Lecture 18

The exterior Schwarzschild solution I

We seek a spherically-symmetric solution of the vacuum Einstein equations without cosmological constant. We will later generalise it to $\Lambda \neq 0$ and also a Coulomb-like electric or magnetic field. Hence we have to solve

$$G_{\alpha\beta} = 0 \qquad (18.1)$$

From last lecture we know that for

$$g = e^{2a(t,r)}\, c\,dt \otimes c\,dt - e^{2b(t,r)}\, dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi \right) \qquad (18.2)$$

the only components of $G_{\alpha\beta}$ which are not a priori vanishing are

$$G_{01} = 2\,\frac{\dot{b}}{r}\, e^{-(a+b)} \qquad (18.3a)$$

$$G_{00} = \left(\frac{2b'}{r} - \frac{1}{r^2} \right) e^{-2b} + \frac{1}{r^2} = r^{-2}\left[1 - \left(r e^{-2b} \right)' \right] \qquad (18.3b)$$

$$G_{11} = \left(\frac{2a'}{r} + \frac{1}{r^2} \right) e^{-2b} - \frac{1}{r^2} \qquad (18.3c)$$

$$G_{22} = G_{33}$$

$$= e^{-2b}\left(a'' + a'^2 - a'b' + \frac{a'-b'}{r}\right)$$

$$- e^{-2a}\left(\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right) \qquad (18.3d)$$

So from (18.1) and (18.3a) we get

$$\dot{b} = 0 \iff b = b(r) \qquad (18.4)$$

Note

$$G_{00} + G_{11} = e^{-2b}\,\frac{2}{r}\,(a' + b') \qquad (18.5)$$

hence (18.1) also implies, using (18.4),

$$a' + b' = 0 \iff a(t,r) = -b(r) + f(t) \qquad (18.5)$$

where f is some undetermined function of coord. time t.

As b only depends on r, $G_{00} = 0$ is an ordinary differential equation for $b(r)$:

$$G_{00} = 0 \iff 1 = \left(r\, e^{-2b}\right)'$$

$$\Rightarrow \quad e^{-2b(r)} = 1 - \frac{2m}{r}, \quad (r > 2m) \qquad (18.6)$$

where the constant of integration is called m. It has the dimension of length.

Together with (18.5)

$$e^{2a(t,r)} = e^{2f(t)}\left(1-\frac{2m}{r}\right) \qquad (18.7)$$

(18.6-7) imply $G_{01} = G_{00} = G_{11} = 0$.
But we still have to satisfy $G_{22} = 0$.
In fact (18.5-6) already imply $G_{22} = 0$.
To see this, note that for $\dot{b} = 0$ and $a' = -b'$ G_{22} reduces to

$$G_{22} = e^{-2b}\left(-b'' + 2b'^2 - 2\, b'/r\right) \qquad (18.8)$$

so that we need to show

$$b'' - 2b'^2 + 2\frac{b'}{r} = 0 \qquad (18.9)$$

We have

$$\begin{aligned}(e^{-2b})'' &= (-2b' e^{-2b})' \\ &= (-2b'' + 4b'^2)\, e^{-2b} \\ &= -2e^{-2b}(b'' - 2b'^2) \qquad (18.10)\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad b'' - 2b'^2 &= -\frac{1}{2} e^{2b} (e^{-2b})'' \\ &= -\frac{1}{2}\left(1-\frac{2m}{r}\right)^{-1}\left(-\frac{4m}{r^3}\right) \\ &= \frac{2m}{r^3}\left(1-\frac{2m}{r}\right)^{-1} \qquad (18.11)\end{aligned}$$

On the other hand

$$2\frac{b'}{r} = \frac{2}{r}\left(-\frac{1}{2}\right)\left[\ln\left(1-\frac{2m}{r}\right)\right]'$$

$$= -\frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1}\frac{2m}{r^2}$$

$$= -\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)^{-1} \qquad (18.12)$$

(18.11-12) show that (18.9) and hence $G_{22} = 0$.

As a result we have determined all solutions of the form (18.2):

$$g = \left(1-\frac{2m}{r}\right)e^{2f(t)}\, c\,dt \otimes c\,dt$$

$$- \left(1-\frac{2m}{r}\right)^{-1} dr \otimes dr$$

$$- r^2\left(d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi\right) \qquad (18.13)$$

The function f of t remains undetermined. However, the original form (18.2) does not specify. We can redefine

$$t \mapsto t' = h(t) \qquad (18.14)$$

so that $\quad dt = dt' / h'(t) \qquad (18.15)$

which merely results in a redefinition of a:

$$e^{a(t,r)} \rightarrow e^{a'(t',r)} := e^{a(r,t)} / h'(t) \qquad (18.16)$$

Hence h can be chosen so as to remove the $f(t)$ - part in a:

$$h'(t) = e^{f(t)} \qquad (18.17)$$

$$\Leftrightarrow \quad t' = h(t) = \int_0^t d\tilde{t}\, e^{f(\tilde{t})} \qquad (18.18)$$

Then, renaming t' for t again, we get (18.13) without explicit time dependent term $f(t)$:

$$g = \left(1 - \frac{2m}{r}\right) c\,dt \otimes c\,dt$$

$$- \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr$$

$$- r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi\right) \qquad (18.19)$$

(Schwarzschild 1916)

This metric has <u>four</u> Killing vector fields, not just three, as could be anticipated from the requirement of spherical symmetry

$$\left.\begin{aligned} X_1 &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \\ X_2 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ X_3 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \end{aligned}\right\} \quad (18.20)$$

for
$$\left.\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}\right\} \quad (18.21)$$

The fourth Killing vector field is (new t coord.)

$$X_4 = \frac{\partial}{\partial t} \qquad (18.22)$$

which we get "for free" as a result of a theorem that we have just proven

Theorem (Jebsen 1921, Birkhoff 1923):
If a metric of the form (18.2) satisfies the vaccum Einstein equation without cosmological constant (i.e. is Ricci flat) it is static. This means it possesses a timelike, hypersurface orthogonal killing vector field.

Since the vacuum Einstein equations for $\Lambda = 0$ are equivalent to vanishing Ricci-Tensor, the exterior Schwarzschild solution is "Ricci flat". Hence all curvature is in the Weyl-Tensor (compare (8.46)):

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}. \qquad (18.23)$$

The non zero components of the Riemann-Tensor follow from eqns. (17.28a-28j), of which c, e, g, and i are zero due to $\dot{b} = 0$. Using that $a = -b$ and that

$$e^{2a} = \left(1 - \frac{2m}{r}\right) = e^{-2b} \qquad (18.24)$$

$$a = \frac{1}{2} \ln\left(1 - \frac{2m}{r}\right) \qquad (18.25)$$

$$a' = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} = \frac{m}{r^2} e^{-2a} \qquad (18.26)$$

$$\rightsquigarrow \quad a' e^{2a} = a' e^{-2b} = -b' e^{-2b} = \frac{m}{r^2} \qquad (18.27)$$

Using this, we can now write down all non-vanishing components of the Riemann Tensor in an orthonormal frame. We note the covariant components. The only little calculation left is that for R_{0101}

$$R_{0101} = -e^{-2b}\left(a'' + 2a'^2\right)$$

$$\left(e^{2a}\right)'' = \left(2a' e^{2a}\right)' = 2\left(a'' + 2a'^2\right) e^{2a}$$

$$= 2\left(a'' + 2a'^2\right) e^{-2b} \qquad (18.28)$$

$$\Rightarrow R_{0101} = -\frac{1}{2}\left(e^{2a}\right)'' = -\frac{1}{2}\left(-\frac{2m}{r}\right)''$$

$$= \frac{2m}{r^3} \qquad (18.29a)$$

Further:

$$R_{0202} = -\frac{a'}{r} e^{-2b} = -\frac{m}{r^3} \qquad (18.29b)$$

$$R_{0303} = -\frac{a'}{r} e^{-2b} = -\frac{m}{r^3} \qquad (18.29c)$$

$$R_{1212} = -\frac{b'}{r} e^{-2b} = +\frac{m}{r^3} \qquad (18.29d)$$

$$R_{1313} = -\frac{b'}{r} e^{-2b} = +\frac{m}{r^3} \qquad (18.29e)$$

$$R_{2323} = -r^{-2}\left(1 - e^{-2b}\right) = -\frac{2m}{r^3} \qquad (18.29f)$$

or

$$\left.\begin{aligned} \frac{m}{r^3} &= \frac{1}{2} R_{0101} = -\frac{1}{2} R_{2323} \\ &= -R_{0202} = -R_{0303} \\ &= +R_{1212} = +R_{1313} \end{aligned}\right\} \qquad (18.30)$$

The quadratic curvature skalar

$$K := R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$$

is called the Kretschmann-Skalar.
Taking all symmetries into account, which here amount to

$$K = R_{0101} R^{0101} + R_{1001} R^{1001} + R_{0110} R^{0110} + R_{1010} R^{1010} + \cdots$$

$$= 4 \left(R_{0101} \right)^2 + \cdots \qquad (18.31)$$

We set

$$K = 4 \left(R_{0101}^2 + R_{0202}^2 + R_{0303}^2 \right.$$
$$\left. R_{1212}^2 + R_{1313}^2 + R_{2323}^2 \right)$$

$$= 4 \frac{m^2}{r^6} \left(4 + 1 + 1 + 1 + 1 + 4 \right)$$

$$= 48 \left(\frac{m}{r^3} \right)^2 \qquad (18.32)$$

$\Rightarrow$ No curvature singularities except for $r = 0$. But, at this moment, we only take the solution seriously for $r > 2m$, for only then is $e^{2a} = e^{-2b} = \left(1 - \frac{2m}{r}\right) > 0$.

As the solution is stationary, we can calculate the acceleration of a stationary observer who stays at a fixed distance from the central object.

The timelike Killing field is

$$K = \frac{\partial}{\partial t} \tag{18.33}$$

corresponding to an observer-four-velocity

$$u = c\,\frac{K}{[g(K,K)]^{1/2}} = f\,K \tag{18.34}$$

where

$$f := c\,[g(K,K)]^{-1/2} = \left(1-\frac{2m}{r}\right)^{-1/2} \tag{18.35}$$

We wish to calculate the four-acceleration

$$a := \nabla_u u \tag{18.36}$$

<u>First method:</u> coordinate free.

Have for any $X \in ST^1_0(M)$

$$X(f) = -\frac{c}{2}\,[g(K,K)]^{-3/2}\, X(g(K,K))$$

$$= -\frac{1}{2c^2}\, f^3\, 2\, g(\nabla_X K, K)$$

$$= -(f^3/c^2)\, g(\nabla_K X + [X,K], K)$$

zero torsion

$$= -(f^3/c^2)\, g(\nabla_K X - L_K X, K) \tag{18.37}$$

But

$$g(\nabla_K X, K) = K[g(X,K)] - g(X, \nabla_K K)$$

$$g(L_K X, K) = K[g(X,K)] - (L_K g)(X,K)$$

(since $L_K K = [K,K] = 0$). Using that K is Killing, i.e. $L_K g = 0$, we get

$$X(f) = (f^3/c^2)\, g(\nabla_K K, X) \tag{18.38}$$

And since this holds for all $X \in ST^1_0(M)$

$$df = (f^3/c^2)\,(\nabla_K K)^\downarrow \tag{18.39}$$

This equation holds generally for all Killing fields. It can also be written as

$$d(g(K,K)) = -2\,(\nabla_K K)^\downarrow \tag{18.40}$$

Now, for the acceleration we have

$$a = \nabla_u u = \nabla_{fK}(fK)$$

$$= f\, \nabla_K (fK) = f^2\, \nabla_K K \tag{18.41}$$

obvious due to K = Killing

The second equation follows from

$$\nabla_K f = L_K f = L_K \, c \, (g(K,K))^{-1/2}$$

$$= -\frac{c}{2} (g(K,K))^{-3/2} \underbrace{L_K(g(K,K))}_{=0} = 0 \qquad (18.42)$$

for $L_K g = 0$ and $L_K K = 0$.

Equations (18.39) and (18.41) together imply

$$a^{\flat} = c^2 \, df/f$$

$$= c^2 \, d(\ln(f))$$

$$= -\frac{c^2}{2} \, d(\ln(f^{-2}))$$

$$= -\frac{c^2}{2} \, d\left(g(K,K)/c^2\right) \qquad (18.43)$$

This is still a general equation, valid for any Killing field on a Lorentzian manifold. If we now apply this to our special case where f is given by (18.35), $f = (1 - 2m/r)^{-1/2}$, we get

$$a^{\flat} = c^2 \, d \ln\left[\left(1 - \frac{2m}{r}\right)^{-1/2}\right]$$

$$= -\frac{c^2}{2} \left(1 - \frac{2m}{r}\right)^{-1} \frac{2m}{r^2} \, dr \qquad (18.44)$$

And since

$$\theta^r = \left(1-\frac{2m}{r}\right)^{-1/2} dr \qquad (18.45)$$

$$a^\downarrow = -\frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}\theta^r$$

$$= \frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2}(e_r)^\downarrow \qquad (18.46)$$

where $$e_r = \left(1-\frac{2m}{r}\right)^{1/2}\frac{\partial}{\partial r} \qquad (18.47)$$

and $$e_r^\downarrow = g(e_r, \cdot) = -\theta^r \qquad (18.48)$$

Applying the $\uparrow$ - isomorphism to (18.46) implies

$$a = \frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2} e_r \qquad (18.49)$$

This says that the stationary observer accelerates <u>outward</u> in radial direction with modulus

$$\|a\| = \left[|g(a,a)|\right]^{1/2} = \frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right) \qquad (18.50)$$

Note : 1) This diverges for $r \downarrow 2m$

2) This turns into Newtonian value $\frac{GM}{r^2}$ for $r \to \infty$ if $m = \frac{GM}{c^2}$

The last observation gives us an interpretation of the parameter m (integration constant (compare (18.6))) in terms of the Newtonian mass M:

$$m = \frac{GM}{c^2} \qquad (18.51)$$

The acceleration diverges if the radii approach the critical value

$$r = 2m = 2GM/c^2 =: r_s \qquad (18.52)$$

which is called the "<u>Schwarzschild Radius</u>" an which coincides in value with what we previously called the gravitational radius R_g (which was just a formal assignment of a quantity with the physical dimension of a length proportional to the mass. Note that

$$g(K,K) = \left(1 - \frac{2m}{r}\right) c^2 \qquad (18.53)$$

so K turns lightlike for $r = 2m$.

$\Rightarrow$ Only the region $r > 2m$ is stationary. If we considered the Schwarzschild metric (18.19) valid for $r < 2m$ it would <u>not</u> be stationary. For $r < 2m$ t and r change rôles: $\partial/\partial t$ is spacelike and $\partial/\partial r$ is timelike. Hence $r \searrow 0$ occurs "in time", that is: $r=0$ is a "moment in time", not a place "in space". Still put differently: The curvature singularity $r \downarrow 0$ is "sometime" not "somewhere". But for the moment we do not care about $r \leq 2m$.

Just to show that coordinate based calculations can be quicker than coordinate free ones, we show how to arrive at (18.49) by direct coordinate computation. We have

$$u = \left(1-\frac{2m}{r}\right)^{-1/2} \frac{\partial}{\partial t} \qquad (18.54)$$

$$\nabla_u u = \left(1-\frac{2m}{r}\right)^{-1/2} \nabla_{\frac{\partial}{\partial t}} \left(1-\frac{2m}{r}\right)^{-1/2} \frac{\partial}{\partial t}$$

$$= \left(1-\frac{2m}{r}\right)^{-1} \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t}$$

$$= \left(1-\frac{2m}{r}\right)^{-1} \Gamma^{\lambda}_{tt} \frac{\partial}{\partial x^\lambda}$$

$$= \left(1-\frac{2m}{r}\right)^{-1} \frac{1}{2} g^{\lambda\sigma}\left(-g_{tt,\sigma} + 2 \overset{0}{\cancel{g_{\sigma t,t}}}\right) \frac{\partial}{\partial x^\lambda}$$

$$\left(1-\frac{2m}{r}\right)^{-1} \frac{1}{2} g^{rr}\left(-g_{tt,r}\right) \frac{\partial}{\partial r} \qquad (18.55)$$

$$g_{tt} = c^2\left(1-\frac{2m}{r}\right) \curvearrowright g_{tt,r} = c^2 \frac{2m}{r^2} \qquad (18.56)$$

$$g^{rr} = (g_{rr})^{-1} = -\left(1-\frac{2m}{r}\right) \qquad (18.57)$$

Hence

$$\nabla_u u = \frac{mc^2}{r^2}\frac{\partial}{\partial r} = a^r \frac{\partial}{\partial r} \qquad (18.58)$$

$$\Rightarrow \quad a^r = \frac{mc^2}{r^2} = \frac{GM}{r^2} \qquad \left(m = \frac{GM}{c^2}\right). \qquad (18.59)$$

This seems perfectly regular at $r \searrow 2m$.
But note that

$$g_{rr} = g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -\left(1-\frac{2m}{r}\right)^{-1} \qquad (18.60)$$

diverges at $r \searrow 2m$ which means that $\partial/\partial r$ becomes "infinitely long" in that limit. Writing

$$e_r = \left(1-\frac{2m}{r}\right)^{1/2}\frac{\partial}{\partial r} \qquad (18.61)$$

so that e_r is normalised, we have

$$a = a^r \frac{\partial}{\partial r} = \hat{a}^r e_r \qquad (18.62)$$

where
$$\hat{a}^r = \left(1-\frac{2m}{r}\right)^{-1/2} a^r$$

$$= \frac{mc^2}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2} \qquad (18.63)$$

now shows that the modulus of a at $2m$ diverges. $\Rightarrow$ Moral: Regular coefficients do not always imply singularity free behaviour.

Finally note a conceptual point:
The acceleration of the worldline of a stationary (in fact static) observer is outward pointing:

$$a = \frac{GM}{r^2}\left(1-\frac{2m}{r}\right)^{-1/2} e_r$$

This is because you have to accelerate in an outward direction in order to not fall onto the central mass, i.e. in order to stay on a stationary orbit. If you "let go", i.e. switch-off the agent that is accelerating you, you will be force free and fall towards smaller radii. This is quite the opposite way of how one speaks in Newtonian theory, where one says that you fall onto a mass because the mass pulls you towards it. In GR, gravity is not a force but part of an inertial structure. Forces in GR cause deviations from geodesics. The geodesic here is the trajectory that falls towards the mass. In GR "forces" are the causes for not falling towards the mass, the free fall is "without cause", like inertial motion in Newtonian physics, there is no "pull" towards the mass.