Lecture 16

Redshift, gravitational lenses
Shapiro time-delay

I.) Redshift.

General Theory: Let k be the field of wave vectors for an electromagnetic wave.

We have that k is null and geodesic

$$g(k,k) = 0 \qquad (16.1)$$

$$\nabla_k k = 0 \qquad (16.2)$$

An observer with four-velocity u,

$$g(u,u) = c^2$$

measures at $p \in M$ the frequency

$$g(u,k)|_p = \omega_p \qquad (16.3)$$

Assume spacetime to be stationary, i.e. admit a timelike Killing field K

$$L_K g = 0 \qquad (16.4)$$

Then, because $\nabla g = 0$ and $\nabla_k k = 0$

$$k\,(g(k,K)) = g(k, \nabla_k K) \qquad (16.5)$$

$$T = 0 \Rightarrow \nabla_k K = \nabla_K k + [k, K] \qquad (16.6)$$

hence

$$k\,(g(k,K)) = \underbrace{g(k, \nabla_K k)}_{K\,\frac{1}{2}(g(k,k)) = 0} - g(k, [K,k])$$

$$= - g(k, L_K k) \qquad (16.7)$$

But

$$0 = K\,(g(k,k)) = (\cancel{L_K g})^{0}(k,k) + 2 g(k, L_K k)$$

hence $\quad k\,(g(k,K)) = 0 \qquad (16.8)$

$\Rightarrow$ $g(k,K)$ is constant along the light rays $\quad (16.9)$

If observer u moves along integral lines of K, i.e.

$$u = c\,\frac{K}{\|K\|} \qquad (16.10)$$

where $\|K\| = [g(K,K)]^{1/2} \qquad (16.11)$

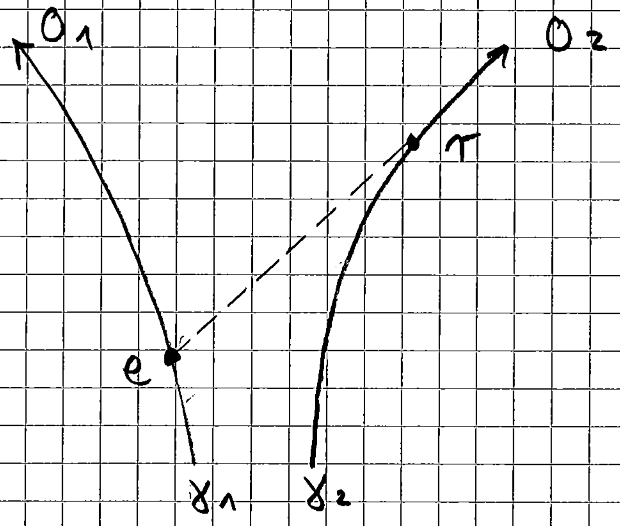
then

$$g(k,u)\,\|k\| = \text{const} \tag{16.12}$$

along light ray, or, using (16.3):

$$\omega\,\|k\| = \text{const}. \tag{16.13}$$

Now consider two observers $O_{1,2}$ moving along two different integral lines of K and a light signal sent from O_1 at $e \in \gamma_1$ to $r \in \gamma_2$



then $\omega(e)\,\|K\|_e = \omega(r)\,\|K\|_r$, or

$$\omega(r) = \omega(e)\,\frac{\|K\|_e}{\|K\|_r}$$

$$= \omega(e)\left[\frac{g_e(K,K)}{g_r(K,K)}\right]^{1/2} \tag{16.14}$$

This is exactly valid in any stationary spacetime.

Note that here the frequencies $\omega(r)$ and $\omega(e)$ refer to $g(k,u)|_r$ and $g(k,u)|_e$, i.e. are "frequencies" with respect to proper time of the stationary observers.

In adapted coordinates, where

$$K = \frac{\partial}{\partial x^0} \tag{16.15}$$

$$\begin{aligned} g = {} & g_{00}(\vec{x})\, dx^0 \otimes dx^0 \\ & + g_{0a}(\vec{x})\,(dx^0 \otimes dx^a + dx^a \otimes dx^0) \\ & + g_{ab}(\vec{x})\, dx^a \otimes dx^b \end{aligned} \tag{16.16}$$

We have

$$g(k,k)(\vec{x}) = g_{00}(\vec{x}) \tag{16.17}$$

hence

$$\omega(\vec{x}_r) = \omega(\vec{x}_e) \left[\frac{g_{00}(\vec{x}_e)}{g_{00}(\vec{x}_r)} \right]^{1/2} \tag{16.18}$$

In the linear approximation we had

$$g_{00}(\vec{x}) = 1 + \frac{2\phi(\vec{x})}{c^2} \tag{16.19}$$

Neglecting terms of order $(\phi/c^2)^2$ and higher

We get

$$\omega(\vec{x}_r) = \omega(\vec{x}_e)\left[\frac{1+\frac{2\phi(\vec{x}_e)}{c^2}}{1+\frac{2\phi(\vec{x}_r)}{c^2}}\right]^{1/2} \quad (16.20)$$

$$= \omega(\vec{x}_e)\left[1+\frac{\phi(\vec{x}_e)-\phi(\vec{x}_r)}{c^2}\right]+O(2)$$

or

$$\omega(\vec{x}_e) = \omega(\vec{x}_r)\left[1-\frac{\phi(\vec{x}_e)-\phi(\vec{x}_r)}{c^2}\right]+O(2) \quad (16.21)$$

The "redshift factor z" is usually defined by

$$z := \frac{\omega_e-\omega_r}{\omega_r} = \frac{\phi_r-\phi_e}{c^2} = \frac{\Delta\phi}{c^2} \quad (16.22)$$

Note: $z > 0$ if $\omega_r < \omega_e$, i.e. if the received frequency is smaller than the emitted, which is the case if $\phi_r > \phi_e$, i.e. the gravitational potential is higher at the point of reception. In terms of wavelength $\lambda_{e,r} \sim 1/\omega_{e,r}$ have

$$z = \frac{\lambda_r-\lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e}-1 \quad (16.23)$$

Using (16.14) the exact expression for z is

$$Z = \left[\frac{g_r(k,k)}{g_e(k,k)} \right]^{1/2} - 1 \qquad (16.24)$$

II. Defraction of light by the static gravitational field

Let g be a static metric; in adapted coordinates we have

$$g = \phi^2 c^2 dt \otimes dt - \bar{g}_{ab}\, dx^a \otimes dx^b \qquad (16.25)$$

where ϕ and $\bar{g}_{ab}$ are independent of t.

From Problem-Sheet 5 we know that the spatial projections of lightlike geodesics are geodesics in the optical metric

$$\hat{g}_{ab} = \bar{g}_{ab} / \phi^2 \qquad (16.26)$$

We assume $\bar{g}_{ab}$ – and hence $\hat{g}_{ab}$ – to be conformally flat (as is, e.g., always the case if $\bar{g}_{ab}$ and ϕ are spherically symmetric).

$$\hat{g}_{ab}(\vec{x}) = n^2(\vec{x})\, \delta_{ab} \qquad (16.27)$$

Geodesics in $\hat{g}$ obey (lightlike)

$$g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta = 0 . \qquad (16.28)$$

$\Rightarrow$

$$\dot{t} = \frac{1}{c}\left[\hat{g}_{ab}(\vec{X}(\lambda))\,\dot{X}^a(\lambda)\,\dot{X}^b(\lambda)\right]^{1/2} \tag{16.29}$$

The spatial arc length in opt. metric is

$$\left.\begin{aligned} l &= \int \left[\hat{g}_{ab}(\vec{X}(\lambda))\dot{X}^a(\lambda)\dot{X}^b(\lambda)\right]^{1/2} d\lambda \\ &= c\int dt = cT \end{aligned}\right\} (16.30)$$

$\Rightarrow$ Fermat's principle of "shortest arrival".

$$\delta \int \left[\hat{g}_{ab}(\vec{X}(\lambda))\,\dot{X}^a(\lambda)\,\dot{X}^b(\lambda)\right]^{1/2} d\lambda = 0 \tag{16.31}$$

For positive definite metrics, stationary points of the energy-functional (i.e. geodesics) are also stationary points of the length functional. The latter is reparametrisation invariant and contains all reparametrised versions of the geodesics, i.e. all autoparallels, as stationary points.

For (16.27) get

$$\delta \int n(\vec{X}(\lambda))\,\left(\dot{\vec{X}}(\lambda)\cdot\dot{\vec{X}}(\lambda)\right)^{1/2} d\lambda = 0 \tag{16.32}$$

Looks like in ordinary optics for light rays in diffractive medium with index of diffraction $n(\vec{x})$.

In the linearised theory have for static cases

$$
\left.\begin{aligned}
g = &\left(1 + \frac{2\phi(\vec{x})}{c^2}\right) c^2\, dt \otimes dt \\
&- \left(1 - \frac{2\phi(\vec{x})}{c^2}\right) d\vec{x} \otimes d\vec{x}
\end{aligned}\right\} \quad (16.33)
$$

$$\Rightarrow \quad \hat{g}_{ab}(\vec{x}) = \left(1 - \frac{4\phi(\vec{x})}{c^2}\right) \delta_{ab}$$

$$= n^2\, \delta_{ab}$$

$$\Rightarrow \quad n(\vec{x}) = \left(1 - \frac{2\phi(\vec{x})}{c^2}\right) \qquad (16.34)$$

As regards spatial light rays, a Newtonian gravitational potential acts like an optically active medium with index of refraction given by (16.34).
As if the "velocity of light" were space-point dependent

$$c(\vec{x}) = \frac{c_0}{n(\vec{x})} = c_0 \left(1 + \frac{2\phi(\vec{x})}{c^2}\right) \qquad (16.35)$$

But: This is a coordinate dependent statement !

Back to (16.32): The variation of the "optical path length" is

$$\delta \int n(\vec{x}) \left(\dot{\vec{x}} \cdot \dot{\vec{x}}\right)^{1/2} d\lambda$$

$$= \int \left\{ \delta \vec{x} \cdot \vec{\nabla} n \, (\cdot\cdot)^{1/2} + n \, (\cdot\cdot)^{-1/2} \dot{\vec{x}} \cdot \delta \dot{\vec{x}} \right\} d\lambda$$

$$= \int (\cdot\cdot)^{1/2} \delta \vec{x} \cdot \left\{ \vec{\nabla} n - (\cdot\cdot)^{-1/2} \frac{d}{d\lambda} \left[(\cdot\cdot)^{-1/2} n \dot{\vec{x}} \right] \right\} d\lambda$$

$$= \int (\cdot\cdot)^{-1/2} \delta \vec{x} \cdot \left\{ \vec{\nabla} n - \frac{d}{ds} \left(n \frac{d\vec{x}}{ds} \right) \right\} d\lambda \qquad (16.36)$$

where $\quad ds = \left(\dot{\vec{x}}(\lambda) \cdot \dot{\vec{x}}(\lambda) \right)^{1/2} d\lambda \qquad (16.37)$

is the differential in the arc length with respect to euclidean length (defined by coordinates $\vec{x}$ wrt which metric takes the manifestly conformally flat form (16.27)

Hence the variation vanishes if and only if

$$\frac{d}{ds} \left(n(\vec{x}(s)) \frac{d\vec{x}}{ds} \right) = \vec{\nabla} n(\vec{x}(s)) \qquad (16.38)$$

(Eikonal - Equation)

(16.38) gives

$$n \frac{d^2\vec{x}}{ds^2} + \left(\frac{d\vec{x}}{ds}\cdot\vec{\nabla} n\right)\frac{d\vec{x}}{ds} = \vec{\nabla} n$$

or

$$\left.\begin{aligned} n \frac{d^2\vec{x}}{ds^2} &= \vec{\nabla}_\perp n \\ \vec{\nabla}_\perp &= \vec{\nabla} - \frac{d\vec{x}}{ds}\left(\frac{d\vec{x}}{ds}\cdot\vec{\nabla}\right) \end{aligned}\right\} \quad (16.39)$$

= projection of $\vec{\nabla}$ perpendicular to direction of ray.

In the Newtonian case we get for

$$n = 1 - 2\phi/c^2 \qquad (16.40)$$

using $\vec{e} := d\vec{x}/ds$ (16.41)

= spatial tangent vector

$$\frac{d\vec{e}}{ds} = \frac{1}{n}\vec{\nabla}_\perp n \qquad (16.42)$$

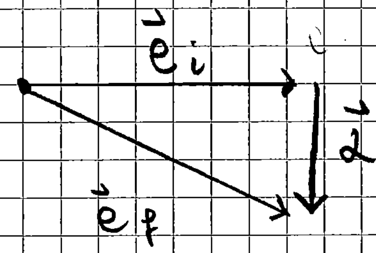
$$\overset{(1)}{=} -\frac{2}{c^2}\vec{\nabla}_\perp \phi \qquad (16.43)$$

here we neglected terms $(\phi/c^2)^2$ and higher order. Note: (16.42) is still exact.

Integration of (16.43) along the ray

$$\int_{s_i}^{s_f} ds \, \frac{d\vec{e}}{ds} = \vec{e}_f - \vec{e}_i =: \vec{\alpha}$$

$$= -\frac{2}{c^2} \int \vec{\nabla}_\perp \, \phi(\vec{x}(s)) \, ds \qquad (16.44)$$



$\vec{\alpha}$ describes the deviation of the direction of the light ray in "euclidean space".

Since ϕ obeys

$$\Delta \phi = 4\pi G \rho$$

$$\rightsquigarrow \quad \phi(\vec{x}) = -G \int \frac{\rho(\vec{x}') \, d^3x'}{\|\vec{x} - \vec{x}'\|} \qquad (16.45)$$

using

$$\vec{\nabla}_\perp \frac{1}{\|\vec{x} - \vec{x}'\|} = -\frac{(\vec{x} - \vec{x}')_\perp}{\|\vec{x} - \vec{x}'\|^3} \qquad (16.46)$$

$$\Rightarrow \quad \boxed{\vec{\alpha} = +\frac{2G}{c^2} \int d^3x' \, \rho(\vec{x}') \int ds \, \frac{(\vec{x}(s) - \vec{x}')_\perp}{\|\vec{x}(s) - \vec{x}'\|^3}} \qquad (16.47)$$

For small deviations the integral may be evaluated along undisturbed (i.e. straight) path to get leading order contribution for deviation vector $\vec{\alpha}$

Let the undisturbed path be along the z-axis

$$\left.\begin{aligned} \vec{X}(s) &= s\,\vec{e}_z + \vec{\xi} \\ \vec{\xi} &= x\,\vec{e}_x + y\,\vec{e}_y \end{aligned}\right\} \quad 16.48$$

Setting
$$\left.\begin{aligned} \vec{X}' &= z'\,\vec{e}_z + \vec{\xi}' \\ \vec{\xi}' &= x'\,\vec{e}_x + y'\,\vec{e}_y \end{aligned}\right\} \quad (16.49)$$

have
$$\int_{-\infty}^{\infty} ds \, \frac{(\vec{X}(s) - \vec{X}')_\perp}{\|\vec{X}(s) - \vec{X}'\|^3}$$

$$= \int_{-\infty}^{\infty} ds \, \frac{\vec{\xi} - \vec{\xi}'}{\left[(s - z')^2 + (\vec{\xi} - \vec{\xi}')^2\right]^{3/2}}$$

$$= \frac{\vec{\xi} - \vec{\xi}'}{\|\vec{\xi} - \vec{\xi}'\|^3} \int_{-\infty}^{\infty} ds \left[1 + \frac{(s - z')^2}{(\vec{\xi} - \vec{\xi}')^2}\right]^{-3/2}$$

$$= \frac{\vec{\xi} - \vec{\xi}'}{\|\vec{\xi} - \vec{\xi}'\|^2} \int_{-\infty}^{\infty} \frac{dk}{(1 + k^2)^{3/2}} \qquad (16.50)$$

Where $k := (s - z') / \|\vec{\xi} - \vec{\xi}'\|$

Using

$$\int_{-\infty}^{+\infty} \frac{dk}{(1+k^2)^{3/2}} = \frac{k}{(1+k^2)^{1/2}} \Bigg|_{-\infty}^{\infty} = 2 \qquad (16.51)$$

get

$$\int_{-\infty}^{\infty} ds \, \frac{(\vec{x}(s) - \vec{x}')_\perp}{\| \vec{x} - \vec{x}' \|^3} = 2 \, \frac{\vec{\xi} - \vec{\xi}'}{\| \vec{\xi} - \vec{\xi}' \|^2} \qquad (16.52)$$

And further, using

$$\left. \begin{aligned} & d^3 x' = d^2 \xi' \, dz' \\ \text{and} \quad & \int_{\mathbb{R}^3} d^3 x' = \int_{\mathbb{R}^2} d^2 \xi' \int_{-\infty}^{\infty} dz' \end{aligned} \right\} (16.53)$$

we get for $\vec{\alpha}$

$$\begin{aligned} \vec{\alpha}(\vec{\xi}) &= -\frac{4G}{c^2} \int d^2 \xi' \, \frac{\vec{\xi} - \vec{\xi}'}{\| \xi - \vec{\xi}' \|^2} \int dz' \, \rho(\vec{\xi}', z') \\ &= -\frac{4G}{c^2} \int_{\mathbb{R}^2} d^2 \xi' \, \Sigma(\xi') \, \frac{\vec{\xi} - \vec{\xi}'}{\| \vec{\xi} - \vec{\xi}' \|^2} \end{aligned} \qquad (16.54)$$

where

$$\Sigma(\vec{\xi}') := \int_{-\infty}^{\infty} dz' \, \rho(\vec{\xi}', z') \qquad (16.55)$$

is effective 2-d mass density $\perp$ to ray direction.

We will see in Problem 4 of Sheet 9 how (16.54-55) can be used to derive a "lens map", that is, a map from the lens plane to the "object-" or "source plane". This has important applications in cosmology.

As a simple application of (16.54-55) we take ρ to represent a point source:

$$\rho(\vec{x}) = M\,\delta^{(3)}(\vec{x}). \qquad (16.56)$$

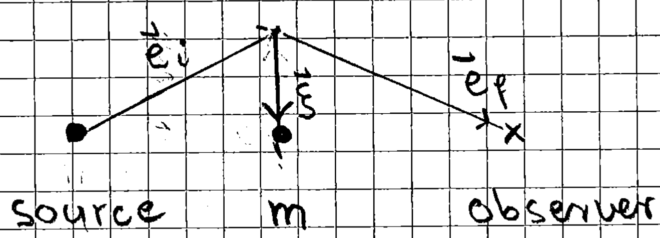
Then

$$\Sigma(\vec{\xi}) = M\,\delta^{(2)}(\vec{\xi}) \qquad (16.57)$$

and

$$\vec{\alpha}(\vec{\xi}) = -\frac{4G}{c^2} M \frac{\vec{\xi}}{\|\vec{\xi}\|^2} \qquad (16.58)$$

The negative sign shows that $\vec{e}_f - \vec{e}_i$ is directed oppositely to $\vec{\xi}$, which means



that the light is deflected inwards, i.e. oppositely to $\vec{\xi}$.

Taking the norm of (16.58) we get

$$\|\vec{\alpha}(\vec{\xi})\| = \frac{4GM}{c^2} \cdot \frac{1}{d} = \frac{2R_g}{R} \qquad (16.59)$$

where $\quad R := \|\vec{\xi}\| \qquad (16.60)$

and $\quad R_g := 2GM/c^2 \qquad (16.61)$

Now

$$\|\vec{\alpha}\| = \left[(\vec{e}_f - \vec{e}_i)^2\right]^{1/2}$$
$$= \left[2(1-\cos(\varphi))\right]^{1/2} \qquad (16.62)$$

where $\quad \cos(\varphi) = \vec{e}_f \cdot \vec{e}_i$

and $\quad \varphi = \measuredangle(\vec{e}_f, \vec{e}_i) \qquad \Big\} (16.63)$

is the deflection angle. We assumed φ to be small. Then

$$\cos(\varphi) = 1 - \frac{\varphi^2}{2} + \cdots \qquad (16.64)$$

$$\rightsquigarrow \left[2(1-\cos(\varphi))\right]^{1/2} \cong \varphi + O(\varphi^3)$$

Therefore we have the light-deflection formula:

$$\boxed{\varphi = \|\vec{\alpha}\| = \frac{4GM}{c^2 R} = \frac{2R_g}{R}} \qquad (16.65)$$

which is twice the value (2.18).

If we take

$$M = M_\odot = 1.988 \times 10^{30}\ \text{kg}$$

$$R = R_\odot = 6.957 \times 10^{5}\ \text{km}$$

$$\leadsto \quad R_g = \frac{2GM}{c^2} = 2.95\ \text{km} \qquad (16.66)$$

$$\Rightarrow \quad \varphi = \|\vec{\alpha}\| = 8.48 \times 10^{-6}$$

in radians (Bogenmaß). In arc seconds this is

$$\left((8.48 \times 10^{-6}) \cdot \frac{360}{2\pi} \cdot 3600\right)''$$

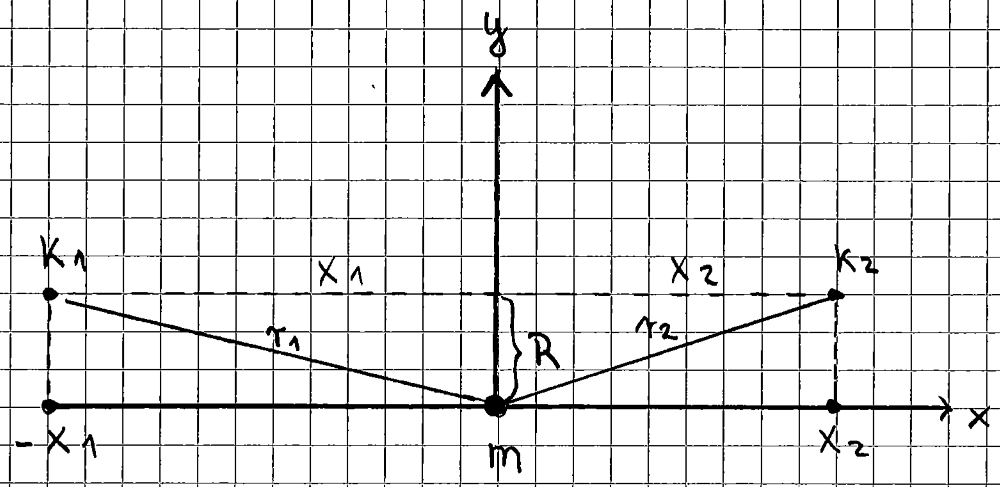
$$= 1,75'' \qquad (16.67)$$

Which is what is observed for light just grazing the suns disc.

The famous expeditions to Principe (West African Island) and Sobral (Brasil) in 1919 (so called "Edington experiment") led to mean values of $(1.61)''$ and $(1.98)''$ respectively (see original paper on homepage). There were many error-sources in those observations. Today's accuracy is 10^{-5} !

III.

Shapiro time delay



(16.68)

A light signal is exchanged between to bodies K_1 and K_2, e.g. the Earth and the Venus. A third body with mass m produces a gravitational field. The masses of K_1 and K_2 and their associated gravitational field is not considered.

The mass m produces an effective index of refraction

$$n(\vec{x}) = 1 - \frac{2\phi(\vec{x})}{c^2}$$

$$= 1 + \frac{2Gm}{c^2 r} \qquad (16.69)$$

where $r = \|\vec{x}\|$.

The coordinate timespan T for the 2-way (fourth and back) light travel from K_1 to K_2 and back is to leading order (i.e. along the unperturbed straight path)

$$T = 2\int_{-x_1}^{x_2} \frac{dx}{c(\vec{x})} = \frac{2}{c_0}\int_{-x_1}^{x_2} n(x\vec{e}_x + R\vec{e}_y)\,dx$$

$$= \frac{2}{c_0}\int_{-x_1}^{x_2}\left\{1 + \frac{2Gm}{c_0^2\sqrt{x^2+R^2}}\right\} dx$$

$$\left.\begin{aligned} &= \frac{2}{c_0}\left\{ (x_1 + x_2) \right. \\ &\quad \left. + \frac{2Gm}{c_0^2}\left[\sinh^{-1}\left(\frac{x_1}{R}\right) + \sinh^{-1}\left(\frac{x_2}{R}\right)\right]\right\} \end{aligned}\right\} \quad (16.70)$$

The first term corresponds to the light's travel time without presence of the mass m. The time-delay (in coordinate time) due to the mass is therefore

$$\Delta T = \frac{4Gm}{c_0^3}\left[\sinh^{-1}\left(\frac{x_1}{R}\right) + \sinh^{-1}\left(\frac{x_2}{R}\right)\right] \quad (16.71)$$

We have (see figure)

$$\frac{x_{1,2}}{R} = \left(\frac{r_{1,2}^2}{R^2} - 1\right)^{1/2} \qquad (16.72)$$

By $\sinh^{-1}$ we mean the inverse function for the positive branch of the sinh-function:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = y$$

$$e^x - 2y - e^{-x} = 0$$

$$e^{2x} - 2e^x y - 1 = 0$$

$$e^x = y + (y^2+1)^{1/2}$$

↑ no negative sign here

$$\left.\begin{aligned} x &= \ln\left\{y + (y^2+1)^{1/2}\right\} \\ &= \sinh^{-1}(y) \end{aligned}\right\} \qquad (16.73)$$

Hence, using (16.72)

$$\sinh^{-1}\left(\frac{x_{1,2}}{R}\right) = \ln\left\{\frac{r_{1,2}}{R} + \left(\frac{r_{1,2}^2}{R^2} - 1\right)^{1/2}\right\}$$

$$= \ln\left\{\frac{r_{1,2}}{R}\left(1 + \left[1 - \left(\frac{R}{r_{1,2}}\right)^2\right]^{1/2}\right)\right\} \qquad (16.74)$$

We assume

$$R \ll r_{1,2} \tag{16.75}$$

and expand in powers of $R/r_{1,2}$

$$\sinh\left(\frac{x_{1,2}}{R}\right) =$$

$$\ln\left\{\frac{r_{1,2}}{R}\left(1+1-\frac{1}{2}\left(\frac{R}{r_{1,2}}\right)^2+\left(\frac{R}{r_{1,2}}\right)^{n\geq 3}\right)\right\}$$

$$= \ln\left\{\frac{2r_{1,2}}{R}\left(1-\frac{1}{4}\left(\frac{R}{r_{1,2}}\right)^2+\left(\frac{R}{r_{1,2}}\right)^{n\geq 3}\right)\right\}$$

$$= \ln\left(\frac{2r_{1,2}}{R}\right) + \ln\left(1 + \text{Terms}\left(\frac{R}{r}\right)^{n\geq 2}\right)$$

$$= \ln\left(\frac{2r_{1,2}}{R}\right) + \text{Terms}\left(\frac{R}{r}\right)^{n\geq 2} \tag{16.76}$$

Therefore

$$\sinh^{-1}\left(\frac{x_1}{R}\right) + \sinh^{-1}\left(\frac{x_2}{R}\right)$$

$$= \ln\left(\frac{4r_1 \cdot r_2}{R^2}\right) + \text{Terms}\left(\frac{R}{r}\right)^{n\geq 2} \tag{16.77}$$

Hence, to leading order, we get

$$\boxed{\begin{aligned}\Delta T &= 2 \cdot \frac{1}{c} \cdot \frac{2Gm}{c^2} \cdot \ln\left(\frac{4 r_1 r_2}{R^2}\right) \\ &= 2 \cdot \frac{R_g}{c} \cdot \ln\left(\frac{4 r_1 r_2}{R^2}\right)\end{aligned}} \qquad (16.78)$$

where, as usual,

$$R_g = 2Gm/c^2 \qquad (16.79)$$

and where we replaced c_0 with c = vacuum velocity of light again (there is now no danger of confusion with $c(\vec{x})$ which will not occur anymore).
Note that R_g/c is just the time light takes to travel accross the gravitational radius R_g.

Since this time-delay is measured with one and the same clock, say that moving with K_1, we just need to multiply (16.78) with $[g_{00}(\vec{x}_1)]^{1/2}$ in order to get the <u>proper-time delay</u>

$$\Delta\tau(\vec{x}_1) = \Delta T \left(g_{00}(\vec{x}_1)\right)^{1/2} \qquad (16.80)$$

This just corresponds to a constant factor (constant because we always measure at $\vec{x}_1$). In any case, this factor goo will cancel in the actual observable we consider below.

Let us calculate the number of a relevant example

r_1 = Orbital mean radius of Earth

$\cong 1.5 \times 10^{11}$ m

r_2 = Orbital mean radius of Venus

$\cong 1.08 \times 10^{11}$ m

R = solar radius

$\cong 7 \times 10^{8}$ m

m = solar mass

$\cong 2 \cdot 10^{30}$ kg

$$\Rightarrow \Delta T(\text{Earth-Venus}) = 2.3 \times 10^{-4} \, s \qquad (16.81)$$

This corresponds to an apparent increase in the distance Earth-Venus of

$$\Delta L = \frac{1}{2} c \, \Delta T = 35 \text{ km} \qquad (16.82)$$

If instead of Venus we had taken Mercury, so that

$$r_2' = \text{radius Mercury orbit}$$

$$\simeq 5.8 \times 10^{10}\ \text{m} \qquad (16.83)$$

(roughly half the value of r_2 for Venus) we would have obtained

$$\Delta T(\text{Earth-Mercury}) = 2.2 \times 10^{-4}\ \text{s}$$

which is very close to the value of Venus, showing the mere logarithmic dependence of ΔT on r_2.

The actual measured quantity is not ΔT; for that to make sense we would have to know the location of Venus (or whatever the reflecting body is) much more accurate than 35 km. That is not the case, leaving alone the uncertainty of where exactly the signal is reflected (mean penetration depth of electromagnetic signal in Venus' atmosphere).

Instead one measures the time derivative of ΔT as the sun approaches the line of sight between Earth and Venus. This is a dimensionless quantity and is also independent as to whether one calculates the coordinate-time derivative of the coordinate-time delay, $d\Delta T/dt$, or the proper-time derivative of the proper-time delay, $d\Delta\tau/d\tau$. Since

$$\left.\begin{aligned} \Delta\tau &= (g_{00}(\vec{x}_1))^{1/2}\,\Delta T \\ d\tau &= (g_{00}(\vec{x}_1))^{1/2}\,dt \end{aligned}\right\} \quad (16.84)$$

we have

$$\frac{d\Delta T}{dt} = \frac{d\Delta\tau}{d\tau} \qquad (16.85)$$

It is the quantity on the left that we are calculating and the quantity on the right that we are interested in (since it is $d\Delta\tau/d\tau$ that is measured by a clock at $\vec{x}_1$).

We have

$$\frac{d\,\Delta T}{dt} = \frac{4Gm}{c^3}\,\frac{d}{dt}\ln\left[\frac{4\,r_1(t)\,r_2(t)}{R^2(t)}\right] \qquad (16.86)$$

Where
$$r_{1,2}^2(t) = R^2(t) + \underbrace{x_{1,2}^2}_{\text{indep. of } t} \qquad (16.87)$$

$$\rightsquigarrow\quad r_{1,2}\,\dot r_{1,2} = R\,\dot R$$

$$\rightsquigarrow\quad \dot r_{1,2} = \frac{R}{r_{1,2}}\,\dot R \ll \dot R \qquad (16.88)$$

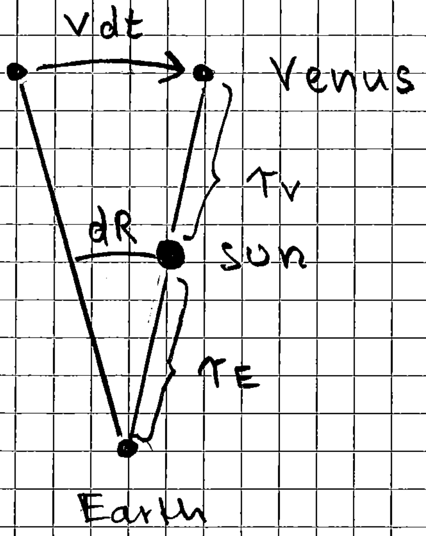
Hence, to leading order, we only need to consider the time dependence of R:

$$\frac{d\,\Delta T}{dt} = -\frac{8Gm}{c^3}\,\frac{\dot R}{R}\left(1 + \text{Terms}\left(\frac{R}{r_{1,2}}\right)\right)$$

$$\overset{(1)}{=} -\frac{8Gm}{c^3}\,\frac{\dot R}{R} \qquad (16.89)$$

Which is independent of $r_{1,2}$!

But what is $\dot R / R$?



Venus is close to "superior conjunction" and just about to disapper behind the sun
v is Venus' orbital velocity

Have

$$\frac{v\,dt}{dR} = \frac{r_V + r_E}{r_E} \qquad (16.90)$$

$$\Rightarrow \quad \dot{R} = v\,\frac{r_E}{r_E + r_V} \qquad (16.91)$$

v = orbital velocity of Venus

$$= \frac{2\pi r_E}{T_{Venus}} = \frac{2\pi \cdot 1.08 \times 10^{11}\,\mathrm{m}}{224.7\ \mathrm{days}}$$

$$= 3.5 \times 10^{4}\ \mathrm{m \cdot s^{-1}}$$

$$\Rightarrow \quad \dot{R} = 2 \times 10^{4}\ \mathrm{m \cdot s^{-1}}$$

$$\dot{R}/c = 6.7 \times 10^{-5} \qquad (16.92)$$

The modulus of the time rate of change of the delay ΔT is then

$$\left|\frac{d\Delta T}{dt}\right| = 4\left(\frac{2Gm}{c^2 R}\right)\cdot\left(\frac{\dot{R}}{c}\right)$$

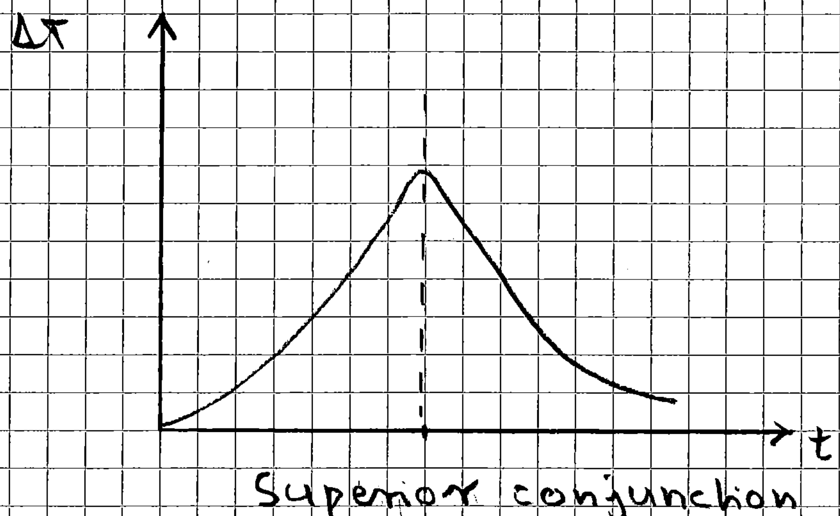
$$= 4\cdot\left(\frac{R_g^{(sun)}}{R_\odot}\right)\left(\frac{\dot{R}}{c}\right) \qquad (16.93)$$

Valid at the moment where $R = R_\odot$

$$= 4\cdot 4.2\times 10^{-6}\cdot 6.7\times 10^{-5}$$

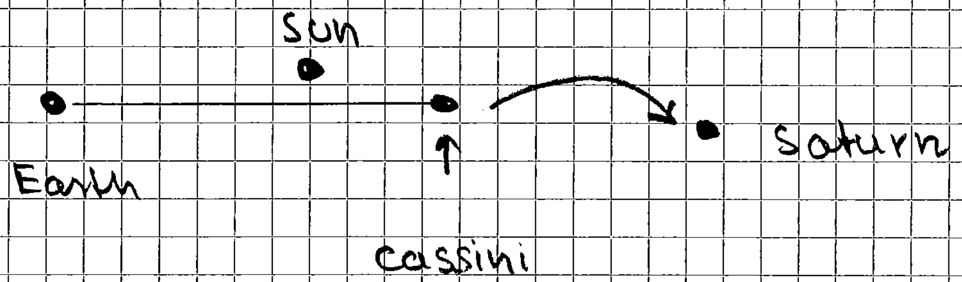
$$= 1.13\times 10^{-9}$$

$$\cong 97\ \mu s/\text{day} \qquad (16.94)$$



(16.95)

Time delay measurements are among the most accurate to test GR in the solar system. The Cassini misson used a satellite "Cassini" as the reflecting object K_2, which has the advantage to have a well defined reflection surface.



The Cassini Mission confirmed GR-predictions at

$$10^{-5} = 10^{-3}\,\% \quad \text{(Cassini)} \qquad (16.96)$$

level.

In October 2010 time delay measurements were performed on the binary system

$$\text{PSR J } 1614-2230 \qquad (16.97)$$

consisting of a neutron star pulsar ($\nu = 317\ s^{-1}$) of mass $1.97\ M_\odot$ (heavy!) and a much lighter white dwarf of $0.5\ M_\odot$. The system is at a distance of 1200 pc.

Time-delay measurements of the pulses - delayed by the gravitational field of the white dwarf.

Measurements of the Keplerian parameters yield the "mass function"

$$f(m_1, m_2) = \frac{[m_W \sin(i)]^3}{(m_{NS} + m_W)^2} \qquad (16.98)$$

where i = inclination angle (= ∡ between line of sight and normal to orbital plane). Time delay measurement yield m_W, so that m_{NS} can be deduced. In this case one obtained

$$m_{NS} = (1.97 \pm 0.04)\, m_\odot \qquad (16.99)$$

(Demorest et. al. Oct 29 2010, arXiv: 1010.5788)

Which is one of the heaviest neutron stars known. Currently the heaviest NS known is PSR J0740+6620 discovered in 2019 with

$$\boxed{m_{NS} = 2.14^{+0.10}_{-0.09}\, m_\odot} \qquad (16.100)$$

also calculated by Shapiro time delay on a compagnion white dwarf.