Lecture 13

Linearized Einstein equation: Action, energy-momentum, quadrupole formula

In order to calculate the power of radiating sources, we have to determine the energy-momentum distribution in gravitational waves. For that it is best to determine the Lagrangian density whose Euler-Lagrange equation give the linearized Einstein equation

We recall from Lecture 10, formula (10.6):

$$\overset{(1)}{R}_{\alpha\beta} = -\frac{1}{2}\left(\Box h_{\alpha\beta} + \partial_\alpha \partial_\beta h - \partial_\alpha \partial^\lambda h_{\lambda\beta} - \partial_\beta \partial^\lambda h_{\lambda\alpha}\right) \tag{13.1}$$

$$\overset{(1)}{R} = -\Box h + \partial^\alpha \partial^\beta h_{\alpha\beta} \tag{13.2}$$

$$\begin{aligned}\overset{(1)}{G}_{\alpha\beta} &= \overset{(1)}{R}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\overset{(1)}{R} \\ &= -\frac{1}{2}\left(\Box h_{\alpha\beta} + \partial_\alpha\partial_\beta h - \partial_\alpha \partial^\lambda h_{\lambda\beta} - \partial_\beta \partial^\lambda h_{\lambda\alpha}\right) \\ &\quad - \frac{1}{2}\eta_{\alpha\beta}\left(-\Box h + \partial^\alpha \partial^\beta h_{\alpha\beta}\right)\end{aligned} \tag{13.3}$$

We want to find a Lagrange density $\mathcal{L}_g$ whose variation – up to divergence terms – gives field equations

$$\delta \mathcal{L} \sim \left(\overset{(1)}{G}_{\alpha\beta} - \kappa T_{\alpha\beta} \right) \delta h^{\alpha\beta} \qquad (13.4)$$

Have

$$\overset{(1)}{G}_{\alpha\beta}\, \delta h^{\alpha\beta}$$

$$= \underbrace{-\frac{1}{2} \Box h_{\alpha\beta}\, \delta h^{\alpha\beta}}_{1} \underbrace{- \frac{1}{2} \partial_\alpha \partial_\beta h\, \delta h^{\alpha\beta}}_{2}$$

$$+ \underbrace{\partial_\alpha \partial^\lambda h_{\lambda\beta}\, \delta h^{\alpha\beta}}_{3}$$

$$+ \underbrace{\frac{1}{2} \Box h\, \delta h}_{4} \underbrace{- \frac{1}{2} \partial^\alpha \partial^\beta h_{\alpha\beta}\, \delta h}_{2} \qquad (13.5)$$

$$= \delta \left\{ \frac{1}{4} \partial_\lambda h_{\alpha\beta}\, \partial^\lambda h^{\alpha\beta} + \frac{1}{2} \partial_\beta h\, \partial_\alpha h^{\alpha\beta} - \frac{1}{2} \partial^\lambda h_{\lambda\beta}\, \partial_\alpha h^{\alpha\beta} - \frac{1}{4} \partial_\alpha h\, \partial^\alpha h \right\}$$

$$+ \text{Terms} \sim \partial_\lambda V^\lambda \qquad (13.6)$$

Hence the Lagrange-density can be read off, up to an overall constant

$$\mathcal{L} = C\Big\{ \frac{1}{4}\big[\partial_\lambda h_{\alpha\beta}\partial^\lambda h^{\alpha\beta} - \partial_\alpha h \partial^\alpha h + 2\partial_\beta h \partial_\alpha h^{\alpha\beta} - 2\partial^\lambda h_{\lambda\beta}\partial_\alpha h^{\alpha\beta}\big] - \kappa T^{\alpha\beta} h_{\alpha\beta}\Big\} \tag{13.7}$$

The constant is determined by

$$\int d^3x\, \mathcal{L} = E_{kin} - E_{pot.} \tag{13.8}$$

This can be read-off the interaction term $\sim T^{\alpha\beta} h_{\alpha\beta}$ in Newtonian limit

$$T^{\alpha\beta} h_{\alpha\beta} \rightarrow T^{00} h_{00} = c^2 \rho \frac{2\phi}{c^2} = 2\rho\phi \tag{13.9}$$

This is twice the Newtonian potential energy. Hence

$$C = \frac{1}{2\kappa} \tag{13.10}$$

and

$$\mathcal{L} = \frac{1}{2\kappa}\Big\{ \frac{1}{4}\big[\partial_\lambda h_{\alpha\beta}\partial^\lambda h^{\alpha\beta} - \partial_\lambda h \partial^\lambda h + 2\partial_\beta h \partial_\alpha h^{\alpha\beta} - 2\partial^\lambda h_{\lambda\beta}\partial_\alpha h^{\alpha\beta}\big] - \kappa T^{\alpha\beta} h_{\alpha\beta}\Big\} \tag{13.11}$$

(Pauli-Fierz Lagrangian, Spin 2, Mass 0)

Action

$$S[h,\Omega,T] = \frac{1}{2k}\int dt\wedge d^3x \Big\{$$

$$\frac{1}{4}\Big[\partial_\lambda h_{\alpha\beta}\partial^\lambda h^{\alpha\beta} - \partial_\lambda h \partial^\lambda h$$

$$+ 2\partial_\beta h \partial_\alpha h^{\alpha\beta} - 2\partial^\lambda h_{\lambda\beta}\partial_\alpha h^{\alpha\beta}\Big]$$

$$- k T^{\alpha\beta} h_{\alpha\beta}\Big\} \tag{13.12}$$

Note the physical dimensions:

$$\left.\begin{aligned} \left[\frac{1}{k}\right] &= N = Kg\,\frac{m}{s^2} \\ [dt\wedge d^3x] &= m^3\cdot s \\ [\partial h\,\partial h] &= m^{-2} \end{aligned}\right\} Kg\,\frac{m^2}{s} \tag{13.13}$$

We write

$$\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_{int} \tag{13.14}$$

where $\mathcal{L}_{grav}$ = part containing only h

$$\mathcal{L}_{int} = \frac{1}{2k}\left(-k T^{\alpha\beta} h_{\alpha\beta}\right)$$

$$= -\frac{1}{2} T^{\alpha\beta} h_{\alpha\beta}. \tag{13.15}$$

In the DeDonder gauge

$$\partial^{\alpha} \bar{h}_{\alpha\beta} = 0 \iff \partial^{\alpha} h_{\alpha\beta} = \frac{1}{2} \partial_{\beta} h \qquad (13.16)$$

the gravitational Lagrange-density simplifies

$$\mathcal{L}_{grav} = \frac{1}{8\kappa} \Big(\partial_{\lambda} h_{\alpha\beta} \partial^{\lambda} h^{\alpha\beta} - \partial_{\lambda} h \partial^{\lambda} h$$

$$+ 2 \partial_{\beta} h \partial_{\alpha} h^{\alpha\beta} - 2 \partial^{\lambda} h_{\lambda\beta} \partial_{\alpha} h^{\alpha\beta} \Big)$$

$$= \frac{1}{8\kappa} \Big(\partial_{\lambda} h_{\alpha\beta} \partial^{\lambda} h^{\alpha\beta} - \cancel{\partial_{\lambda} h \partial^{\lambda} h}$$

$$+ \cancel{2 \partial_{\beta} h \frac{1}{2} \partial^{\beta} h} - \frac{1}{2} \partial_{\beta} h \partial^{\beta} h \Big)$$

$$= \frac{1}{8\kappa} \Big(\partial_{\lambda} h_{\alpha\beta} \partial^{\lambda} h^{\alpha\beta} - \frac{1}{2} \partial_{\alpha} h \partial^{\alpha} h \Big) \qquad (13.17)$$

This is the Lagrangian-density of a Poincaré-invariant field theory. There is a straightforward way to calculate from it the energy-momentum tensor, which combines the four Noether currents that correspond to space-time translation symmetry

Recall from Hamiltonian mechanics the prescription that leads from a Lagrangian $L(q,\dot{q})$ to the Energy-Function:

$$E(q,\dot{q}) = \frac{\partial L}{\partial \dot{q}^a}\dot{q}^a - L(q,\dot{q}) \qquad (13.18)$$

Similarly, the Energy-Momentum-Tensor $t^\mu{}_\nu$ for a Lagrangian-Density $\mathcal{L}_{grav}(h,\partial h)$ is given by

$$t^\mu{}_\nu = \frac{\partial \mathcal{L}_{grav}}{\partial(\partial_\mu h_{\alpha\beta})}\partial_\nu h_{\alpha\beta} - \delta^\mu{}_\nu \mathcal{L}_{grav} \qquad (13.19)$$

In De Donder gauge this follows from (13.17) for the grav. field:

$$t^\mu{}_\nu = \frac{1}{4\kappa}\Big[\partial^\mu h^{\alpha\beta}\partial_\nu h^{\alpha\beta} - \frac{1}{2}\partial^\mu h\,\partial_\nu h - \frac{1}{2}\delta^\mu{}_\nu\Big(\partial_\lambda h_{\alpha\beta}\partial^\lambda h^{\alpha\beta} - \frac{1}{2}\partial_\alpha h\,\partial^\alpha h\Big)\Big] \qquad (13.20)$$

or

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\Big\{\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} - \frac{1}{2}\partial_\mu h\,\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}\Big(\partial_\alpha h_{\beta\gamma}\partial^\alpha h^{\beta\gamma} - \frac{1}{2}\partial_\alpha h\,\partial^\alpha h\Big)\Big\} \qquad (13.21)$$

For a monochromatic wave where, e.g.,

$$k^\mu = \frac{\omega}{c}(e_0 + e_3) \qquad (13.22)$$

we have

$$k = \|\vec{k}\| = \frac{\omega}{c} = \frac{2\pi}{\lambda} \qquad (13.23)$$

and

$$\partial_\mu h_{\alpha\beta} \sim k_\mu h_{\alpha\beta} \qquad (13.24)$$

Averaging for such waves the corresponding $t_{\mu\nu}$ over space-time regions Ω with linear dimensions

$$L \gg \lambda \qquad (13.25)$$

then the second term in $t_{\mu\nu}$ that is proportional to $\eta_{\mu\nu}$ becomes

$$\int_\Omega (\partial_\mu h_{\alpha\beta}\, \partial^\mu h^{\alpha\beta})\, d^4x$$

$$= \int_{\partial\Omega} n^\mu (h_{\alpha\beta}\partial_\mu h^{\alpha\beta})\, dO$$

$$- \int_\Omega h^{\alpha\beta}\, \Box h_{\alpha\beta}\, d^4x \qquad (13.26)$$

But outside the source we have

$$\Box h_{\alpha\beta} = 0 \qquad (13.27)$$

This means that the integral over the second term of $t_{\mu\nu}$ is a surface term. But this is supressed by a factor

$$\frac{1}{k \cdot L} = \frac{\lambda}{L} \qquad (13.28)$$

as compared to the integrals of the term

$$\int_\Omega (\partial h)(\partial h) \; d^4 x \qquad (13.29)$$

from the first term of $t_{\mu\nu}$. This is true because it has one space-time-integration less, hence one factor L less, and one differentiation less, hence one factor k less. Therefore

$$(\text{surface integral}) \cong \frac{1}{kL} (\text{bulk integral}) \qquad (13.30)$$

If the space-time average is denoted by $\langle \ldots \rangle$ then for $L \gg \lambda$ we have in leading order

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{32\pi G} \langle \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} - \frac{1}{2} \partial_\mu h \partial_\nu h \rangle \qquad (13.31)$$

For

$$h_{\alpha\beta} = A_{\alpha\beta} \exp(i k_\mu X^\mu) + cc$$

$$\leadsto \partial_\mu h_{\alpha\beta} = -i k_\mu A_{\alpha\beta} \exp(-i k_\mu X^\mu) + cc \qquad (13.32)$$

then averaging $\langle \cdot\cdot \rangle$ kills all terms in which the exponential function survives in the squares $(\partial h)(\partial h)$.

$$\partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta}$$

$$= \Big(-i k_\mu A_{\alpha\beta} \exp(-i k\cdot x) + i k_\mu \bar{A}_{\alpha\beta} \exp(i k\cdot x)\Big)$$

$$\Big(-i k_\nu A^{\alpha\beta} \exp(-i k\cdot x) + i k_\nu \bar{A}^{\alpha\beta} \exp(i k\cdot x)\Big)$$

$$= 2 k_\mu k_\nu A_{\alpha\beta} \bar{A}^{\alpha\beta} + \text{Terms} \sim \exp(\pm 2 i k\cdot x) \qquad (13.33)$$

Likewise

$$\langle \partial_\mu h \, \partial_\nu h \rangle = 2 k_\mu k_\nu \left| \eta^{\alpha\beta} A_{\alpha\beta} \right|^2$$

$$+ \text{Terms} \sim \exp(\pm 2 i k \cdot x) \qquad (13.34)$$

Hence

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{16\pi G} k_\mu k_\nu \left[A_{\alpha\beta} \bar{A}^{\alpha\beta} - \frac{1}{2} |A|^2 \right] \qquad (13.35)$$

where $A = \eta^{\alpha\beta} A_{\alpha\beta}$

In TT - gauge $A = 0$ and

$$\left.\begin{aligned} A_{11} &= - A_{22} =: A_+ \\ A_{12} &= A_{21} =: A_\times \end{aligned}\right\} \qquad (13.36)$$

and all other $A_{\alpha\beta}$ vanish. Then

$$A_{\alpha\beta} \bar{A}^{\alpha\beta} = 2 \left(|A_+|^2 + |A_\times|^2 \right) \qquad (13.37)$$

Hence

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{8\pi G} k_\mu k_\nu \left(|A_+|^2 + |A_\times|^2 \right)$$

$$(13.38)$$

If $A_{\alpha\beta}$ is real, so that

$$h_{\alpha\beta} = A_{\alpha\beta} \exp(-ik \cdot x) + A_{\alpha\beta} \exp(ik \cdot x)$$

$$= 2 A_{\alpha\beta} \cos(k \cdot x)$$

$$=: H_{\alpha\beta} \cos(k \cdot x) \qquad (13.39)$$

where

$$\left.\begin{aligned} 2A_+ &=: H_+ \\ 2A_\times &=: H_\times \end{aligned}\right\} \quad (13.40)$$

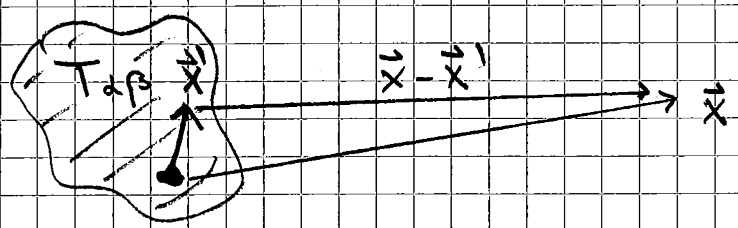
i.e. H are the real amplitudes of h, then

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{32\pi G} k_\mu k_\nu \left(H_+^2 + H_\times^2 \right) \qquad (13.41)$$

Generation of gravitational waves

We had

$$\bar{h}_{\alpha\beta}(t,\vec{x}) = -\frac{\kappa}{2\pi}\int \frac{T_{\alpha\beta}\left(t-\frac{\|\vec{x}-\vec{x}'\|}{c}, \vec{x}'\right)}{\|\vec{x}-\vec{x}'\|}\, d^3x' \tag{13.42}$$



The point $\vec{x}$ is far outside the spatial support of the source. The origin of the coordinate system is inside the support. Then

$$r = \|\vec{x}\| \gg r' = \|\vec{x}'\| \tag{13.43}$$

We have

$$\begin{aligned} \|\vec{x}-\vec{x}'\| &= \left(r^2 + r'^2 - 2\,\vec{x}\cdot\vec{x}'\right)^{1/2} \\ &= r\left(1 - 2\,\vec{n}\cdot\vec{x}'/r + O(2)\right)^{1/2} \\ &= r - \vec{n}\cdot\vec{x}' + O(1/r) \end{aligned} \tag{13.44}$$

where $\vec{n} := \frac{\vec{x}}{r}$ (13.45)

To leading order we can then replace $\|\vec{x}-\vec{x}'\|^{-1}$ in the integrand with $\frac{1}{r}$

$$\frac{1}{\|\vec{x}-\vec{x}'\|} = \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{13.46}$$

The longest-ranging part $\sim 1/r$ of the amplitude is (using $\kappa = 8\pi G/c^4$):

$$h_{\alpha\beta}(t,\vec{x}) = -\frac{4G}{c^4}\frac{1}{r}\int T_{\alpha\beta}\left(t-\frac{r}{c},\vec{x}'\right) d^3x' \tag{13.47}$$

[Note: This is just like in Electrodynamics.]

What are the integrals over the components $T_{\alpha\beta}$? Here, too, we proceed very similar to ED.

We have the integrability condition

$$\partial_\mu T^{\mu\nu} = 0 \tag{13.48}$$

$$\Rightarrow \begin{cases} \partial_0 T^{00} + \partial_k T^{k0} = 0 & (13.49) \\ \partial_0 T^{0k} + \partial_l T^{lk} = 0 & (13.50) \end{cases}$$

$$\Rightarrow \partial_0^2 T^{00} = -\partial_k \partial_0 T^{k0}$$

$$= \partial_l \partial_k T^{lk} \tag{13.51}$$

Multiplication of (13.51) with $x^a x^b$ gives

$$\partial_0^2 \left(x^a x^b T^{00}\right)$$

$$= x^a x^b \partial_\ell \partial_k T^{\ell k}$$

$$= \partial_\ell \partial_k \left(x^a x^b T^{\ell k}\right)$$

$$- \partial_\ell \left(T^{\ell a} x^b + T^{\ell b} x^a\right) - 2 T^{ab} \tag{13.52}$$

Hence

$$T^{ab} = -\frac{1}{2} \partial_0^2 \left(T_{00} x^a x^b\right)$$

$$+ \frac{1}{2} \partial_\ell \partial_k \left(T^{\ell k} x^a x^b\right)$$

$$- \frac{1}{2} \partial_\ell \left(T^{\ell a} x^b + T^{\ell b} x^a\right) \tag{13.53}$$

In the integral of T^{ab} over spatial $\Omega \subset \mathbb{R}^3$ containing the support of $T_{\alpha\beta}$ the divergence terms do not contribute due to Gauß' law. Hence, using $\partial_0 = \frac{1}{c} \partial_t$, we get

$$\int T^{ab} (\tau - \tfrac{r}{c}, \vec{x}') \, d^3 x'$$

$$= -\frac{1}{2c^2} \frac{d^2}{dt^2} \int T^{00} \left(t - \tfrac{r}{c}, \vec{x}'\right) x'^a x'^b \, d^3 x'$$

$$= -\frac{1}{2} \frac{d^2}{dt^2} \int \rho \left(t - \tfrac{r}{c}, \vec{x}'\right) x'^a x'^b \, d^3 x' \tag{13.54}$$

where

$$\rho := T_{00} / c^2 \qquad (13.55)$$

Hence (13.47) becomes

$$\bar{h}_{ab}(t, \vec{x}) = \frac{2G}{c^4} \cdot \frac{1}{r} \cdot \ddot{I}_{ab}\left(t - \frac{r}{c}\right) \qquad (13.56)$$

where

$$I_{ab}(t) := \int \rho(t, \vec{x}') \, x'^a \, x'^b \, d^3x' \qquad (13.57)$$

is the second moment of the mass-distribution.

The direction of propagation from the source point $\vec{x}'$ to the detection point $\vec{x}$ is, to leading order, given by $\vec{n} = \vec{x} / r$ (we do not take into consideration the spatial extent of the source, which we approximate by a point).

We are interested in the components of $\bar{h}_{ab}$ transversal to $\vec{n}$ (TT-gauge). These are projected out by

$$P^a{}_b = \delta^a{}_b - n^a n_b \qquad (13.58)$$

$$\bar{h}_{ab} \mapsto \bar{h}^T_{ab} = P^c{}_a P^d{}_b \bar{h}_{cd} \qquad (13.59)$$

The transverse-traceless parts are

$$\bar{h}^{TT}{}_{ab} = \bar{h}^{T}{}_{ab} - \frac{1}{2} P_{ab} \bar{h}^{c}{}_{c} \qquad (13.60)$$

Exacty that projection we apply to $\ddot{I}_{ab}$. The trace free part of I_{ab} is called

$$\tilde{Q}_{ab} := I_{ab} - \frac{1}{3} \delta_{ab} I^{c}{}_{c}$$
$$= \frac{1}{3} Q_{ab} \qquad (13.61)$$

Here Q_{ab} are the components of the quadrupole moment and $\tilde{Q}_{ab}$ that of the <u>reduced quadrupole moment</u>.

Since the TT-projection of δ_{ab} vanishes, we have $\bar{h}^{TT}{}_{ab} = h^{TT}{}_{ab}$ and

$$\boxed{h^{TT}_{ab}(t, \vec{x}) = \frac{2G}{c^4} \cdot \frac{1}{r} \cdot \ddot{\tilde{Q}}^{TT}_{ab}\left(t - \frac{r}{c}\right)} \qquad (13.62)$$

Important: The TT-projection

$$P^{a}{}_{b} = \delta^{a}{}_{b} - n^{a} n_{b} \qquad (13.63)$$

depends on $\vec{x}$ through

$$n^{a} = x^{a}/r \qquad (13.64)$$

This dependence enters $\tilde{Q}^{TT}{}_{ab}$ in (13.62) !

Now back to (13.31)

$$\langle t_{\mu\nu} \rangle = \frac{c^4}{32\pi G} \langle \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} - \frac{1}{2} \partial_\mu h \partial_\nu h \rangle$$

$$= \frac{c^4}{32\pi G} \langle \partial_\mu h^{ab} \partial_\nu h_{ab} \rangle \qquad (13.65)$$

in gauge where $h_{0\mu} = 0 = h = h^c{}_c$

The radial component of energy-current density $\vec{S}$ is

$$\vec{S} \cdot \vec{n} = -c\, t^{0k} n_k = -c\, t_{0k} n^k \qquad (13.66)$$

This we need for $r \to \infty$ in order to integrate over 2-sphere of $r \to \infty$

$$\vec{S} \cdot \vec{n} = -c\, n^k \frac{c^4}{32\pi G} \langle \partial_0 h^{TT}_{ab} \partial_k h^{TT\,ab} \rangle$$

Now, $\partial_k h^{TT\,ab}(t, \vec{x}) \sim \partial_k \left[\frac{1}{r} \left(\ddot{Q}^{ab}(t - r/c) \right) \right]$

$$= \underbrace{-\frac{n_k}{r^2} \ddot{Q}^{TT\,ab}}_{\text{from } \frac{1}{r}'} + \underbrace{\frac{1}{r} \left(-\frac{n_k}{c} \right) \dddot{Q}^{TT\,ab}}_{\text{from retardation}} \qquad (13.67)$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$\frac{1}{r^3}, \frac{1}{r^4}$ - Terms $\qquad\qquad$ $\frac{1}{r^2}$ - Terms

Hence

$$\vec{S}\cdot\vec{n} = -c\, n^k \frac{c^4}{32\pi G}$$

$$\Big\langle \frac{2G}{c^4}\frac{1}{r}\underbrace{\frac{1}{c}}_{\text{from } \partial_0 = \frac{1}{c}\partial_t} \dddot{Q}^{TT}_{ab}\cdot\frac{2G}{c^4}\frac{1}{r}\left(-\frac{n_k}{c}\right)\dddot{Q}^{TT ab}\Big\rangle$$

$$+ O(1/r^3)$$

$$\left.\begin{aligned} &= \frac{G}{8\pi c^5}\cdot\frac{1}{r^2}\cdot\Big\langle \mathrm{Trace}\left(\dddot{Q}^{TT}\right)^2\Big\rangle_{ret.} \\ &\quad + O(1/r^3)\end{aligned}\right\} \quad (13.68)$$

This expression we need to integrate over a 2-sphere "at infinity" to get the total radiated power. The angular dependence of the integrand comes from the angular dependence of n^a. We have

$$\int_{S^2_1} d\Omega\, n^a n^b = \delta^{ab}\int d\Omega\, z^2$$

$$= \delta^{ab}\, 2\pi \int_{-1}^{+1} d\sigma\,(1-\sigma^2) = \frac{4\pi}{3}\delta^{ab} \quad (13.69)$$

But we also need 4th-powers of n^a.

$$\int_{S^2_1} d\Omega \, n^a n^b n^c n^d$$

Since there are only 3 components, n^1, n^2, n^3, there are at least 2 equal. If 3 components are the same, the integral vanishes, e.g.,

$$\int n^1 (n^3)^3 \, d\Omega = \underbrace{\int_0^{2\pi} d\varphi \cos(\varphi)}_{0} \underbrace{\int_0^{\pi} d\theta \sin\theta \sin\theta \cos^3\theta}_{\int_{-1}^{+1} d\sigma \underbrace{\sigma^3 (1-\sigma^2)^{1/2}}_{\text{odd}} = 0} \quad (13.70)$$

Hence the integral has non vanishing contributions only if

1.) $a = b \neq c = d$

2.) $a = c \neq b = d$

3.) $a = d \neq b = c$

4.) $a = b = c = d$

In the first 3 cases we have ($\sigma = \cos\theta$)

$$\int d\Omega \, (n^1)^2 (n^3)^2 = \underbrace{\int_0^{2\pi} d\varphi \cos^2(\varphi)}_{\pi} \int_{-1}^{+1} d\sigma \, (1-\sigma^2)\sigma^2 \quad (13.71)$$

$$= \pi \left(\frac{2}{3} - \frac{2}{5} \right) = 4\pi/15$$

For case 4) we have

$$\int d\Omega\, (n^3)^4 = 2\pi \int_{-1}^{1} d\sigma\, \sigma^4$$

$$= \frac{4\pi}{5} = 3 \cdot \frac{4\pi}{15} \qquad (13.72)$$

Together this gives

$$\int_{S_1^2} d\Omega \; n^a n^b n^c n^d$$

$$= \frac{4\pi}{15}\left(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) \qquad (13.73)$$

We now calculate the spherical integration over $Sp(\dddot{Q}^{TT})^2$.

For any 3×3 Matrix Q we have

$$Q^{TT} = PQP - \frac{1}{2} P\, Tr(PQ)$$

$$\rightsquigarrow (Q^{TT})^2 = PQPQP - PQP\, Tr(PQ) + \frac{1}{4} P \left[Tr(PQ)\right]^2 \qquad (13.74)$$

$$Tr(Q^{TT})^2 = Tr(PQPQ) - \frac{1}{2}\left(Tr(PQ)\right)^2 \qquad (13.75)$$

since $Tr(P) = 0$.

The first term is

$$\mathrm{Tr}(PQPQ) = P_a{}^b Q_{bc} P^c{}_d Q^{da}$$

$$= Q_{ab} Q^{ab} - 2 n^a n_b Q_{ac} Q^{bc}$$

$$+ n^a n^b n^c n^d Q_{ab} Q_{cd} \qquad (13.76)$$

and the second term easily follows from

$$\mathrm{Tr}(PQ) = P^{ab} Q_{ab} = - n^a n^b Q_{ab} \qquad (13.77)$$

for traceless Q, i.e. $\delta^{ab} Q_{ab} = 0$,

hence

$$-\frac{1}{2}\left(\mathrm{Tr}(PQ)\right) = -\frac{1}{2} n^a n^b n^c n^d Q_{ab} Q_{cd} \qquad (13.78)$$

Hence (setting again $Q = \dddot{\tilde{Q}}$

$$\left.\begin{aligned} \mathrm{Tr}\left[\left(\dddot{\tilde{Q}}^{TT}\right)^2\right] &= \dddot{\tilde{Q}}^{ab} \dddot{\tilde{Q}}_{ab} \\ &- 2 n^a n^b \dddot{\tilde{Q}}_{ac} \dddot{\tilde{Q}}_b{}^c \\ &+ \frac{1}{2} n^a n^b n^c n^d \dddot{\tilde{Q}}_{ab} \dddot{\tilde{Q}}_{cd} \end{aligned}\right\} (13.79)$$

Angular integration of (13.68) over sphere of radius r then gives in $r \to \infty$ limit:

$$\int_{S^2_1} \vec{S}\cdot\vec{n}\, r^2\, d\Omega = L_{GW} = \text{total GW-luminosity}$$

$$= \frac{G}{8\pi c^5}\int_{S^2_1} d\Omega \left\langle \mathrm{Tr}\left(\dddot{Q}^{TT}\right)^2\right\rangle_{ret}$$

$$= \frac{G}{8\pi c^5}\left\langle \dddot{Q}_{ab}\,\dddot{Q}^{ab}\right\rangle_{ret}$$

$$\times\ 4\pi\ \underbrace{\left[1-\frac{2}{3}+\frac{1}{2}\cdot 2\cdot\frac{1}{15}\right]}_{\frac{15-10+1}{15}=\frac{2}{5}}$$

$$= \frac{G}{5c^5}\left\langle \dddot{Q}_{ab}\,\dddot{Q}^{ab}\right\rangle_{ret} \qquad (13.80)$$

$$= \frac{G}{5c^4}\left\langle \dddot{I}_{ab}\,\dddot{I}^{ab}-\frac{1}{3}\left(\dddot{I}^{c}{}_{c}\right)^2\right\rangle_{ret.} \qquad (13.81)$$

("Quadrupole Formula")