Lecture 14

Linearised Einstein equation: Emission of Gravitational waves by rotating rigid body

The two essential equations we are going to use in this lecture are those for the amplitude and the total power radiated by a gravitational wave (Luminosity):

$$h^{TT}_{ab}(t, \vec{x}) = \frac{2G}{c^4} \cdot \frac{1}{r} \cdot \ddot{\tilde{Q}}^{TT}_{ab}(t - r/c) \qquad (14.1)$$

$$+ \mathcal{O}(1/r^2)$$

(far-field $1/r$ - leading term)

$$L_{GW} = \frac{G}{5c^5} \langle \dddot{\tilde{Q}}_{ab} \dddot{\tilde{Q}}^{ab} \rangle_{ret} \qquad (14.2)$$

$$= \frac{G}{5c^5} \langle \dddot{I}_{ab} \dddot{I}^{ab} - \frac{1}{3} (\dddot{I}^c{}_c)^2 \rangle \qquad (14.3)$$

Where I_{ab} is the second moment and $\tilde{Q}_{ab}$ the reduced quadrupole moment of the mass-distribution:

$$I_{ab} := \int \rho(\vec{x}) \, x^a x^b \, d^3x \qquad (14.4)$$

$$\tilde{Q}_{ab} := I_{ab} - \frac{1}{3} \delta_{ab} I^c{}_c \qquad (14.5)$$

Example 1

As a first example we consider a rigid body that rotates rigidly with constant rate ω about a principal (main) axis of inertia.

Remark: We recall that the moment of inertia tensor and the 2nd moment tensor are not the same

$$I^{ab} = \int \rho(\vec{x})\, x^a x^b \, d^3x \qquad (14.6)$$

$$= \text{2nd moment - tensor}$$

$$\Theta^{ab} = \int \rho(\vec{x})\,(r^2 \delta^{ab} - x^a x^b)\, d^3x \qquad (14.7)$$

$$= \text{moment of inertia - tensor}$$

Hence

$$\Theta^{ab} = \delta^{ab}\, \mathrm{Tr}(I) - I^{ab} \qquad (14.8)$$

$$I^{ab} = \frac{1}{2}\delta^{ab}\, \mathrm{Tr}(\Theta) - \Theta^{ab} \qquad (14.9)$$

The principle axes of both are identical with eigenvalues

$$\left.\begin{aligned} \Theta_1 &= I_2 + I_3 \\ \Theta_2 &= I_1 + I_3 \\ \Theta_3 &= I_1 + I_2 \end{aligned}\right\} \quad (14.10)$$

$$\left.\begin{aligned} I_1 &= \frac{1}{2}(\Theta_2 + \Theta_3 - \Theta_1) \\ I_2 &= \frac{1}{2}(\Theta_3 + \Theta_1 - \Theta_2) \\ I_3 &= \frac{1}{2}(\Theta_1 + \Theta_2 - \Theta_3) \end{aligned}\right\} \quad (14.11)$$

$$\left.\begin{aligned} I_1 - I_2 &= -(\Theta_1 - \Theta_2) \\ I_2 - I_3 &= -(\Theta_2 - \Theta_3) \\ I_3 - I_1 &= -(\Theta_3 - \Theta_1) \end{aligned}\right\} \quad (14.12)$$

The quadrupole moment is 3 times the traceless part of I and the reduced quadrupole moment is $\frac{1}{3}$ of that, i.e. simply the traceless part of that.

Keeping all that in mind we continue to work with the second moment I^{ab}.

So let $\{e'_a : 1,2,3\}$ be a diagonalising basis of I (or θ) that is body fixed

$$\left.\begin{aligned} I'_{ab} &= I(e'_a, e'_b) \\ &= \mathrm{diag}(I'_1, I'_2, I'_3) \end{aligned}\right\} \quad (14.13)$$

We assume the body rotates about the e'_3 axis

$$\underbrace{e'_b(t)}_{\text{body fixed}} = D^a{}_b(t)\, \underbrace{e_a}_{\text{space fixed}} \qquad (14.14)$$

$$\{D^a{}_b(t)\} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (14.15)$$

$$I_{ab}(t) = (D^{-1}(t))^c{}_a\, (D^{-1}(t))^d{}_b\, I'_{cd} \qquad (14.16)$$

or dropping indices (matrix notation) and using $D^{-1} = D^T$ (orthogonal group):

$$I = D I' D^T \qquad (14.17)$$

Writing $\omega t = \varphi$ we get for (12)-sub-matrix

$$\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} I_1' & 0 \\ 0 & I_2' \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} I_1' \cos\varphi & -I_2' \sin\varphi \\ I_1' \sin\varphi & I_2' \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} I_1' \cos^2\varphi + I_2' \sin^2\varphi \,, & (I_1' - I_2') \sin\varphi\cos\varphi \\ (I_1' - I_2') \sin\varphi\cos\varphi \,, & I_1' \sin^2\varphi + I_2' \cos^2\varphi \end{pmatrix} \qquad (14.18)$$

Use

$$\left.\begin{aligned} \sin\varphi\cos\varphi &= \tfrac{1}{2}\sin(2\varphi) \\ \cos(2\varphi) &= \cos^2\varphi - \sin^2\varphi \\ &= 2\cos^2\varphi - 1 \\ &= 1 - 2\sin^2\varphi \\ \rightsquigarrow \quad \cos^2\varphi &= \tfrac{1}{2}(1 + \cos(2\varphi)) \\ \sin^2\varphi &= \tfrac{1}{2}(1 - \cos(2\varphi)) \end{aligned}\right\} \quad (14.19)$$

Then

$$I_{11}(t) = \tfrac{1}{2}(I_1' + I_2') + \tfrac{1}{2}(I_1' - I_2')\cos(2\omega t) \qquad (14.20)$$

$$I_{22}(t) = \tfrac{1}{2}(I_1' + I_2') - \tfrac{1}{2}(I_1' - I_2')\cos(2\omega t) \qquad (14.21)$$

$$I_{12}(t) = I_{21}(t) = \tfrac{1}{2}(I_1' - I_2')\sin(2\omega t) \qquad (14.22)$$

$$I_{33}(t) = I_3' = \text{const.} \qquad (14.23)$$

$$I_{13} = I_{23} = I_{31} = I_{32} = 0 \qquad (14.24)$$

We set

$$\Theta := I_1' + I_2' \tag{14.25}$$

$$\varepsilon := \frac{I_1' - I_2'}{I_1' + I_2'} \tag{14.26}$$

Note from (14.10-12) that in terms of principal moments of inertia there are

$$\Theta = \Theta_3' \tag{14.27}$$

$$\varepsilon = \frac{\Theta_2' - \Theta_1'}{\Theta_3'} \tag{14.28}$$

The time derivatives of $I_{ab}(t)$ are computed from (14.20-24):

$$\left.\begin{aligned} \ddot{I}_{11}(t) &= -2\omega^2\, \varepsilon\Theta \cos(2\omega t) \\ \ddot{I}_{22}(t) &= 2\omega^2\, \varepsilon\Theta \cos(2\omega t) \\ \ddot{I}_{12}(t) &= -2\omega^2\, \varepsilon\Theta \sin(2\omega t) \end{aligned}\right\} \tag{14.29}$$

$$\left.\begin{aligned} \dddot{I}_{11}(t) &= 4\omega^3\, \varepsilon\Theta \sin(2\omega t) \\ \dddot{I}_{22}(t) &= -4\omega^3\, \varepsilon\Theta \sin(2\omega t) \\ \dddot{I}_{12}(t) &= -4\omega^3\, \varepsilon\Theta \cos(2\omega t) \end{aligned}\right\} \tag{14.30}$$

All other time derivatives (2nd or 3rd) vanish. In particular $\dddot{I}_{33} = 0$. Hence

$$\delta^{ab} \dddot{I}_{ab}(t) = \dddot{I}_{11}(t) + \dddot{I}_{22}(t)$$

$$= 0 \qquad (14.31)$$

and same for $\delta^{ab} \ddot{I}_{ab} = 0$. Hence

$$\dddot{I}_{ab} \dddot{I}^{ab} - \frac{1}{3} \left(\dddot{I}^{c}{}_{c} \right)^2$$

$$= \dddot{I}_{11}^{2} + \dddot{I}_{22}^{2} + 2 \dddot{I}_{12}^{2}$$

$$= 2 \left(4 \omega^3 \varepsilon \theta \right)^2 \sin^2(2\omega t)$$

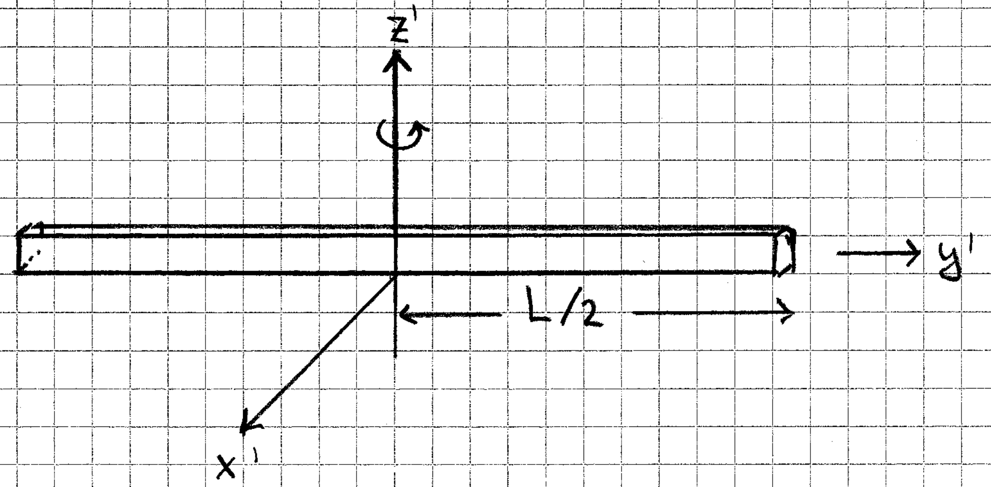
$$+ 2 \left(4 \omega^3 \varepsilon \theta \right)^2 \cos^2(2\omega t)$$

$$= 32 \omega^6 (\varepsilon \theta)^2 \qquad (14.32)$$

independent of t!

$$\Rightarrow \boxed{L_{GW} = \frac{32}{5} \frac{G}{c^5} \omega^6 \varepsilon^2 \theta^2} \qquad (14.33)$$

Take, e.g., a homogeneous rod of length L and mass M which rotates about an axis through its centre perpendicular to the length-direction



The 2nd moments of the mass distribution in the rest frame of the rod are (we idealise the rod to have vanishing extension in x' and z' direction):

$$I_1' = I_3' = 0 \qquad (14.34)$$

$$I_2' = 2 \cdot \int_0^{L/2} y'^2 \, dm(y)$$

$$= 2 \cdot \int_0^{L/2} y^2 \frac{dm}{dy} \, dy$$

but $\frac{dm}{dy} = \frac{M}{L} = \text{const}$

$$\rightarrow I_2' = 2 \frac{M}{L} \frac{1}{3} \left(\frac{L}{2}\right)^3 = \frac{1}{12} M L^2 \qquad (14.35)$$

Note that the moments of inertia are

$$\Theta_1' = \Theta_3' = I_2' = \frac{1}{12} M L^2 \qquad (14.36)$$

$$\Theta_2' = 0 \qquad (14.37)$$

Hence $$\Theta = I_2' = \frac{1}{12} M L^2 \qquad (14.38)$$

$$\varepsilon = - I_2' / I_2' = -1 \qquad (14.39)$$

$$\Rightarrow \boxed{L^{(rod)}_{GW} = \frac{2}{45} \cdot \frac{G}{c^5} \cdot \omega^6 \cdot M^2 L^4} \qquad (14.40)$$

e.g. $$\left.\begin{aligned} & L = 100\,m, \; M = 10^6\,kg \\ & \omega = 2\pi \cdot 3\,Hz \end{aligned}\right\} \qquad (14.41)$$

$$\curvearrowright L^{(rod)}_{GW} = 5{,}5 \cdot 10^{-27} \; J \cdot s^{-1} \qquad (14.42)$$

How do you know that the rod does not break (→ exercises)

We also wish to know the amplitudes and polarisations of the emitted GW. For that we use (14.1):

$$h^{TT}_{ab}(t,\vec{x}) = \frac{2G}{c^4}\frac{1}{r}\ddot{\tilde{Q}}^{TT}_{ab}\left(t-\frac{r}{c}\right) \qquad (14.43)$$

Note: The right-hand side receives a non-trivial dependency on the space direction through the TT-projection.

In our case (14.29) implies

$$\delta^{ab}\,\ddot{I}_{ab}(t) = 0 \qquad (14.44)$$

Hence

$$\{\ddot{\tilde{Q}}_{ab}(t)\} = -2\omega^2\varepsilon\Theta \begin{pmatrix} \cos(2\omega t) & \sin(\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (14.45)$$

and

$$h^{TT}_{ab}(t,\vec{x}) = \frac{-4G}{c^4}\frac{\omega^2\varepsilon\Theta}{r}\begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}^{TT} \qquad (14.46)$$

Or, in tensor notation, introducing

$$\psi_+ := \theta^1\otimes\theta^1 - \theta^2\otimes\theta^2 \qquad (14.47)$$

$$\psi_\times := \theta^1\otimes\theta^2 + \theta^2\otimes\theta^1 \qquad (14.48)$$

where $\{\theta^1,\theta^2,\theta^3\}$ is dual basis to $\{e_1,e_2,e_3\}$:

$$\boxed{h^{TT}(t,\vec{x}) = -\frac{4G}{c^4}\frac{\omega^2\varepsilon\Theta}{r}\left\{\cos(2\omega t)\,\psi_+^{TT} + \sin(2\omega t)\,\psi_\times^{TT}\right\}} \qquad (14.49)$$

Specialised to the rod, where

$$\varepsilon\Theta = -\frac{1}{12} M L^2 \qquad (14.50)$$

we get with more suggestive prefactors

$$\boxed{h^{TT}(t,\vec{x}) = \frac{2}{3}\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \times \left\{\cos(2\omega t)\,\psi_+ + \sin(2\omega t)\,\psi_\times\right\}^{TT}} \qquad (14.51)$$

Note that

$$\left.\begin{aligned} \frac{2GM}{c^2} &= \text{gravitational radius of } M \\ &\quad (= \text{Schwarzschild radius}) \\ \frac{L\omega}{2} &= v = \text{velocity of ends of rod} \end{aligned}\right\} \qquad (14.52)$$

Note also that the basis $\{\theta^a : a=1,2,3\}$ that appears in (14.47-48) is dual to $\{e_a : a=1,2,3\}$ which is orthonormal and adapted to the situation in so far, as e_3 is parallel to the axis of rotation.

The TT-projections are now easily applied by first applying to ψ_+ and $\psi_\times$ the projection

$$\psi_{+\times} \mapsto \psi_{+\times} \circ P_n^\perp \otimes P_n^\perp \tag{14.53}$$

and then taking the traceless part.
Here

$$P_n^\perp : \mathbb{R}^3 \to \{n\}^\perp \subset \mathbb{R}^3 \tag{14.54}$$

is the orthogonal projector onto the orthogonal complement of n.

Here are two particularly simple cases

<u>Case 1: $n = e_3$.</u>

That is, the "line of sight" is parallel to the axis of rotation. Then $P_n^\perp$ = projector onto 12-plane and

$$\theta^1 \circ P_n^\perp = \theta^1 \quad \text{and} \quad \theta^2 \circ P_n^\perp = \theta^2 \tag{14.55}$$

so that

$$\left.\begin{aligned} \psi_+ \circ P_n^\perp \otimes P_n^\perp &= \psi_+ \\ \psi_\times \circ P_n^\perp \otimes P_n^\perp &= \psi_\times \end{aligned}\right\} \tag{14.56}$$

In other words, the expression in $\{\cdot\cdot\}$ in (14.51) is already TT projected and we can write

$$\left.\begin{aligned} h^{TT}(t\vec{x}) &= h_+ \psi_+ + h_\times \psi_\times \\ h_+(t,r) &= \frac{2}{3}\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \cos(2\omega t) \\ h_\times(t,r) &= \frac{2}{3}\,\frac{2GM}{c^2 r}\left(\frac{L\omega}{2c}\right)^2 \sin(2\omega t) \end{aligned}\right\} \quad (14.57)$$

= linear superposition of orthogonal states with equal amplitude and $\pi/4$ phase difference; in oder words: circular polarised state

Another expression of that is

$$h_+ + i h_\times = \frac{2}{3}\,\frac{2GM}{c^2 r}\left(\frac{L\omega}{2c}\right)^2 \exp(2i\omega t) \quad (14.58)$$

Case 2 : $n = e_1$

In this case the line of sight is in the plane of rotation, i.e. perpendicular to the axis of rotation. Then $P_n^\perp$ = projector onto 23-plane and

$$\theta^1 \circ P_n^\perp = 0, \quad \theta^2 \circ P_n^\perp = \theta^2 \quad (14.59)$$

$$\left.\begin{aligned} \psi_+ \circ P_n^\perp \otimes P_n^\perp &= -\,\theta^2 \otimes \theta^2 \\ \psi_\times \circ P_n^\perp \otimes P_n^\perp &= 0 \end{aligned}\right\} \quad (14.60)$$

The traceless part of that in the (23) plane is

$$-\frac{1}{2}\left(\theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3\right) \qquad (14.61)$$

and hence

$$\left.\begin{aligned} h^{TT}(t,\vec{x}) &= h_+ \, \psi'_+ \\ h_+ &= -\frac{1}{3}\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \cos(2\omega t) \\ \psi'_+ &= \left(\theta^2 \otimes \theta^2 - \theta^3 \otimes \theta^3\right) \end{aligned}\right\} \quad (14.62)$$

This is a linear-polarised wave.

Case 3: $n = \cos(\alpha)\, e_3 + \sin(\alpha)\, e_1$

Here the line of sight has an angle of inclination α with the axis of rotation or, put differently, an angle $(\pi - \alpha)$ with the plane of rotation

We have

$$P_n^\perp = id - n \otimes n^\downarrow \tag{14.63}$$

Where $e_a^\downarrow = \theta^a$ since we are in euclidean space

$$P_n^\perp = e_1 \otimes \theta^1 + e_2 \otimes \theta^2 + e_3 \otimes \theta^3$$
$$- (\sin(\alpha)\, e_1 + \cos(\alpha)\, e_3) \otimes$$
$$(\sin(\alpha)\, \theta^1 + \cos(\alpha)\, \theta^3)$$

$$= \cos^2(\alpha)\, e_1 \otimes \theta^1 + e_2 \otimes \theta^2 + \sin^2(\alpha)\, e_3 \otimes \theta^3$$
$$- \sin(\alpha)\cos(\alpha)\, (e_1 \otimes \theta^3 + e_3 \otimes \theta^1) \tag{14.64}$$

Hence

$$\left.\begin{aligned}
P_n^\perp(e_1) &= \cos(\alpha)\,[\cos(\alpha)\, e_1 - \sin(\alpha)\, e_3] \\
&= \cos(\alpha)\, \bar{e}_1 \\
P_n^\perp(e_2) &= e_2 \\
P_n^\perp(e_3) &= \sin(\alpha)\,[\sin(\alpha)\, e_3 - \cos(\alpha)\, e_1] \\
&= -\sin(\alpha)\, \bar{e}_1
\end{aligned}\right\} \text{14.64}$$

Where for $\cos(\alpha) \neq 0$ $\bar{e}_1$ and e_2 form an orthonormal basis of $\{n\}^\perp$, the plane $\perp$ to n.

For the dual basis we have

$$P_n^{\perp}(\theta^\circ) := \theta^\circ \circ P_n^{\perp} \qquad (14.65)$$

hence

$$\left.\begin{aligned} P_n^{\perp}(\theta^1) &= \cos(\alpha)[\cos(\alpha)\,\theta^1 - \sin(\alpha)\,\theta^3] \\ &= \cos(\alpha)\,\bar{\theta}^1 \\ P_n^{\perp}(\theta^2) &= \theta^2 \\ P_n^{\perp}(\theta^3) &= \sin(\alpha)[\sin(\alpha)\,\theta^3 - \cos(\alpha)\,\theta^1] \\ &= -\sin(\alpha)\,\bar{\theta}^1 \end{aligned}\right\} \quad (14.66)$$

where for $\cos(\alpha) \neq 0$ $\bar{\Theta}_1$ and Θ_2 form an orthonormal basis of $(\{n\}^{\perp})^*$

Orthonormal basis of $\{n\}^{\perp}$

$$\left.\begin{aligned} \bar{e}_1 &= \cos(\alpha)\,e_1 - \sin(\alpha)\,e_3 \\ \bar{e}_2 &= e_2 \end{aligned}\right\} \quad (14.67)$$

with dual basis

$$\left.\begin{aligned} \bar{\theta}^1 &= \cos(\alpha)\,\theta^1 - \sin(\alpha)\,\theta^3 \\ \bar{\theta}^2 &= \theta^2 \end{aligned}\right\} \quad (14.68)$$

We can now calculate the TT-projection of ψ_+ and $\psi_\times$. For that we need to form

$$\psi_+^{TT} = (\psi_+ \circ P_n^\perp \otimes P_n^\perp) - \frac{1}{2}(P_n^\perp)^\downarrow \, \mathrm{Tr}(\psi_+ \circ P_n^\perp \otimes P_n^\perp) \tag{14.68}$$

where $(P_n^\perp)^\downarrow$ is the purely covariant version of $(P_n^\perp)$. But

$$P_n^\perp = \mathrm{id} - n \otimes n^\downarrow = \bar{e}_1 \otimes \bar{\theta}^1 + \bar{e}_2 \otimes \bar{\theta}^2 \tag{14.69}$$

hence

$$(P_n^\perp)^\downarrow = \bar{\theta}^1 \otimes \bar{\theta}^1 + \bar{\theta}^2 \otimes \theta^2 \tag{14.70}$$

Now,

$$P_n^\perp \psi_+ = P_n^\perp (\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2) = \cos^2(\alpha)\, \bar{\theta}^1 \otimes \bar{\theta}^1 - \bar{\theta}^2 \otimes \bar{\theta}^2 \tag{14.71}$$

The trace of that is $\cos^2(\alpha) - 1$. Hence

$$\begin{aligned}\psi_+^{TT} &= \cos^2(\alpha)\, \bar{\theta}^1 \otimes \bar{\theta}^1 - \bar{\theta}^2 \otimes \bar{\theta}^2 \\ &\quad - \frac{1}{2}(\bar{\theta}^1 \otimes \bar{\theta}^1 + \bar{\theta}^2 \otimes \bar{\theta}^2)(\cos^2(\alpha) - 1) \\ &= \frac{1}{2}(1 + \cos^2(\alpha)) \left[\bar{\theta}^1 \otimes \bar{\theta}^1 - \bar{\theta}^2 \otimes \bar{\theta}^2 \right] \\ &= \frac{1}{2}(1 + \cos^2(\alpha))\, \bar{\psi}_+ \end{aligned} \tag{14.72}$$

Likewise

$$P_n^{\perp}\,\psi_\times = P_n^{\perp}\left(\theta^1\otimes\theta^2 - \theta^2\otimes\theta^1\right)$$

$$= \cos(\alpha)\left(\bar\theta^1\otimes\bar\theta^2 + \bar\theta^2\otimes\bar\theta^1\right) \qquad (14.73)$$

which is already traceless and therefore already the TT-projection.

$$\psi_\times^{TT} = \cos(\alpha)\left[\bar\theta^1\otimes\bar\theta^2 + \bar\theta^2\otimes\bar\theta^1\right]$$

$$= \cos(\alpha)\,\bar\psi_\times \qquad (14.74)$$

Hence we get for $h^{TT}(t,\vec{x})$ from (14.51): using

$$f(r) := \frac{2}{3}\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \qquad (14.75)$$

$$h^{TT}(t,r) = $$

$$\left.\begin{aligned} & f(r)\,\frac{1+\cos^2(\alpha)}{2}\,\cos(2\omega t)\,\bar\psi_+ \\ & + f(r)\,\cos(\alpha)\,\sin(2\omega t)\,\bar\psi_\times \end{aligned}\right\} \quad (14.76)$$

The vectors $\bar{\varphi}_+$ and $\bar{\varphi}_\times$ in $V^* \otimes V^*$ are orthogonal and of equal norm (2). They define a basis set of linear polarisation states. (14.76) is a superposition of those with amplitude ratio

$$\frac{1+\cos^2(\alpha)}{2} \Big/ \cos(\alpha) = \frac{h_+}{h_\times} \qquad (14.77)$$

and phase difference π. This is an "elliptically polarized wave with eccentricity of ellipse by

$$\varepsilon = \left[\frac{|\bar{h}_+|^2 - |\bar{h}_\times|^2}{|h_+|^2} \right]^{1/2}$$

$$= \left[\frac{\frac{1}{4}(1+\cos^2(\alpha))^2 - \cos^2(\alpha)}{\frac{1}{4}(1+\cos^2(\alpha))^2} \right]^{1/2}$$

$$= \left[\frac{\frac{1}{4}(1-\cos^2(\alpha))^2}{\frac{1}{4}(1+\cos^2(\alpha))^2} \right]^{1/2}$$

$$= \frac{1-\cos^2(\alpha)}{1+\cos^2(\alpha)}, \qquad (14.77)$$

with circular polarisation, $\varepsilon = 0$, for $\alpha = 0$ and linear polarisation, $\varepsilon = 1$, for $\alpha = \pi/2$ as extreme cases.