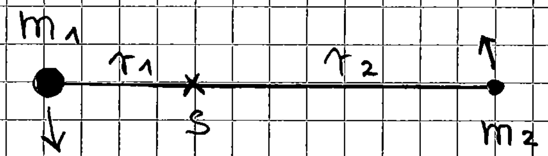
Lecture 15

Linearised Einstein equation: Emission of Gravitational waves by binary star system on circular orbits

Consider two point-like stars of masses m_1 and m_2 in a gravitationally bound state. We make the simplifying assumption that each star moves on a circular (rather than elliptic) orbit around the common centre of mass S



(15.1)

$$r = r_1 + r_2 \, , \qquad m_1 r_1 = m_2 r_2 \qquad (15.2)$$

hence

$$\left.\begin{aligned} r_1 &= r \frac{m_2}{m_1 + m_2} \\ r_2 &= r \frac{m_1}{m_1 + m_2} \end{aligned}\right\} \qquad (15.3)$$

In the co-rotating (body-fixed) frame, if the stars are along the 1-axis, the 2nd moments of the mass distribution are

$$I_1' = r_1^2 m_1 + r_2^2 m_2$$

$$= r^2 \left[\frac{m_1 m_2^2 + m_1^2 m_2}{(m_1 + m_2)^2} \right]$$

$$= r^2 \quad \frac{m_1 m_2}{m_1 + m_2} \tag{15.4}$$

$$I_2' = I_3' = 0 \tag{15.5}$$

$$\Rightarrow \quad \Theta = I_1' + I_2' = I_1' \tag{15.6}$$

$$\varepsilon = \frac{I_1' - I_2'}{I_1' + I_2'} = 1 \tag{15.7}$$

The angular frequency of the circular orbit is determined by Kepler's 3rd law, which follows here from the simple equality of centripetal = gravitational force.
The moduli of the centrifugal forces on both masses are the same. If ω is the angular frequency for each mass, we have

$$\left.\begin{aligned} F_1 &= m_1 \omega^2 r_1 = \omega^2 r \frac{m_1 m_2}{m_1 + m_2} \\ F_2 &= m_2 \omega^2 r_2 = \omega^2 r \frac{m_1 m_2}{m_1 + m_2} \end{aligned}\right\} \quad (15.8)$$

Hence equality with gravitational attraction gives

$$\omega r \frac{m_1 m_2}{m_1 + m_2} = G \frac{m_1 m_2}{r^2} \qquad (15.9)$$

$$\Rightarrow \quad \boxed{\omega^2 = G \frac{m_1 + m_2}{r^3}} \qquad (15.10)$$

Kepler's 3rd Law

If we insert this into equation (14.33) for the luminosity, which also applies here since the system rotates like a rigid body of two point masses at fixed distance r, we get

$$L_{GW} = \frac{32}{5} \cdot \frac{G}{c^5} \omega^6 (\Theta \varepsilon)^2$$

$$= \frac{32}{5} \cdot \frac{G}{c^5} \cdot \underbrace{\frac{G^3 (m_1 + m_2)^3}{r^9}}_{\omega^6} \cdot \underbrace{r^4 \left[\frac{m_1 m_2}{m_1 + m_2}\right]^2}_{(\Theta \varepsilon)^2 = (I_1^1)^2}$$

$$L_{GW} = \frac{32}{5} \cdot \frac{G^4}{c^5} \cdot \frac{m_1^2 \cdot m_2^2 \cdot (m_1 + m_2)}{r^5} \qquad (15.11)$$

$$= \frac{1}{5} \frac{c^5}{G} \underbrace{\left[\frac{2 G m_1}{c^2 r}\right]^2 \left[\frac{2 G m_2}{c^2 r}\right]^2 \left[\frac{2 G (m_1 + m_2)}{c^2 r}\right]}_{\text{each dimensionless}} \qquad (15.12)$$

$$= 7.26 \times 10^{51}\ W \left(\frac{R_g^{(1)}}{r}\right)^2 \left(\frac{R_g^{(2)}}{r}\right)^2 \left(\frac{R_g^{(12)}}{r}\right) \qquad (15.13)$$

Where

$$\left.\begin{aligned} R_g^{(1)} &:= 2 G m_1 / c^2 \\ R_g^{(2)} &:= 2 G m_2 / c^2 \\ R_g^{(12)} &:= 2 G (m_1 + m_2) / c^2 \end{aligned}\right\} \qquad (15.14)$$

are the gravitational (Schwarzschild) radii for the masses m_1, m_2 and the total mass $m_1 + m_2$, respectively.

The amplitude for, say, the circular polarized wave seen in the line of sight perpendicular to the orbital plane at a distance R follows from (14.49), (where r must be replaced by R since r is already used for the distance of the masses)

$$|h| = \frac{4G}{c^4} \cdot \frac{1}{R} \cdot \omega^2 \cdot |\varepsilon\theta|$$

$$= \frac{4G}{c^4} \cdot \frac{1}{R} \cdot G \frac{m_1 + m_2}{r^3} \cdot r^2 \frac{m_1 m_2}{m_1 + m_2}$$

$$= \frac{r}{R} \left(\frac{2 G m_1}{c^2 r} \right) \left(\frac{2 G m_2}{c^2 r} \right) \qquad (15.15)$$

$$= \frac{r}{R} \left(\frac{R_g^{(1)}}{r} \right) \left(\frac{R_g^{(2)}}{r} \right) \qquad (15.16)$$

Where r = distance of masses

R = distance of observer to system (i.e. centre of mass) assuming $R \gg r$

Equation (15.16) provides a quick and intuitive way to estimate GW amplitudes from coalescing binaries.

Due to the loss of energy into gravitational radiation the system changes its orbital parameters, here the mass seperation r and the angular frequency. These are not difficult to calculate.

We recall the Virial Theorem for potential-forces $\vec{F} = -\vec{\nabla} V$, where

$$V(r) = \alpha\, r^n \tag{15.17}$$

Then the time averages

$$\langle A \rangle := \lim_{T \to \infty} \left\{ \frac{1}{T} \int_0^T dt\, A(t) \right\} \tag{15.18}$$

for the kinetic energy T and the potential energy V obey

$$\langle T \rangle = \frac{n}{2} \langle V \rangle \tag{15.19}$$

Since $E = T + V$, have

$$\left.\begin{aligned} E &= \langle T \rangle + \langle V \rangle \\ &= \left(1 + \frac{n}{2}\right) \langle V \rangle \\ &= \left(1 + \frac{2}{n}\right) \langle T \rangle \end{aligned}\right\} \tag{15.20}$$

For the Kepler problem have $n = -1$; hence

$$\begin{aligned} E &= \frac{1}{2} \langle V \rangle \\ &= \frac{1}{2} G\, m_1 m_2 \left\langle \frac{1}{r} \right\rangle \end{aligned} \tag{15.21}$$

Since we consider circular orbits (r and $1/r$ do not depend on t and T and V are separately constant.
Then

$$E = \frac{G}{2} \frac{m_1 m_2}{r} \qquad (15.22)$$

As long as the energy-loss due to gravitational radiation is adiabatic, which means that the systems evolves in a sequence of equilibrium states, in particular states for each of which Kepler's 3rd law holds between r and ω, we have

$$-\frac{dE}{dt} = L_{GW} \iff \qquad (15.23)$$

$$-\frac{G}{2} \frac{m_1 m_2}{r^2} \dot{r} = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 (m_1 + m_2)}{r^5}$$

$$\Rightarrow r^3 \dot{r} = -\frac{64}{5} \frac{G^3}{c^5} m_1 m_2 (m_1 + m_2) \qquad (15.24)$$

and integrated between initial and final times t_i, t_f and distances r_i and r_f, we get

$$r_f^4 - r_i^4 = -\frac{256}{5} \frac{G^3}{c^5} m_1 m_2 (m_1 + m_2) [t_f - t_i]$$

(15.25)

If initially $t_i = 0$ and $r_i = r$, we get for $r_f = 0$ (collapse) the decay time t_f which we call <u>decay time</u> T_d

$$r^4 = \frac{32}{5} R_g^{(1)} R_g^{(2)} R_g^{(12)} (T_d c) \qquad (15.26)$$

or solved for the decay time

$$T_d = \frac{5}{32} \left(\frac{r}{R_g^{(1)}}\right) \left(\frac{r}{R_g^{(2)}}\right) \left(\frac{r}{R_g^{(12)}}\right) \left(\frac{r}{c}\right) \qquad (15.27)$$

In case the masses are equal, so that $R_g^{(1)} = R_g^{(2)} = \frac{1}{2} R_g^{(12)} =: R_g$

$$T_d = \frac{5}{64} \left(\frac{r}{R_g}\right)^3 \left(\frac{r}{c}\right) \qquad (15.28)$$

Examples $\quad m_1 = m_2 = M_\odot = 2 \cdot 10^{30}$ kg

$$R_g^\odot = \frac{2 G M_\odot}{c^2} = 2.97 \ 10^3 \text{ m}$$

$$\cong 3 \text{ km} \qquad (15.29)$$

1.) $r = R_\odot$ = solar radius

$$= 6.96 \ 10^5 \text{ km} \cong 7 \cdot 10^5 \text{ km}$$

$$\Rightarrow r / R_g = \frac{7}{3} \cdot 10^5$$

$$r / c = \frac{7}{3} \text{ s}$$

Hence

$$T_d = \frac{5}{64}\left(\frac{7}{3}\right)^4 10^{15}\,s$$

$$\left.\begin{aligned} &= 2.3 \times 10^{15}\,s \\ &= 7.3 \times 10^{7}\,y \end{aligned}\right\} \quad (15.30)$$

which is a long time, even though the system is already quite compact (two stars of solar masses orbiting each other at a distance corresponding to the solar radius).

2.) $r = 50\,km$

$$\rightsquigarrow \quad \frac{r}{c} = 1.67 \times 10^{-4}\,s$$

$$\frac{r}{R_g} = 16.6$$

$$\Rightarrow \quad T_d = 59\,ms \qquad (15.31)$$

In this last case, the amplitude is according to (15.16)

$$|h| = \frac{r}{R}\left(\frac{R_g}{r}\right)^2 = \frac{50\,km}{R}\,(16.6)^{-2}$$

$$= 5.9 \times 10^{-21} \,/\, R\,[Mpc] \qquad (15.32)$$

i.e. $\cong 6 \cdot 10^{-21}$ at a distance of one Mpc.

Note:

$$1\,ly = 2.9979 \cdot 10^5\ km \cdot s^{-1}$$
$$\times\ 3600 \cdot 24 \cdot 365\ s$$

$$= 9.45 \cdot 10^{12}\ km \qquad (15.33)$$

$$1\ pc = 3{,}26\ ly \qquad (15.34)$$

$$1\ Mpc = 10^6\ pc = 3{,}08 \cdot 10^{19}\ km \qquad (15.35)$$

$$\curvearrowright\ 1\ km = 3{,}24 \times 10^{-20}\ Mpc$$

$$\curvearrowright\ \frac{50\ km}{R}\ (16.6)^{-2}$$

$$= \frac{50}{(16.6)^2} \cdot 3{,}24 \times 10^{-20} \left(\frac{R}{Mpc}\right)^{-1}$$

$$= 5.89 \times 10^{-21} \left(\frac{R}{Mpc}\right)^{-1} \qquad (15.36)$$

The centre of the Virgo cluster (1300 – 2000 galaxies) has a distance of approx. 16.5 Mpc from us. So if two solar mass stars (most likely Neutron stars) emitted gravitational waves in the final stage of coalescence, we should be able to detect it in a distance of up to a few tens of Mpc. Current detector sensitivities reach up to $|h| \sim 10^{-22}$, possibly 10^{-23}.

Change of period due to emission of gravitational waves

Adiabatic hypothesis implies the validity of 3rd Kepler law throughout the emission process. Let T be the period, then

$$\omega^2 = G \frac{m_1 + m_2}{r^3} = \left(\frac{2\pi}{T}\right)^2$$

or $$r^3(t) = T^2(t)\, G \frac{m_1 + m_2}{4\pi^2} \qquad (15.37)$$

where we indicated the t-dependence of r (mutual distance) and T (period).

Differentiation of (15.37) gives

$$3r^2 \dot{r} = 2\dot{T}\, T \quad G \frac{m_1 + m_2}{4\pi^2} \qquad (15.38)$$

Division of that by (15.37)

$$3 \frac{\dot{r}}{r} = 2 \frac{\dot{T}}{T}$$

or $$\dot{T} = \frac{3}{2} T \frac{\dot{r}}{r} \qquad (15.39)$$

From (15.24) we get

$$\frac{\dot{r}}{r} = -\frac{8}{5} \left(\frac{c}{r}\right)\left(\frac{R_g^{(1)}}{r}\right)\left(\frac{R_g^{(2)}}{r}\right)\left(\frac{R_g^{(12)}}{r}\right) \qquad (15.40)$$

Comparison with (15.27) shows that the right-hand side is $-\frac{1}{4}(1/T_d)$

$$\frac{\dot{r}}{r} = -\frac{1}{4T_d} \tag{15.41}$$

Hence

$$\dot{T} = -\frac{3}{8}\frac{T}{T_d} \tag{15.42}$$

where, we repeat,

$$T_d = \frac{5}{32}\left(\frac{r}{c}\right)\left(\frac{r}{R_g^{(1)}}\right)\left(\frac{r}{R_g^{(2)}}\right)\left(\frac{r}{R_g^{(12)}}\right) \tag{15.43}$$

Application to PSR 1913+16
(aka: "Hulse-Taylor Pulsar")

PSR 1913+16 is a compact binary-star system in the constellation Aquila at

right ascension : 19 h 30 m 12.4655 s
declination : 16° 01' 08.189"
distance R : 2.1×10^4 ly = 6.4 kpc

It consists of a radio-visible pulsar (neutron star) of mass m_1 and an invisible companion (likely neutron star as well) of mass m_2

It is found that

$$\left.\begin{aligned} m_1 &= (1.4414 \pm 2\cdot 10^{-4})\, m_\odot \\ m_2 &= (1.3867 \pm 2\cdot 10^{-4})\, m_\odot \end{aligned}\right\} \quad (15.44)$$

The pulse period of the first is

$$\tau_1 = 59.029\,997\,929\,88 \text{ ms} \quad (15.45)$$

and the orbital period is

$$T = (0.322\,997\,448\,930 \pm 4\cdot 10^{-12})\, d \quad (15.46)$$

The mutual distance r that follows from Keplers 3rd law (15.37) is then, with

$$m_1 + m_2 = 5.625 \times 10^{30} \text{ kg}$$

$$T(t) = 2.79 \times 10^4 \text{ s}$$

$$\rightsquigarrow \quad \left.\begin{aligned} r(t) &= 1.95 \times 10^9 \text{ m} \\ &= 1.95 \times 10^6 \text{ km} \end{aligned}\right\} \quad (15.47)$$

Compare with solar diameter $D_\odot = 2R_\odot$

$$D_\odot = 1.39 \times 10^6 \text{ km} \quad (15.48)$$

With
$$\left.\begin{aligned} R_g^{(1)} &= \frac{2Gm_1}{c^2} = 4.28 \text{ km} \\ R_g^{(2)} &= \frac{2Gm_2}{c^2} = 4.12 \text{ km} \end{aligned}\right\} \quad (15.49)$$

$$R_g^{(12)} = 8.4 \text{ km} \tag{15.49}$$

We get, using

$$T_d = \frac{5}{32} \underbrace{\left(\frac{r}{R_g^{(1)}}\right)\left(\frac{r}{R_g^{(2)}}\right)\left(\frac{r}{R_g^{(12)}}\right)}\underbrace{\left(\frac{r}{c}\right)}$$

$$= \frac{5}{32} \times 5 \cdot 10^{16} \cdot 6.5 \text{ s}$$

$$\left.\begin{aligned} &= 5.1 \times 10^{16} \text{ s} \\ &= 1.6 \times 10^{9} \text{ y} \end{aligned}\right\} \tag{15.50}$$

The total GW-luminosity follows from (15.13):

$$L_{GW} = 7.26 \times 10^{51} \text{ W} \left(\frac{R_g^{(1)}}{r}\right)^2 \left(\frac{R_g^{(2)}}{r}\right)^2 \left(\frac{R_g^{(12)}}{r}\right)$$

$$= 6.73 \cdot 10^{23} \text{ W} \tag{15.51}$$

Compare with electromagnetic solar luminosity

$$L_{EM}^{(sun)} = 3.8 \times 10^{26} \text{ W} \tag{15.52}$$

or that of the Crab-Pulsar, which is $0.9 \times L_{EM}^{(sun)}$.

We can also calculate the amplitude, using (15.16)

$$|h| = \frac{r}{R}\left(\frac{R_g^{(1)}}{r}\right)\left(\frac{R_g^{(2)}}{r}\right) = \frac{R_g^{(1)} \cdot R_g^{(2)}}{R \cdot r}$$

and $R = 2.1 \times 10^4\ ly$

$$= 1.98 \times 10^{17}\ km \qquad (15.53)$$

$$\rightsquigarrow \quad |h| = 4.88 \times 10^{-23} \qquad (15.54)$$

Note: Our estimates for L_{GW} and $|h|$ are based on the formulae for <u>circular</u> orbits. But, in fact, the orbits of the Hulse Taylor pulsar are highly eccentric:

$$\varepsilon = 0.6171334 \qquad (15.55)$$

This means that we underestimated both quantities since eccentricity enhances the output in GW. For example, for L_{GW} we have instead of (15.11)

$$L_{GW} = \frac{32}{5}\,\frac{G^4}{c^5}\,\frac{m_1^2 \cdot m_2^2 \cdot (m_1 + m_2)}{r^5}\, f(\varepsilon)$$

where
$$f(\varepsilon) := \frac{1 + \frac{73}{24}\varepsilon^2 + \frac{37}{96}\varepsilon^4}{(1-\varepsilon^2)^{7/2}} \qquad (15.56)$$

↑ See Straumann Chap. 5.5

This is $f(\varepsilon)$ times the value calculated above. For the Hulse-Taylor value (15.56) we have

$$f(\varepsilon = 0.6171334) = 11.86 \qquad (15.57)$$

This results in

$$L_{GW}^{(HT)} = 7.98 \cdot 10^{24}\ W \qquad (15.58)$$

This also means that the amplitude (15.54) has to be corrected, though not by the same factor (more like $[f(\varepsilon)]^{1/2} \cong 3,4$).

The Hulse-Taylor pulsar was the first system on which an indirect evidence was produced for the existence of gravitational waves. The measured quantity was $\dot{T}$, i.e. the decrease in the orbital period due to energy-loss in gravitational waves. Note that with (15.42) and (15.50) our naive calculation would have given (circular orbits)

$$\dot{T} = -\frac{3}{8}\frac{T}{T_d} = -2.05 \times 10^{-13} \qquad (15.59)$$

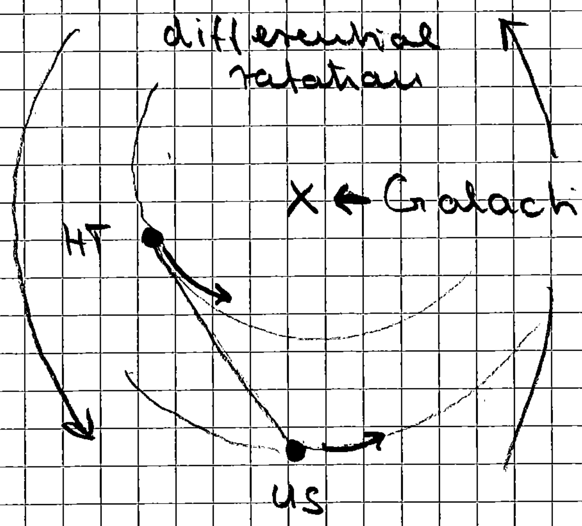
The improved calculation multiplies this again with $f(\varepsilon)$ [which is obvious, since we deduced $\dot{T}$ from the virial-theorem setting the energy loss equal to L_{GW}]. Hence the improved value is

$$\dot{T}^{(HT)} = -11.86 \cdot 2.05 \times 10^{-13}$$

$$= -2.4329 \; 10^{-12} \qquad (15.60)$$

The accurate values are

$$\left.\begin{aligned} \dot{T}_{GR} &= -(2.40242 \pm 2 \cdot 10^{-5}) \cdot 10^{-12} \\ \dot{T}_{obs} &= -(2.4184 \pm 9 \cdot 10^{-4}) \cdot 10^{-12} \\ \dot{T}_{Gal} &= -(0.0128 \pm 5 \cdot 10^{-3}) \cdot 10^{-12} \end{aligned}\right\} \quad (15.61)$$



$\dot{T}_{Gal}$ comes from an accelerated approach of PSR 1913+16 towards us (acc. Doppler Effect)

So the observation corresponds to the GR-effect plus the Doppler contribution from the <u>differential rotation</u> of the <u>galactic disk</u>.

$$\dot{T}_{GR} \cong \dot{T}_{obs} - \dot{T}_{Gal} \qquad (15.62)$$

$\downarrow$

✓ within errors !

This won Russel Hulse and Joseph Taylor the Nobel Prize for the year 1993 " For the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation".

The current accuracy is [Weisberg & Huang ApJ 2016 , arXiv: 1606.02744]

$$\frac{\dot{T}_{obs} - \dot{T}_{Gal}}{\dot{T}_{GR}} = 0.9983 \pm 0.0016 \qquad (15.63)$$

based on new (2014) galactic parameters

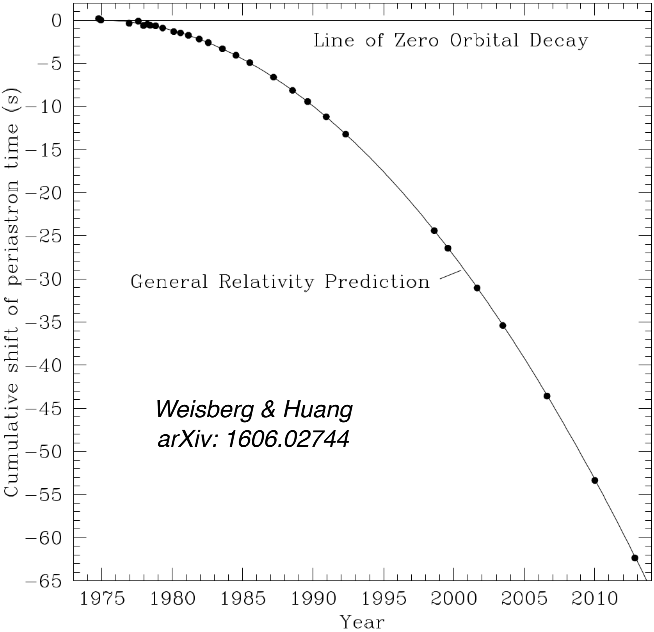


Figure 3. The orbital decay of PSR B1913+16 as a function of time. The curve represents the orbital phase shift expected from gravitational wave emission according to General Relativity. The points, with error bars too small to show, represent our measurements.