Lecture 17

Spherically symmetric metrics. Calculation of Riemann-, Ricci-, and Einstein-Tensors using the Cartan structure equations.

Eventually we seek solutions to Einstein's equations for the idealised case of a spherically symmetric star; inside ($T_{\mu\nu} \neq 0$) and outside ($T_{\mu\nu} = 0$) the star. But first we need to define "spherical symmetry".

Definition: Let (M, g) be a 4-dimensional spacetime and

$$\left.\begin{aligned} \phi : \; & SO(3) \to \mathrm{Diff}(M) \\ & h \longmapsto \phi_h \end{aligned}\right\} \qquad (17.1)$$

be an isometric action of $SO(3)$; that is

$$\left.\begin{aligned} \phi_e &= \mathrm{id}|_M \, , \\ \phi_{h'} \circ \phi_h &= \phi_{h' \cdot h} \end{aligned}\right\} \text{ "action"} \qquad (17.2)$$

with

$$\left.\begin{aligned} \phi_h^* g = g \end{aligned}\right\} \text{ "isometric"} \qquad (17.3)$$

$$\forall \, h \in SO(3)$$

We further require the orbits of this action to be spacelike S^2 (2-spheres); that is

$$\mathrm{Orb}_p(SO(3)) = \bigcup_{h \in SO(3)} \phi_h(p) \qquad (17.4)$$

is an embedded spacelike topological 2-sphere whose geometry is neccessarily "round", i.e. isometric to that of a standard 2-sphere in $\mathbb{R}^3$ of some radius. This is because of our previous requirement (17.3) that $SO(3)$ acts by isometries.

A spacetime admitting an isometric action of $SO(3)$ with spacelike S^2-orbits is called <u>spherically symmetric</u>.

Theorem: A spherically symmetric spacetime admits local charts in which

$$\left.\begin{aligned} g = e^{2a} c^2\, dt \otimes dt - e^{2b}\, dr \otimes dr \\ - r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi \right) \end{aligned}\right\} \quad (17.5)$$

where a and b depend on (t, r) but not on θ, φ.

The SO(3) orbits are given by the 2-spheres of constant t and r, which are isometric to the 2-spheres of metric radius r in $\mathbb{R}^3$. Their area in (M,g) is

$$A(r) = 4\pi r^2 \tag{17.6}$$

just like for a sphere in $\mathbb{R}^3$ of euclidean radius r. It is for this reason that our coordinate r is called "<u>area radius</u>"
It may be geometrically defined by

$$r : M \to \mathbb{R}_+$$

$$p \mapsto r(p) := \left(\frac{A(p)}{4\pi}\right)^{1/2} \tag{17.7}$$

where

$$A(p) = \text{Area}\left(\text{Orb}_p(SO(3))\right). \tag{17.8}$$

Now, for the theorem above to be valid we must assume that this function is a coordinate, which is only true iff

$$dr \neq 0 \tag{17.9}$$

This condition must be added to the list of hypotheses for our theorem.

We shall consider the general case in an exercise.

Given a metric of the form (17.5) we can now write it in terms of orthonormal co-basis:

$$g = \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \qquad (17.10)$$

$$\left.\begin{aligned} \theta^0 &= e^a\, c\,dt = e^a\, dx^0 \\ \theta^1 &= e^b\, dr \\ \theta^2 &= r\, d\theta \\ \theta^3 &= r \sin\theta\, d\varphi \end{aligned}\right\} \qquad (17.11)$$

where $a = a(t, r)$, $b = b(t, r)$.

We shall write

$$\left.\begin{aligned} \dot{a} &:= \frac{\partial a}{\partial x^0}, \quad a' := \frac{\partial a}{\partial r} \\ \dot{b} &:= \frac{\partial b}{\partial x^0}, \quad b' := \frac{\partial b}{\partial r} \end{aligned}\right\} \qquad (17.12)$$

We now employ the Cartan Structure Equations to calculate all connection coefficients and all curvature components with respect to this orthonormal frame.

The 1st structure equation is $(T=0)$

$$d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta = 0 \qquad (17.13)$$

where for $\omega_{\alpha\beta} = \eta_{\alpha\gamma}\,\omega^\gamma{}_\beta$ we have

$$\omega_{\alpha\beta} = -\,\omega_{\beta\alpha} \qquad (17.14)$$

which expresses the metricity of the connection.

We proceed by evaluating (17.13) for $\alpha = 0, 1, 2, 3$. Using (17.11):

$\alpha = 0)$

$$d\theta^0 = e^a(\dot{a}\,dx^0 + a'\,dr) \wedge dx^0$$

$$= -\,a' e^{-b}\,\theta^0 \wedge \theta^1$$

$$= -\,\omega^0{}_a \wedge \theta^a \quad (\text{note } \omega^0{}_0 = 0)$$

$$\Rightarrow \quad \omega^0{}_1 = a' e^{-b}\,\theta^0 + \text{Terms} \sim \theta^1 \qquad (17.15a)$$

$$\omega^0{}_2 = \text{Terms} \sim \theta^2 \qquad (17.15b)$$

$$\omega^0{}_3 = \text{Terms} \sim \theta^3 \qquad (17.15c)$$

$\alpha = 1)$

$$d\theta^1 = e^b(\dot{b}\,dx^0 + b'\,dr) \wedge dr$$
$$= -\dot{b}\,e^{-a}\,\theta^1 \wedge \theta^0$$
$$= -\omega^1{}_0 \wedge \theta^0 - \omega^1{}_2 \wedge \theta^2 - \omega^1{}_3 \wedge \theta^3$$

$$\Rightarrow \quad \omega^1{}_0 = \dot{b}\,e^{-a}\,\theta^1 + \text{Terms} \sim \theta^0 \qquad (17.16a)$$
$$\omega^1{}_2 = \text{Terms} \sim \theta^2 \qquad (17.16b)$$
$$\omega^1{}_3 = \text{Terms} \sim \theta^3 \qquad (17.16c)$$

$\alpha = 2)$

$$d\theta^2 = dr \wedge d\theta = \frac{1}{r} e^{-b}\,\theta^1 \wedge \theta^2$$
$$= -\omega^2{}_0 \wedge \theta^0 - \omega^2{}_1 \wedge \theta^1 - \omega^2{}_3 \wedge \theta^3$$

$$\Rightarrow \quad \omega^2{}_0 = \text{Terms} \sim \theta^0 \qquad (17.17a)$$
$$\omega^2{}_1 = \frac{1}{r} e^{-b}\,\theta^2 + \text{Terms} \sim \theta^1 \qquad (17.17b)$$
$$\omega^2{}_3 = \text{Terms} \sim \theta^3 \qquad (17.17c)$$

$\alpha = 3)$

$$d\theta^3 = \sin\theta\, dr \wedge d\varphi + r\cos\theta\, d\theta \wedge d\varphi$$
$$= \frac{1}{r} e^{-b}\,\theta^1 \wedge \theta^3 + \frac{1}{r}\cot\theta\;\theta^2 \wedge \theta^3$$
$$= -\omega^3{}_0 \wedge \theta^0 - \omega^3{}_1 \wedge \theta^1 - \omega^3{}_2 \wedge \theta^2$$

$$\Rightarrow \quad \omega^3{}_0 = \text{Terms} \sim \theta^0 \qquad (17.18a)$$

$$\omega^3{}_1 = \frac{1}{r} e^{-b}\, \theta^3 + \text{Terms} \sim \theta^1 \qquad (17.18b)$$

$$\omega^3{}_2 = \frac{1}{r} \cot\theta\, \theta^3 + \text{Terms} \sim \theta^2 \qquad (17.18c)$$

Using

$$\omega^0{}_a = \omega^a{}_0$$

and $\qquad \omega^a{}_b = -\omega^b{}_a$

we get from

(17.15a) and (17.16a):

$$\omega^0{}_1 = \omega^1{}_0 = a' e^{-b}\, \theta^0 + \dot{b} e^{-a}\, \theta^1 \qquad (17.19a)$$

(17.15b) and (17.17a):

$$\omega^0{}_2 = \omega^2{}_0 = 0 \qquad (17.19b)$$

(17.15c) and (17.18a):

$$\omega^0{}_3 = \omega^3{}_0 = 0 \qquad (17.19c)$$

(17.16b) and (17.17b):

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{1}{r} e^{-b}\, \theta^2 \qquad (17.19d)$$

(17.16c) and (17.18b):

$$\omega^1{}_3 = -\omega^3{}_1 = -\frac{1}{r} e^{-b}\, \theta^3 \qquad (17.19e)$$

(17.17c) and (17.18c)

$$\omega^2{}_3 = -\omega^3{}_2 = -\frac{1}{r}\cot\theta\ \theta^3 \tag{17.19f}$$

To summarise we get for the connection coefficients

$$\omega^0{}_1 = \omega^1{}_0 = a' e^{-b}\theta^0 + \dot{b} e^{-a}\theta^1 \tag{17.20a}$$

$$= a' e^{a-b} dx^0 + \dot{b} e^{b-a} dr \tag{17.20b}$$

$$\omega^0{}_2 = \omega^0{}_2 = 0 \tag{17.21}$$

$$\omega^0{}_3 = \omega^3{}_0 = 0 \tag{17.22}$$

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{1}{r} e^{-b}\theta^2 \tag{17.23a}$$

$$= -e^{-b} d\theta \tag{17.23b}$$

$$\omega^1{}_3 = -\omega^3{}_1 = -\frac{1}{r} e^{-b}\theta^3 \tag{17.24a}$$

$$= -e^{-b}\sin\theta\, d\varphi \tag{17.24b}$$

$$\omega^2{}_3 = -\omega^3{}_2 = -\frac{1}{r}\cot\theta\ \theta^3 \tag{17.25a}$$

$$= -\cos\theta\, d\varphi \tag{17.25b}$$

The Curvature is now calculated by the 2nd Cartan Structure equation:

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta$$

01)

$$d\omega^0{}_1 = (a' e^{a-b})' \, dr \wedge dx^0$$
$$+ (\dot{b} e^{b-a})^{\cdot} \, dx^0 \wedge dr$$
$$= e^{-(a+b)} \left[(\dot{b} e^{b-a})^{\cdot} - (a' e^{a-b})' \right] \theta^0 \wedge \theta^1$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_1 = \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1 = 0$$

$$\Rightarrow \Omega^0{}_1 = e^{-(a+b)} \left[(\dot{b} e^{b-a})^{\cdot} - (a' e^{a-b})' \right] \theta^0 \wedge \theta^1 \qquad (17.26a)$$

02)

$$d\omega^0{}_2 = 0$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_2 = \omega^0{}_1 \wedge \omega^1{}_2$$
$$= (a' e^{-b} \theta^0 + \dot{b} e^{-a} \theta^1) \wedge \left(-\frac{1}{r} e^{-b}\right) \theta^2$$
$$= -\frac{a'}{r} e^{-2b} \theta^0 \wedge \theta^2 - \frac{\dot{b}}{r} e^{-(a+b)} \theta^1 \wedge \theta^2$$

$$\Rightarrow \Omega^0{}_2 = -\frac{a'}{r} e^{-2b} \theta^0 \wedge \theta^2 - \frac{\dot{b}}{r} e^{-(a+b)} \theta^1 \wedge \theta^2 \qquad (17.26b)$$

03)

$$d\omega^0{}_3 = 0$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_3 = \omega^0{}_1 \wedge \omega^1{}_3$$

$$= \left(a' e^{-b} \theta^0 + \dot{b} e^{-a} \theta^1\right)\left(-\tfrac{1}{r} e^{-b} \theta^3\right)$$

$$= -\frac{a'}{r} e^{-2b} \theta^0 \wedge \theta^3 - \frac{\dot{b}}{r} e^{-(a+b)} \theta^1 \wedge \theta^3$$

$$\Rightarrow \Omega^0{}_3 = -\frac{a'}{r} e^{-2b} \theta^0 \wedge \theta^3 - \frac{\dot{b}}{r} e^{-(a+b)} \theta^1 \wedge \theta^3 \qquad (17.26c)$$

12)

$$d\omega^1{}_2 = e^{-b}\left(\dot{b}\, dx^0 + b' dr\right) \wedge d\theta$$

$$= \frac{\dot{b}}{r} e^{-(a+b)} \theta^0 \wedge \theta^2 + \frac{b'}{r} e^{-2b} \theta^1 \wedge \theta^2$$

$$\omega^1{}_\lambda \wedge \omega^\lambda{}_2 = \omega^1{}_0 \wedge \cancel{\omega^0{}_2} + \omega^1{}_3 \wedge \omega^3{}_2$$

$= 0$ since $\omega^1{}_3 \sim \theta^3$ and $\omega^3{}_2 \sim \theta^3$

$$\Rightarrow \Omega^1{}_2 = \frac{\dot{b}}{r} e^{-(a+b)} \theta^0 \wedge \theta^2 + \frac{b'}{r} e^{-2b} \theta^1 \wedge \theta^2 \qquad (17.26d)$$

13)

$$d\omega^1{}_3 = e^{-b}\left(\dot{b}\, dx^0 + b' dr\right) \wedge \sin\theta\, d\varphi$$

$$- e^{-b} \cos\theta \; d\theta \wedge d\varphi$$

$$= \frac{\dot{b}}{r} e^{-(a+b)} \theta^0 \wedge \theta^3 + \frac{b'}{r} e^{-2b} \theta^1 \wedge \theta^3$$

$$- \frac{\cot\theta}{r^2} e^{-b} \theta^2 \wedge \theta^3$$

$$\omega^1{}_\lambda \wedge \omega^\lambda{}_3 = \omega^1{}_0 \wedge \cancel{\omega^0{}_3}^{\,0} + \omega^1{}_2 \wedge \omega^2{}_3$$

$$= \left(-\frac{1}{r} e^{-b} \theta^2\right) \wedge \left(-\frac{1}{r} \cot\theta \, \theta^3\right)$$

$$= + \frac{\cot\theta}{r^2} e^{-b} \theta^2 \wedge \theta^3$$

$$\Rightarrow \Omega^1{}_3 = \frac{\dot{b}}{r} e^{-(a+b)} \theta^0 \wedge \theta^3 + \frac{b'}{r} e^{-2b} \theta^1 \wedge \theta^3 \qquad (17.26e)$$

23)

$$d\omega^2{}_3 = \sin\theta \, d\theta \wedge d\varphi$$

$$= \frac{1}{r^2} \theta^2 \wedge \theta^3$$

$$\omega^2{}_\lambda \wedge \omega^\lambda{}_3 = \omega^2{}_0 \wedge \omega^0{}_3 + \omega^2{}_1 \wedge \omega^1{}_3$$

$$= \frac{1}{r} e^{-b} \theta^2 \wedge \left(-\frac{1}{r} e^{-b} \theta^3\right)$$

$$= -\frac{1}{r^2} e^{-2b} \theta^2 \wedge \theta^3$$

$$\Rightarrow \Omega^2{}_3 = \frac{1}{r^2} \left[1 - e^{-2b}\right] \theta^2 \wedge \theta^3 \qquad (17.26f)$$

From

$$\Omega^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\mu\nu}\, \theta^\mu \wedge \theta^\nu \qquad (17.27)$$

with $R^\alpha{}_{\beta\mu\nu} = - R^\alpha{}_{\beta\nu\mu}$, we read off all curvature components in the ortho-normal frame θ^α:

$$R^0{}_{101} = R_{0101}$$

$$= e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{b}\dot{a}\right) - e^{-2b}\left(a'' + a'^2 - a'b'\right) \qquad (17.28a)$$

$$R^0{}_{202} = R_{0202} = -\frac{a'}{r} e^{-2b} \qquad (17.28b)$$

$$R^0{}_{212} = R_{0212} = -\frac{\dot{b}}{r} e^{-(a+b)} \qquad (17.28c)$$

$$R^0{}_{303} = R_{0303} = -\frac{a'}{r} e^{-2b} \qquad (17.28d)$$

$$R^0{}_{313} = R_{0313} = -\frac{\dot{b}}{r} e^{-(a+b)} \qquad (17.28e)$$

$$R^1{}_{212} = -R_{1212} = \frac{b'}{r} e^{-2b} \qquad (17.28f)$$

$$R^1{}_{202} = -R_{1202} = \frac{\dot{b}}{r} e^{-(a+b)} \qquad (17.28g)$$

$$R^1{}_{313} = -R_{1313} = \frac{b'}{r} e^{-2b} \qquad (17.28h)$$

$$R^1{}_{303} = -R_{1303} = \frac{\dot{b}}{r} e^{-(a+b)} \qquad (17.28i)$$

$$R^2{}_{323} = -R_{2323} = \frac{1}{r^2}\left(1 - e^{-2b}\right) \qquad (17.28j)$$

All other components vanish!

The non vanishing components of the Ricci-Tensor are

$$R_{00} = R^{\lambda}{}_{0\lambda 0}$$

$$= -R_{1010} - R_{2020} - R_{3030}$$

$$= e^{-2b}(a'' + a'^2 - a'b') - e^{-2a}(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b})$$

$$+ \frac{2a'}{r} e^{-2b}$$

$$= e^{-2b}\left(a'' + a'^2 - a'b' + \frac{2a'}{r}\right)$$

$$- e^{-2a}(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}) \qquad (17.29a)$$

$$R_{01} = R^{\lambda}{}_{0\lambda 1} = -R_{2021} - R_{3031}$$

$$= 2\frac{\dot{b}}{r} e^{-(a+b)} \qquad (17.29b)$$

$$R_{02} = R^{\lambda}{}_{0\lambda 2} = -R_{1012} - R_{3032} = 0 \qquad (17.29c)$$

$$R_{03} = R^{\lambda}{}_{0\lambda 3} = -R_{1013} - R_{2023} = 0 \qquad (17.29d)$$

$$R_{11} = R^{\lambda}{}_{1\lambda 1} = R_{0101} - R_{2121} - R_{3131}$$

$$= e^{-2a}(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}) - e^{-2b}(a'' + a'^2 - a'b')$$

$$+ 2\frac{b'}{r} e^{-2b}$$

$$= e^{-2a}(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}) - e^{-2b}\left(a'' + a'^2 - a'b' - 2\frac{b'}{r}\right) \qquad (17.29.e)$$

$$R_{12} = R^{\lambda}{}_{1\lambda 2} = R_{0102} - R_{3132} = 0 \tag{17.29f}$$

$$R_{13} = R^{\lambda}{}_{1\lambda 3} = R_{0103} - R_{2123} = 0 \tag{17.29g}$$

$$R_{22} = R^{\lambda}{}_{2\lambda 2} = R_{0202} - R_{1212} - R_{3232}$$

$$= -\frac{a'}{r} e^{-2b} + \frac{b'}{r} e^{-2b} + \frac{1}{r^2}\left(1 - e^{-2b}\right)$$

$$= (b' - a')\frac{1}{r} e^{-2b} + \frac{1}{r^2}\left(1 - e^{-2b}\right) \tag{17.29h}$$

$$R_{23} = R^{\lambda}{}_{2\lambda 3} = R_{0203} - R_{1213} = 0 \tag{17.29i}$$

$$R_{33} = R^{\lambda}{}_{3\lambda 3} = R_{0303} - R_{1313} - R_{2323}$$

$$= -\frac{a'}{r} e^{-2b} + \frac{b'}{r} e^{-2b} + \frac{1}{r^2}\left(1 - e^{-2b}\right)$$

$$= (b' - a')\frac{1}{r} e^{-2b} + \frac{1}{r^2}\left(1 - e^{-2b}\right) \tag{17.29j}$$

The Ricci-scalar is

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = R_{00} - R_{11} - R_{22} - R_{33}$$

$$= 2 e^{-2b}\left(a'' + a'^2 - a'b' + \frac{a' - b'}{r}\right)$$

$$- 2 e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right)$$

$$- 2 (b' - a')\frac{1}{r} e^{-2b} - \frac{2}{r^2}\left(1 - e^{-2b}\right)$$

$$= 2\left\{ e^{-2b}\left(a'' + a'^2 - a'b' + 2\frac{a' - b'}{r} + \frac{1}{r^2}\right)\right.$$

$$\left. - e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right) - \frac{1}{r^2}\right\} \tag{17.30}$$

The components of the Einstein-Tensor are easier to determine directly instead of using this long expression for the Ricci Scalar; i.e.

$$G_{00} = R_{00} - \frac{1}{2}\eta_{00}(R_{00} - R_{11} - R_{22} - R_{33})$$

$$= \frac{1}{2}(R_{00} + R_{11} + R_{22} + R_{33})$$

$$= \frac{1}{2}\left(2\frac{a'+b'}{r}e^{-2b} + 2\frac{b'-a'}{r}e^{-2b} + \frac{2}{r^2}(1-e^{-2b})\right)$$

$$= \left(\frac{2b'}{r} - \frac{1}{r^2}\right)e^{-2b} + \frac{1}{r^2}$$

$$= \frac{1}{r^2}\left[1 - \left(re^{-2b}\right)'\right] \qquad (17.31a)$$

$$G_{11} = R_{11} - \frac{1}{2}\eta_{11}(R_{00} - R_{11} - R_{22} - R_{33})$$

$$= \frac{1}{2}(R_{11} + R_{00} - R_{22} - R_{33})$$

$$= \frac{1}{2}\left(2\frac{a'+b'}{r}e^{-2b} - 2\frac{b'-a'}{r}e^{-2b} - \frac{2}{r^2}(1-e^{-2b})\right)$$

$$= \left(\frac{2a'}{r} + \frac{1}{r^2}\right)e^{-2b} - \frac{1}{r^2} \qquad (17.31b)$$

Looking ahead, we note that

$$G_{00} + G_{11} = (a+b)'\,\frac{2}{r}\,e^{-2b}. \qquad (17.31c)$$

$$G_{22} = R_{22} - \frac{1}{2}\eta_{22}\left(R_{00} - R_{11} - R_{22} - R_{33}\right)$$

$$= \frac{1}{2}\left(R_{00} + \cancel{R_{22}} - R_{11} - \cancel{R_{33}}\right)$$

$$= \frac{1}{2}\left(R_{00} - R_{11}\right)$$

$$= e^{-2b}\left(a'' + a'^2 - a'b' + \frac{a' - b'}{r}\right)$$

$$- e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right) \qquad (17.31d)$$

$$G_{33} = G_{22} \qquad (17.31e)$$

$$G_{01} = R_{01} = 2\,\frac{\dot{b}}{r}\,e^{-(a+b)} \qquad (17.31f)$$

$$\left.\begin{aligned} G_{02} &= R_{02} = 0 \\ G_{03} &= R_{03} = 0 \\ G_{12} &= R_{12} = 0 \\ G_{13} &= R_{13} = 0 \\ G_{23} &= R_{23} = 0 \end{aligned}\right\} (17.31g)$$

Hence the only non-diagonal component of the Einstein Tensor is G_{01} (radial energy current density). This is obvious from spherical symmetry, which cannot allow for non-zero vector-field tangent to S^2.