11.66-67) are exact <u>dipol fields</u>

To see the physical interpretation of a non-vanishing h_{0a} we imagine a ficticious metric that deviates from a Minkowski metric just by those h_{0a}:

$$g = \eta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta + h_{0a}\,(c\,dt \otimes dx^a + dx^a \otimes c\,dt) \qquad (11.69)$$

where the h_{0a} are independent of time. The Christoffel symbols (linear in $\vec{h}$) are then

$$\Gamma^0_{00} = \frac{1}{2}\eta^{00}(-h_{00,0} + h_{00,0} + h_{00,0}) = 0$$

$$\Gamma^0_{0a} = \Gamma^0_{a0} = \frac{1}{2}\eta^{00}(-h_{0a,0} + h_{00,a} + h_{a0,0}) = 0$$

$$\Gamma^a_{00} = \frac{1}{2}\eta^{ab}(-h_{00,a} + h_{a0,0} + h_{0a,0}) = 0$$

$$\Gamma^0_{ab} = \frac{1}{2}\eta^{00}(-h_{ab,0} + h_{0a,b} + h_{b0,a}$$

$$= \frac{1}{2}(h_{0a,b} + h_{0b,a}) \qquad (11.70)$$

$$\Gamma^a_{0b} = \Gamma^a_{b0} = \frac{1}{2}\eta^{ab}(-h_{0b,a} + h_{a0,b} + h_{ba,0})$$

$$= \frac{1}{2}(\partial_a h_b - \partial_b h_a) \qquad (11.71)$$

$$\Gamma^a_{bc} = 0$$

Now we recall the transformation

$$\bar{X}^\lambda(q) := X^\lambda(q) - X^\lambda(p) + \frac{1}{2}\Gamma^\lambda_{\alpha\beta}(p)\left(X^\alpha(q) - X^\alpha(p)\right)\left(X^\beta(q) - X^\beta(p)\right) \quad (11.72)$$

of which we have shown that it transforms the $\bar{\Gamma}^\lambda_{\alpha\beta}(p)$ to zero
($\rightarrow$ Riemann normal coordinates at p)
Hence the $\bar{X}$ - coordinates are locally inertial and hence, in particular, <u>non - rotating</u> at p.

In our case for (11.69) we get for the non - rotating spatial coordinates setting $X^\lambda(p) = 0$,

$$\begin{aligned}
\bar{X}^a &= X^a + \Gamma^a_{0b}(p)\, X^0 X^b \\
&= X^a + \frac{1}{2}\left(\partial_a h_b - \partial_b h_a\right) X^0 X^b \\
&= X^a + \frac{c}{2}\left(\partial_a h_b - \partial_b h_a\right) t\, X^b \\
&= X^a + \frac{c}{2}\, t \left(\vec{X} \times (\vec{\nabla} \times h)\right)_a \\
&= X^a + t\, \frac{1}{2}\left(-c\,(\vec{\nabla} \times \vec{h}) \times \vec{X}\right)_a \\
&= X^a + t\left(\frac{\vec{B}}{2} \times \vec{X}\right)^a \qquad (11.73)
\end{aligned}$$

This shows that with respect to the coordinates X^a the inertial frame is rotating with angular velocity

$$\vec{\omega} := -\frac{\vec{B}}{2} \qquad (11.74)$$

I a fixed $\bar{X}$ coordinate corresponds to

$$\frac{d}{dt}\vec{X} = -\left(\frac{\vec{B}}{2} \times \vec{X}\right) =: \vec{\omega} \times \vec{X}. \qquad (11.75)$$

Hence we may say that the effect of non-zero $\vec{h}$ is to make local inertial frames rotate against our coordinate system with angular velocity $-\vec{B}/2$. This is called the Thirring - Lense precession. From (11.66 - 67) we get

$$\boxed{\begin{aligned} \vec{\omega}_{TL}(\vec{x}) &= \frac{2}{5}\frac{G}{c^2} MR^2 \frac{3(\vec{\Omega}\cdot\vec{n})\vec{n} - \vec{\Omega}}{r^3} \\ &= \frac{G}{c^2}\frac{3(\vec{J}\cdot\vec{n})\vec{n} - \vec{J}}{r^3} \end{aligned}} \qquad (11.76)$$

(Thirring - Lense - Precession)