Lecture 12

Linearized Einstein equation: Gauge fixing and helicity states of gravitational waves

We recall the linearized Einstein equations in DeDonder gauge (10.49):

$$\left.\begin{aligned} \Box \bar{h}_{\alpha\beta} &= -2\kappa T_{\alpha\beta} \\ \partial^\alpha \bar{h}_{\alpha\beta} &= 0 \end{aligned}\right\} \quad (12.1)$$

which we compared with Maxwell equations in Lorenz gauge (10.66):

$$\left.\begin{aligned} \Box A_\alpha &= \mu_0 J_\alpha \\ \partial^\alpha A_\alpha &= 0. \end{aligned}\right\} \quad (12.2)$$

Let us discuss the Maxwell equations first, which are a little simpler. We are interested in the field outside the source, i.e. $J_\alpha = 0$. Then

$$\left.\begin{aligned} \Box A_\alpha &= 0 \\ \partial^\alpha A_\alpha &= 0 \end{aligned}\right\} \quad (12.3)$$

We assume A_α to admit a Fourier-transform (which is a non-trivial assumption)

$$\left.\begin{aligned} \tilde{A}_\alpha(k) &:= \frac{1}{(2\pi)^2}\int d^4x\, e^{-ik\cdot x}\, A_\alpha(x) \\ A_\alpha(x) &:= \frac{1}{(2\pi)^2}\int d^4k\, e^{ik\cdot x}\, \tilde{A}_\alpha(k) \end{aligned}\right\} \quad (12.4)$$

where $k\cdot x := k_\mu x^\mu$. (12.5)

Note that generally, under Fourier-transformation

$$\partial_\alpha \longmapsto i k_\alpha \qquad (12.6)$$

that is

$$\partial_\alpha A_\beta(x) = \frac{1}{(2\pi)^2}\int d^4x\, e^{ik\cdot x}\left(ik_\alpha \tilde{A}_\beta(k)\right) \qquad (12.7)$$

So that the Fourier-transform of $\partial_\alpha A_\beta$ is $ik_\alpha \tilde{A}_\beta$.

Hence the Fourier transformation of Maxwell's equations (12.3) is

$$-(k_\lambda k^\lambda)\, \tilde{A}_\alpha(k) = 0 \qquad (12.8)$$

$$k^2\, \tilde{A}_\alpha(k) = 0 \qquad (12.9)$$

(12.8) says that

$$\tilde{A}_\alpha(k) = 0 \text{ if } k^\lambda k_\lambda \neq 0. \qquad (12.10)$$

In other words: The support of $\tilde{A}_\alpha$ is confined to the light cone:

$$\mathrm{Supp}(\tilde{A}_\alpha) \subseteq \{ k \in V : \eta(k,k) = 0 \}. \qquad (12.11)$$

(12.9) says that the value $\tilde{A}_\alpha(k)$ is η-orthogonal to k:

$$\tilde{A}(k) \in \{k\}^\perp := \{ v \in V : \eta(v,k) = 0 \} \qquad (12.12)$$

Note that since $\tilde{A}(k) = 0$ if k is not lightlike, (12.12) is only relevant for lightlike k, in which case

$$k \in \{k\}^\perp \quad (k \text{ lightlike}) \qquad (12.13)$$

That is: $\tilde{A}(k)$ takes values in a lightlike hyperplane in V that contains k.

We already mentioned that the Lorenz gauge condition still allows for residual gauge transformations

$$A_\alpha \mapsto A'_\alpha := A_\alpha + \partial_\alpha \Lambda \qquad (12.14)$$

$$\text{with} \quad \Box \Lambda = 0 \qquad (12.15)$$

For the Fourier-transform this means

$$\tilde{A}_\alpha(k) \mapsto \tilde{A}'_\alpha(k) := \tilde{A}_\alpha + i k_\alpha \tilde{\Lambda}(k) \qquad (12.16)$$

$$\text{with} \quad \mathrm{supp}(\tilde{\Lambda}) \subseteq \text{Lightcone} \qquad (12.17)$$

where

$$\left.\begin{aligned} \tilde{\Lambda}(k) &:= \frac{1}{(2\pi)^2} \int d^4x \, e^{-ik\cdot x} \Lambda(x) \\ \Lambda(x) &= \frac{1}{(2\pi)^2} \int d^4k \, e^{ik\cdot x} \tilde{\Lambda}(k) \end{aligned}\right\} (12.18)$$

This means that the residual gauge-transformations only modify those values $\tilde{A}(k)$ where k is lightlike. But outside matter, i.e. if $J_\alpha = 0$, these are all non-zero amplitudes. So if $J_\alpha = 0$ the residual gauge transformations can be used to remove one of the 3 Amplitudes $\tilde{A}_\alpha(k)$ with $k^\mu \tilde{A}_\alpha(k) = 0$, namely that parallel to k. The remaining two-dimensional space of field-values $\tilde{A}_\alpha(k)$ per k

can then be naturally identified with the quotient space

$$Q_k := \{k\}^{\perp} / \mathrm{Span}\{k\} \qquad (12.19)$$

Alternatively, and this is what is actually done in practice, this quotient is represented by any 2-dimensional vector-space of $\{k\}^{\perp}$ transversal to $\mathrm{Span}\{k\}$. Such a transversal space may be identified as follows: Choose an arbitrary timelike vector $v \in V$ and consider the intersection of the spacelike hyperplane $\{v\}^{\perp}$ with the lightlike hyperplane $\{k\}^{\perp}$:

$$H_k := \{k\}^{\perp} \cap \{v\}^{\perp} \qquad (12.20)$$

which is a 2-dimensional spacelike hyperplane in $\{k\}^{\perp}$ so that the restriction of the natural projection

$$\left.\begin{aligned} \pi : \{k\}^{\perp} &\to Q_k \\ k &\mapsto [k] \end{aligned}\right\} \qquad (12.21)$$

to H_k:

$$\pi|_{H_k} : H_k \longrightarrow Q_k \qquad (12.22)$$

is an isomorphism of vector spaces.

Analytically this means that in addition to

$$k^\alpha \tilde{A}_\alpha(k) = 0 \qquad (12.19)$$

we may impose another condition

$$V^\alpha \tilde{A}_\alpha(k) = 0 \qquad (12.20)$$

with any fixed timelike V. The statement is that if $\tilde{A}_\alpha$ does not satisfy (12.20) then a change by a residual gauge transformation

$$\tilde{A}'_\alpha(k) = \tilde{A}_\alpha(k) + i k_\alpha \tilde{\Lambda}(k) \qquad (12.21)$$

can be found, such that $\tilde{A}'_\alpha$ will satisfy (12.20). Indeed:

$$V^\alpha \tilde{A}'_\alpha(k) = V^\alpha \tilde{A}_\alpha(k) + i(k\cdot V)\tilde{\Lambda}(k) \stackrel{!}{=} 0 \qquad (12.22)$$

$$\Leftrightarrow \qquad \tilde{\Lambda}(k) = i\,\frac{(V\cdot \tilde{A}(k))}{V\cdot k} \qquad (12.23)$$

This determines $\tilde{\Lambda}$ (and hence Λ) uniquely and hence corresponds to a <u>complete gauge</u>. We also see why we had to restrict V to be timelike, for otherwise the denominator $(V\cdot k)$ in (12.23) would be zero for some k. For k lightlike and V timelike, however, $V \cdot k \neq 0$. (if $k \neq 0$).

Hence we have the

Proposition: A complete gauge for the vacuum Maxwell field A_α is given by

$$\tilde{A}_\alpha(k)\, k^\alpha = 0 \qquad (12.24)$$

$$\tilde{A}_\alpha(k)\, V^\alpha = 0 \qquad (12.25)$$

for any fixed timelike $V \in \mathbb{V}$. Or

$$\partial^\alpha A_\alpha(x) = 0 \qquad (12.26)$$

$$V^\alpha A_\alpha(x) = 0 \qquad (12.27)$$

One says that A has two independ. states and means by that that the vector space in which $\tilde{A}(k)$ takes its values is a 2-dimensional (real) vectorspace:

$$H_k = \{k\}^\perp \cap \{V\}^\perp \qquad (12.28)$$

that is spanned by two (helicity) basis vectors.

For example, if

$$k = \frac{\omega}{c}\,(e_0 + e_3) \qquad (12.29)$$

(plane wave of frequency ω propagating in e_3-direction)

and

$$V = e_0 \qquad (12.30)$$

then $H_k = \mathrm{span}\{e_1, e_2\}$. $\qquad (12.31)$

One says that $\tilde{A}$ has only transversal amplitudes. But note that H_k is only <u>representing</u> Q_k and that this representation depends on V. Had we chosen

$$V = e_0 + a e_1 \quad , |a| < 1 \qquad (12.32)$$

then

$$\left.\begin{aligned} k\cdot\tilde{A}(k) = 0 &\Leftrightarrow \tilde{A}_0(k) - \tilde{A}_3(k) = 0 \\ V\cdot\tilde{A}(k) = 0 &\Leftrightarrow \tilde{A}_0(k) - a\tilde{A}_1(k) = 0 \end{aligned}\right\} (12.33)$$

$$\Rightarrow \tilde{A}(k) = (a\tilde{A}_1(k), \tilde{A}_1(k), \tilde{A}_2(k), a\tilde{A}_1(k)) \qquad (12.34)$$

i.e. the value-space of $\tilde{A}(k)$ is spanned by $(a, 1, 0, a)$ and $(0, 0, 1, 0)$. It makes no sense to address any of these possible representations of Q_k by a H_k as "more" "physical" than another one.

Likewise, it makes no sense to distinguish the components of $\tilde{A}(k)$ into "physical" and "unphysical" ones, except with reference to a particular gauge. The fact that $\tilde{A}_0$ can always be "gauged away" (by choosing $V = e_0$) - outside matter - does not mean that $\tilde{A}_0$ cannot acquire a good physical interpretation. For example the Coulomb field

$$\left.\begin{aligned} \vec{E}(\vec{x}) &= \frac{Q}{4\pi\varepsilon_0}\frac{\vec{x}}{r^3} \\ &= -\vec{\nabla}\varphi - \dot{\vec{A}} \end{aligned}\right\} \quad (12.35)$$

can either be represented by

$$\vec{A} = 0, \quad \varphi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} \qquad (12.36)$$

or

$$\varphi = 0, \quad \vec{A}(\vec{x}) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{x}}{r^3}\, t\,. \qquad (12.37)$$

Where the latter corresponds to the $A_0 = 0$ gauge.

Now, for linearized gravity, we also assume $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$ to admit Fourier transformations:

$$\left.\begin{aligned} \tilde{\bar{h}}_{\alpha\beta}(k) &:= \frac{1}{(2\pi)^2}\int d^4x\, e^{-ik\cdot x}\, \bar{h}_{\alpha\beta}(x) \\ \bar{h}_{\alpha\beta}(x) &:= \frac{1}{(2\pi)^2}\int d^4k\, e^{ikx}\, \tilde{\bar{h}}_{\alpha\beta}(k) \end{aligned}\right\} \quad (12.38)$$

and identically for $\tilde{h}_{\alpha\beta}(k)$ and $h_{\alpha\beta}(x)$.

The DeDonder gauge condition reads for $\tilde{\bar{h}}_{\alpha\beta}$:

$$k^{\alpha}\,\tilde{\bar{h}}_{\alpha\beta}(k) = 0. \qquad (12.39)$$

which means that the value of $\tilde{\bar{h}}$ at k is an element of $\{k\}^{\perp} \otimes \{k\}^{\perp}$:

$$\tilde{\bar{h}}(k) \in \{k\}^{\perp} \otimes \{k\}^{\perp} \qquad (12.40)$$

A residual gauge transformation

$$\tilde{\bar{h}}_{\alpha\beta}(k) \mapsto \tilde{\bar{h}}'_{\alpha\beta}(k) = \qquad (12.41)$$

$$\tilde{\bar{h}}_{\alpha\beta}(k) + i k_{\alpha}\tilde{\Lambda}_{\beta}(k) + i k_{\beta}\tilde{\Lambda}_{\alpha}(k) - i\eta_{\alpha\beta}k^{\lambda}\Lambda_{\lambda}(k)$$

with $\mathrm{supp}(\tilde{\Lambda}_{\alpha}) \subseteq$ lightcone maintains (12.39) and can be used to eliminate further components of $\tilde{\bar{h}}_{\alpha\beta}(k)$.

<u>Proposition</u>: Let $\operatorname{Supp}(\tilde{\bar{h}}_{\alpha\beta}) \subseteq$ light cone and $V \in \mathbb{V}$ timelike. Then residual gauge transformations can be used to achieve

$$\tilde{\bar{h}}(k) = \eta^{\alpha\beta}\, \tilde{\bar{h}}_{\alpha\beta}(k) = 0 , \qquad (12.42)$$

$$V^\alpha\, \tilde{\bar{h}}_{\alpha\beta}(k) = 0 , \qquad (12.43)$$

in addition to De Donder condition (12.39). These three conditions fix $\tilde{\Lambda}_\alpha(k)$ completely, i.e. they constitute a complete gauge.

<u>Proof</u>: Contracting (12.41) with $\eta^{\alpha\beta}$ we get [we drop the arguments (k) now]

$$\tilde{\bar{h}}' = \tilde{\bar{h}} + 2i(k\cdot\tilde{\Lambda}) - 4i(k\cdot\tilde{\Lambda})$$

$$= 0 \iff$$

$$(k\cdot\tilde{\Lambda}) = -\frac{i}{2}\,\tilde{\bar{h}} \qquad (12.44)$$

Contracting (12.41) with V^α we get:

$$V^\alpha\, \tilde{\bar{h}}'_{\alpha\beta} = V^\alpha\, \tilde{\bar{h}}_{\alpha\beta} + i(k\cdot V)\,\tilde{\Lambda}_\beta + i(\tilde{\Lambda}\cdot V)\, k_\beta$$

$$- i\, V_\beta\, (k\cdot\tilde{\Lambda})$$

$$= 0 \qquad (12.45)$$

Contracting (12.41) with $V^\alpha V^\beta$ we get:

$$V^\alpha V^\beta \tilde{h}'_{\alpha\beta} = V^\alpha V^\beta \tilde{h}_{\alpha\beta} + 2i(k\cdot V)(\tilde{\Lambda}\cdot V) - iV^2(k\cdot\tilde{\Lambda})$$

$$= 0 \qquad (12.46)$$

This we solve for $(\tilde{\Lambda}\cdot V)$:

$$(\tilde{\Lambda}\cdot V) = \frac{i}{2(k\cdot V)}\left[V^\alpha V^\beta \tilde{h}_{\alpha\beta} - iV^2(k\cdot\tilde{\Lambda})\right]$$

$$= \frac{i}{2(k\cdot V)}\left[V^\alpha V^\beta \tilde{h}_{\alpha\beta} - \frac{V^2}{2}\tilde{h}\right] \qquad (12.47)$$

where we used (12.44) to eliminate $(k\cdot\tilde{\Lambda})$ on the right-hand side. In this way we now have $(\tilde{\Lambda}\cdot V)$ as function of $\tilde{h}_{\alpha\beta}$ and V. The product $(k\cdot\tilde{\Lambda})$ is already given as function of $\tilde{h}_{\alpha\beta}$ by (12.44). These two expressions are now inserted into (12.45) which we can use to solve for $\tilde{\Lambda}_\beta$:

$$\tilde{\Lambda}_\beta = \frac{i}{(k\cdot V)}\left[V^\alpha \tilde{h}_{\alpha\beta} + i(\tilde{\Lambda}\cdot V)k_\beta - iV_\beta(k\cdot\tilde{\Lambda})\right] \qquad (12.48)$$

$$\tilde{\Lambda}_\beta = \frac{i}{(k\cdot V)}\Big\{ V^\alpha \tilde{\bar{h}}_{\alpha\beta}$$

$$- \frac{k_\beta}{2\,(k\cdot V)}\left(V^\alpha V^\gamma \tilde{\bar{h}}_{\alpha\gamma} - \frac{V^2}{2}\tilde{\bar{h}}\right)$$

$$- \frac{V_\beta}{2}\tilde{\bar{h}} \Big\} \qquad (12.49)$$

This is our solution for $\tilde{\Lambda}_\beta(k)$ in terms of $\tilde{\bar{h}}_{\alpha\beta}(k)$ and V^α.

It satisfies (12.45) by construction, using (12.44) and (12.47) and, conversely, implies (12.44) and (12.47), as one can check keeping in mind that $\tilde{\bar{h}}_{\alpha\beta}$ satisfies the de Donder condition (12.39) and $k^\beta k_\beta = 0$:

$$\tilde{\Lambda}_\beta \cdot k^\beta = \frac{i}{(k\cdot V)}\left(- \frac{k\cdot V}{2}\tilde{\bar{h}}\right) = -\frac{i}{2}\tilde{\bar{h}} \quad ✓$$

$$\tilde{\Lambda}_\beta \cdot V^\beta = \frac{i}{k\cdot V}\Big\{ V^\alpha V^\beta \tilde{\bar{h}}_{\alpha\beta}$$

$$- \frac{1}{2}\left(V^\alpha V^\beta \tilde{\bar{h}}_{\alpha\beta} - \frac{V^2}{2}\tilde{\bar{h}}\right)$$

$$- \frac{V^2}{2}\tilde{\bar{h}}\Big\}$$

$$= \frac{i}{2(k\cdot V)}\left\{ V^\alpha V^\beta \tilde{\bar{h}}_{\alpha\beta} - \frac{V^2}{2}\tilde{\bar{h}}\right\}. \quad ✓$$

Finally we mention that (12.49) takes a simpler form if expressed in terms of

$$\tilde{\bar{h}}_{\alpha\beta} = \tilde{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\tilde{h} \qquad (12.50)$$

i.e. the Fourier transform of the original $h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta}$:

$$\tilde{\Lambda}_\beta(k) = \frac{i}{(k\cdot v)}\Big\{ v^\alpha \tilde{\bar{h}}_{\alpha\beta}(k) - \frac{k_\beta}{2(k\cdot v)} v^\alpha v^\gamma \tilde{\bar{h}}_{\alpha\gamma}\Big\} \qquad (12.51)$$

Since $h_{\alpha\beta}(x)$ is real, the Fourier amplitudes satisfy

$$\tilde{h}_{\alpha\beta}(-k) = [h_{\alpha\beta}(k)]^*$$

But then from (12.51)

$$\tilde{\Lambda}_\beta(-k) = [\tilde{\Lambda}_\beta(k)]^* \qquad (12.52)$$

i.e $\Lambda_\beta(x)$ is also real.

Note: As we can achieve

$$\tilde{h}(k) = \eta^{\alpha\beta}\,\tilde{h}_{\alpha\beta}(k) = 0$$

we have

$$\tilde{\bar{h}}_{\alpha\beta}(k) = \tilde{h}_{\alpha\beta}(k)$$

or $$\bar{h}_{\alpha\beta}(x) = h_{\alpha\beta}(x)$$

In this gauge the difference between $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$ disappears and we have as full gauge conditions

$$k^{\alpha}\,\tilde{h}_{\alpha\beta}(k) = 0$$

$$V^{\alpha}\,\tilde{h}_{\alpha\beta}(k) = 0$$

$$\eta^{\alpha\beta}\,\tilde{h}_{\alpha\beta}(k) = 0$$

or $$\partial^{\alpha}\,h_{\alpha\beta}(x) = 0$$

$$V^{\alpha}\,h_{\alpha\beta}(x) = 0$$

$$\eta^{\alpha\beta}\,h_{\alpha\beta}(x) = 0$$

For example, if

$$\left.\begin{aligned} V &= e_0 \\ k &= \frac{\omega}{c}(e_0 + e_3) \end{aligned}\right\} \quad (12.53)$$

then

$$\tilde{h}_{\alpha\beta}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11} & h_{12} & 0 \\ 0 & h_{21} & h_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12.54)$$

(perpendicular to e_3 and e_0), where by symmetry $\tilde{h}_{\alpha\beta}(k) = \tilde{h}_{\beta\alpha}(k)$ and $\eta^{\alpha\beta}\tilde{h}_{\alpha\beta} = 0$ we have

$$\tilde{h}_{12} = h_{21}, \quad h_{22} = -h_{11} \quad (12.55)$$

Setting $h_{11} = -h_{22} = h_+$ and $h_{12} = h_{21} = h_\times$, we get

$$\tilde{h}_{\alpha\beta}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12.56)$$

The boxed submatrix can be understood as representing the components of a symmetric traceless tensor in

$$\mathbb{H}^2_k := H_k \otimes H_k$$

Where $H_k = \{k\}^\perp \cap \{V\}^\perp$

(compare (12.28)).

h_+ and $h_\times$ are the two independent amplitudes out of which any other can be superposed.