Lecture 19

Timelike and lightlike geodesic motion in the external Schwarzschild geometry

We set

$$r_s = 2m = \frac{2GM}{c^2} \tag{19.1}$$

and write the metric as

$$\begin{aligned} g = & \left(1 - \frac{r_s}{r}\right) c^2 \, dt \otimes dt \\ & - \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr \\ & - r^2 \left(d\theta \otimes d\theta + \sin^2\theta \, d\varphi \otimes d\varphi \right) \end{aligned} \tag{19.2}$$

The energy-functional is

$$\begin{aligned} E = \frac{1}{2} \int \Big\{ & \left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 \\ & - r^2 \left(\dot{\theta}^2 + \sin^2(\theta) \dot{\varphi}^2 \right) \Big\} \, d\lambda \end{aligned} \tag{19.3}$$

where λ is an affine parameter which we take to be the eigentime in the timelike case. Hence

$$g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = \sigma := \begin{cases} c^2 & \text{for particles} \\ 0 & \text{for light.} \end{cases} \tag{19.4}$$

The Euler-Lagrange equations for θ, t and φ are

$$\left.\begin{aligned} \theta) : \quad & \frac{\partial L}{\partial \dot\theta} = -r^2 \dot\theta \\ & \frac{\partial L}{\partial \theta} = -r^2 \sin\theta \cos\theta \, \dot\varphi^2 \end{aligned}\right\} \quad (19.5)$$

$$\Rightarrow \quad (-r^2\dot\theta)^{\cdot} + r^2 \sin\theta\cos\theta\,\dot\varphi^2 \qquad (19.6)$$

So if initially the motion starts within the equatorial plane $\theta = \frac{\pi}{2}$ and tangentially to it, $\dot\theta = 0$, (19.6) implies $\ddot\theta = 0$ and unique solution is

$$\theta = \frac{\pi}{2} \qquad (19.7)$$

for all time. So motion remains in eqatorial plane. This we shall now assume

$t)$ cyclic coordinate : $\frac{\partial L}{\partial t} = 0$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot t} = \dot t\, c^2 \left(1 - \frac{r_s}{r}\right) = \tilde E = \text{const} \qquad (19.8)$$

For massice particles $\tilde E$ has the interpretation of "Energy / rest mass" $= \tilde E$.

φ) cyclic coordinate : $\frac{\partial L}{\partial \varphi} = 0$

$$\rightarrow \frac{\partial L}{\partial \dot{\varphi}} = r^2 \sin(\theta)\, \dot{\varphi}$$

$$(\theta = \tfrac{\pi}{2}) \qquad = r^2 \dot{\varphi} = l = \text{const.} \qquad (19.9)$$

l is twice the "area constant" known from the Newtonian Kepler problem which for massive particles equals "angular momentum / rest mass".

Instead of Euler-Lagrange equation for r we use the integral (19.4) [just like we use energy conservation in Newtonian Kepler problem instead of equation $\ddot{r} = \cdots$].

$$\left. \begin{aligned} &\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_s}{r}} \\ &\qquad - r^2 \left(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\varphi}^2\right) = \sigma \end{aligned} \right\} (19.10)$$

In this we set $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$ and replace $\dot{t}$ and $\dot{\varphi}$ according to (19.8) and (19.9), respectively. The result is an expression in terms of $\dot{r}$ and r, that is, a single ordinary differential equation of first order for $r(\lambda)$:

$$\left(1-\frac{r_s}{r}\right) c^2 \left[\frac{\tilde{E}}{c^2\left(1-\frac{r_s}{r}\right)}\right]^2 - \frac{\dot{r}^2}{1-\frac{r_s}{r}}$$

$$- r^2\left(\frac{l}{r^2}\right)^2 = \sigma$$

or

$$\dot{r}^2 + V_{eff}(r) = \tilde{E}^2/c^2 \qquad (19.11)$$

where

$$V_{eff}(r) = \left(1-\frac{r_s}{r}\right)\left(\frac{l^2}{r^2}+\sigma\right) \qquad (19.12)$$

$$= \underbrace{\sigma}_{\text{const.}} - \underbrace{\frac{\sigma r_s}{r}}_{\text{Kepler pot.}} + \underbrace{\frac{l^2}{r^2}}_{\text{ang. mom. barrier}} - \underbrace{\frac{r_s l^2}{r^3}}_{\text{GR-correction}}$$

This is mathematically equivalent to a perturbed Kepler problem with potential

$$-\frac{1}{2}\left(\frac{\sigma r_s}{r} + \frac{r_s l^2}{r^3}\right)$$

$$= -\frac{GM}{r} - \frac{GM}{c^2} l^2 \frac{1}{r^3} \qquad (19.13)$$

in case $\sigma = c^2$. In the language of Problem 1 of Sheet 2 this corresponds to

$$V(r) = -\frac{GMm}{r} - \frac{GM}{c^2} m l^2 \frac{1}{r^3} \qquad (19.14)$$

$$= -\alpha/r + \Delta_3 V = -\frac{\alpha}{r} + \delta_3 r^{-3} \qquad (19.15)$$

with

$$\left.\begin{aligned} \alpha &= GMm \\ \delta_3 &= -GMm\frac{\ell^2}{c^2} \end{aligned}\right\} \quad (19.16)$$

leading to a periastron precession according to equation (5b) of Sheet 2:

$$\Delta_3\varphi = -6\pi\,\frac{\delta_3/\alpha}{[a(1-\varepsilon^2)]^2}$$

$$= 6\pi\,\frac{\ell^2/c^2}{[a(1-\varepsilon^2)]^2} = \Delta_{GR}\varphi \quad (19.17)$$

Here $\ell = L/m$. For the Keplerian orbit the semi-latus rectum p obeys

$$p = a(1-\varepsilon^2) = L^2/m\alpha \quad (19.18)$$

(compare eq. 2b on sheet 2). Hence

$$p = (m\ell)^2/m\alpha = m^2\ell^2/mGMm$$

$$= \ell^2/GM,$$

or $$\ell^2 = a(1-\varepsilon^2)\,GM \quad (19.19)$$

Hence

$$\boxed{\Delta_{GR}\varphi = 6\pi\,\frac{GM/c^2}{a(1-\varepsilon^2)}} \quad (19.20)$$

Einstein 1916

Going back to (19.12) we see

$$V_{eff}(r) = \sigma - \frac{\sigma r_s}{r} + \frac{l^2}{r^2} - \frac{r_s l^2}{r^3}$$

$$V'_{eff}(r) = \frac{\sigma r_s}{r^2} - 2\frac{l^2}{r^3} + 3\frac{r_s l^2}{r^4}$$

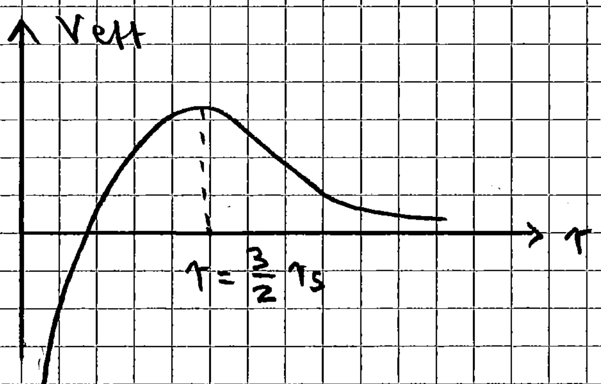
$$= \frac{1}{r^4}\left[r^2 \sigma r_s - 2 r l^2 + 3 r_s l^2\right] \qquad (19.21)$$

We discuss $\sigma = 0$ (light) and $\sigma = c^2$ (particles) separately:

<u>Case 1</u> : $\sigma = 0$ (light)

$$\left.\begin{aligned} &V(r \to \infty) \searrow 0 \\ &V(r \to 0) \searrow -\infty \\ &V'_{eff}(r) = 0 \iff r = \frac{3}{2} r_s \end{aligned}\right\} \qquad (19.22)$$

which must clearly be a maximum



(19.23)

$\Rightarrow$ Unstable circular orbit at

$$r = \frac{3}{2} r_s = 3m \qquad (19.24)$$

<u>Case 2 :</u> $\sigma = c^2$ (massive particles)

$$\left.\begin{aligned} V(r \to \infty) &\nearrow c^2 \\ V(r \to 0) &\searrow -\infty \end{aligned}\right\} \quad (19.25)$$

$$V'_{\text{eff}}(r) = 0 \iff r^2 - 2r\frac{\ell^2}{r_s \sigma} + 3\frac{\ell^2}{\sigma} = 0$$

$$\iff \quad r_{1,2} = \frac{\ell^2}{r_s\sigma} \pm \left[\left(\frac{\ell^2}{r_s\sigma}\right)^2 - 3\frac{\ell^2}{\sigma}\right]^{1/2}$$

$$= \frac{\ell^2}{r_s\sigma}\left\{1 \pm \left[1 - 3\frac{\sigma r_s^2}{\ell^2}\right]^{1/2}\right\} \quad (19.26)$$

Hence, below the critical angular momentum (per rest mass)

$$\ell = \ell_{\text{crit}} := \sqrt{3\sigma}\, r_s = \sqrt{3}\, r_s c \quad (19.27)$$

there are no extrema and V_{eff} is a monotoneously increasing function. The system (particle) may escape to infinity if $\tilde{E}^2/c^2 > \sup(V_{\text{eff}}) = c^2$, i.e. if $\tilde{E} > c^2$, but otherwise just always drops down to $r = 0$.

For $l > l_{crit}$ there exist two extrema

$$\left.\begin{aligned} r_{max} &= 3 r_s \left(\frac{l}{l_{crit}}\right)^2 \left\{ 1 - \left[1 - \left(\frac{l_{crit}}{l}\right)^2 \right]^{1/2} \right\} \\ r_{min} &= 3 r_s \left(\frac{l}{l_{crit}}\right)^2 \left\{ 1 + \left[1 - \left(\frac{l_{crit}}{l}\right)^2 \right]^{1/2} \right\} \end{aligned}\right\} \quad (19.28)$$

Note $\qquad r_{max} \leq r_{min} \qquad$ (19.29)

For $l = l_{crit}$ we have $r_{max} = r_{min} = 3 r_s$

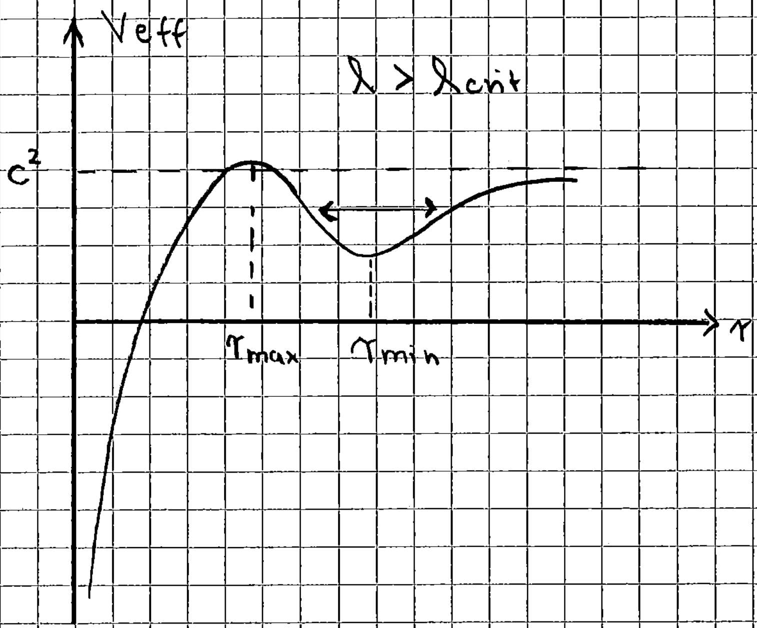
For $l > l_{crit}$ r_{max} and r_{min} decrease and increase, respectively, as function of l

$$r_{max} \xrightarrow[l \to \infty]{} \frac{3}{2} r_s$$

$$r_{min} \xrightarrow[l \to \infty]{} \infty$$

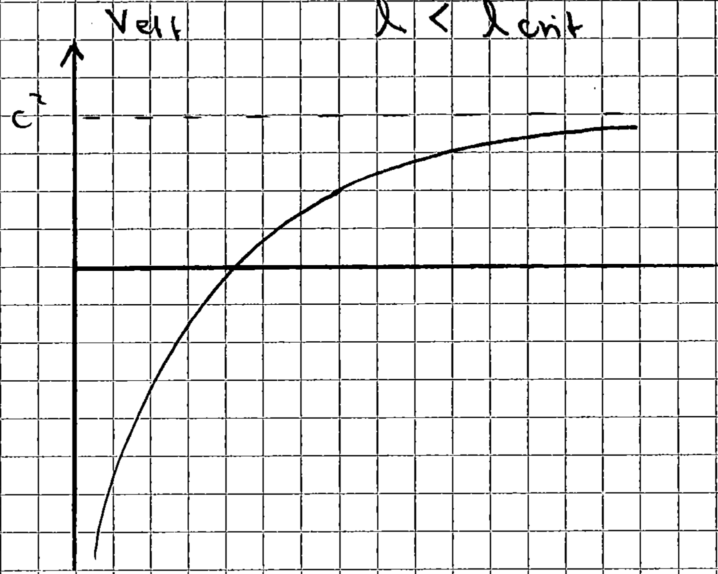
$$\left.\begin{aligned} & r_{max} : \ (l_{crit}, \infty) \longrightarrow (3 r_s, \tfrac{3}{2} r_s) \\ & \qquad \text{monotonically decreasing} \end{aligned}\right\} \quad (19.30)$$

$$\left.\begin{aligned} & r_{min} : \ (l_{crit}, \infty) \longrightarrow (3 r_s, \infty) \\ & \qquad \text{monotonically increasing} \end{aligned}\right\} \quad (19.31)$$



Stable bound orbits exist. Stable circular orbit at r_{min}, unstable at r_{max}.

(19.32)



no stable bound orbits

(19.32)

The "innermost stable circular orbit", so called "ISCO", is the stable circular orbit with smallest r_{min}, which is at $3r_s$

$$r_{ISCO} = 3r_s = 6m \qquad (19.33)$$

But this is not the radius of closest approach. Elliptic orbits oscillating around r_{min} exist down to r_{max}, if $V(r_{max}) < c^2$.

We determine the range for angular momentum, in which

$$V_{eff}(r_{max}) < V_{eff}(r=\infty) = c^2 \qquad (19.34)$$

so that particle does not escape to infinity. (19.34) is equivalent to

$$\sigma - \sigma \frac{r_s}{r_{max}} + \frac{\ell^2}{r_{max}^2} - \frac{r_s \ell^2}{r_{max}^3} < \sigma \qquad (19.35)$$

$$\Leftrightarrow \quad -\sigma \frac{r_s}{r_{max}^3} \left\{ r_{max}^2 - \frac{\ell^2}{\sigma r_s} r_{max} + \frac{\ell^2}{\sigma} \right\} < 0$$

$$\Leftrightarrow \quad r_{max}^2 - \frac{\ell^2}{\sigma r_s} r_{max} + \frac{\ell^2}{\sigma} > 0$$

$$\Leftrightarrow \quad r_{max}^{(1,2)} = \frac{\ell^2}{2\sigma r_s} \pm \left[\left(\frac{\ell^2}{2\sigma r_s} \right)^2 - \frac{\ell^2}{\sigma} \right]^{1/2} \notin \mathbb{R} \qquad (19.36)$$

$$\Leftrightarrow \quad \frac{\ell^4}{4\sigma^2 r_s^2} < \frac{\ell^2}{\sigma} \qquad (19.37)$$

$$\Leftrightarrow \quad \left. \begin{aligned} \ell &< 2\sqrt{\sigma}\, r_s = \frac{2}{\sqrt{3}} \ell_{crit} \\ &= 1.1547 \times \ell_{crit} \end{aligned} \right\} \quad (19.38)$$

Hence $V_{eff}(r_{max}) < V_{eff}(\infty)$ if and only if:

$$\ell_{krit} < \ell < \frac{2}{\sqrt{3}} \ell_{crit} \cong 1.1547 \times \ell_{crit} \qquad (19.39)$$

r_{max} is smaller for $l = \frac{2}{\sqrt{3}} l_{crit}$ since $l \mapsto r_{max}(l)$ is monotonically decreasing. Have from (19.28):

$$r_{max}\left(l = \frac{2}{\sqrt{3}} l_{crit}\right)$$

$$= 3 r_s \left(\frac{2}{\sqrt{3}}\right)^2 \left\{ 1 + \left[1 - \left(\frac{\sqrt{3}}{2}\right)^2 \right]^{1/2} \right\}$$

$$= 4 r_s \frac{1}{2} = 2 r_s = 4m \qquad (19.40)$$

$\Rightarrow$ $\boxed{\text{no stable bound orbit approaches the centre closer than } r = 2 r_s.}$

So we found the following relevant radii:

$$r_s < \frac{3}{2} r_s < 2 r_s < 3 r_s \qquad (19.41)$$

- r_s ↓ "Horizon"
- $\frac{3}{2} r_s$ ↓ unstable circular photon orbit
- $2 r_s$ ↓ Closest approach by stable bound particle orbit
- $3 r_s$ ↓ Closest (innermost) stable circular orbit.

Radial free fall

A. Radial free fall from $r = R > r_s$ to $R > r \geq r_s$

For a radial fall have $\dot{\varphi} = 0$ and hence $l = 0$. Hence the effective potential V_{eff} is according to (19.12):

$$V_{eff}(r) = \sigma \left(1 - \frac{r_s}{r}\right) = c^2\left(1 - \frac{r_s}{r}\right) \qquad (19.42)$$

and (19.11) becomes

$$\dot{r}^2 + c^2\left(1 - \frac{r_s}{r}\right) = \tilde{E}^2/c^2 \qquad (19.43)$$

With initial condition $\dot{r} = 0$ for $r = R$ we get

$$\frac{\tilde{E}^2}{c^2} = c^2\left(1 - \frac{r_s}{R}\right) \qquad (19.44)$$

$$\rightsquigarrow \quad \dot{r}^2 = c^2 r_s \left(\frac{1}{r} - \frac{1}{R}\right)$$

$$= c^2 \frac{r_s}{R}\left(\frac{R}{r} - 1\right) \qquad (19.45)$$

Note that here a dot "$\cdot$" is $\frac{d}{d\tau}$ where τ is Eigentime

Since $\dot{r} < 0$ (infall) we have

$$\dot{r} = - c \left(\frac{r_s}{R}\right)^{1/2} \left[\frac{R}{r} - 1\right]^{1/2} \qquad (19.46)$$

and the Eigentime for free-fall from R to $r < R$ is

$$\tau(R, r) = - \frac{1}{c}\left(\frac{R}{r_s}\right)^{1/2} \int_R^r \frac{dr'}{\left(\frac{R}{r'} - 1\right)^{1/2}}$$

$$= \frac{1}{c}\left(\frac{R^3}{r_s}\right)^{1/2} \int_{\frac{r}{R}}^{1} \frac{dx}{\left[\frac{1}{x} - 1\right]^{1/2}} \qquad (19.47)$$

Where $x := r/R$

Note that $\frac{1}{c}\left(\frac{R^3}{r_s}\right)^{1/2} = \left(\frac{R^3}{2GM}\right)^{1/2}$, which does not depend on c. Indeed, there is exactly the same expression in Newtonian theory as regards Newtonian free-fall time:

$$\frac{1}{2}\dot{r}^2 - G\frac{M}{r} = E \quad ,$$

Using $\dot{r} = 0$ for $r = R \Rightarrow E = -G\frac{M}{R}$

$$\Rightarrow \dot{r}^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$\Rightarrow \quad t = \frac{1}{\sqrt{2GM}} \int_r^R \frac{dr}{\left(\frac{1}{r} - \frac{1}{R}\right)^{1/2}} = \left(\frac{R^3}{2GM}\right)^{1/2} \int_{r/R}^{1} \frac{dx}{\left(\frac{1}{x} - 1\right)^{1/2}} \qquad (19.48)$$

$\Rightarrow$ Eigentime for relativistic free-fall equals Newtonian time for Newtonian free-fall between equal values of the respective radial coordinates (area radius in GR case and euclidean distance in Newtonian case).

In particular the free-fall eigentime is finite for any starting point $r = R > 0$ to any final point $R > r \geq 0$.

An explicit form of the function $\tau(R, r)$ describing the proper time for a radial free-fall between R and r can be obtained by doing the integral in (19.47):

$$\int_{\frac{r}{R}}^{1} \left[\frac{1}{x} - 1\right]^{-1/2} dx \qquad (19.49)$$

In the solutions for Problem 1 of sheet 2 we explained how to do this using cycloids. Here we will do it directly.

$$\int_{\frac{r}{R}}^{1} \frac{dx}{\left(\frac{1}{x} - 1\right)^{1/2}} = \int_{\frac{r}{R}}^{1} \frac{x\,dx}{\left(-x^2 + x\right)^{1/2}} \qquad (19.50)$$

Set $z = 2x - 1 \curvearrowright x = \frac{1}{2}(z+1)$, $x - 1 = \frac{1}{2}(z-1)$

so that $x(1-x) = \frac{1}{4}(1-z^2)$, $x\,dx = \frac{1}{4}(z+1)\,dz$

$$\int_{\frac{r}{R}}^{1} \frac{dx}{\left(\frac{1}{x}-1\right)^{1/2}} = \frac{1}{2}\int_{\left(2\frac{r}{R}-1\right)}^{1} \frac{(1+z)\,dz}{\left[1-z^2\right]^{1/2}}$$

$$= \frac{1}{2}\left\{ \sin^{-1}(z) - \left(1-z^2\right)^{1/2} \right\}\Big|_{\left(2\frac{r}{R}-1\right)}^{1}$$

$$= \frac{1}{2}\left\{ \frac{\pi}{2} + \sin^{-1}\left(1-2\frac{r}{R}\right) + \underbrace{\left(1-\left(2\frac{r}{R}-1\right)^2\right)^{1/2}}_{2\left(\frac{r}{R}\right)^{1/2}\left(1-\frac{r}{R}\right)^{1/2}} \right\}$$

$$= \frac{1}{2}\left\{ \frac{\pi}{2} + \sin^{-1}\left(1-\frac{2r}{R}\right) + 2\left(\frac{r}{R}\right)^{1/2}\left(1-\frac{r}{R}\right)^{1/2} \right\} \quad (19.51)$$

Hence for (19.47) we get for $\tau(R,r)$:

$$\tau(R,r) = \left(\frac{R^3}{c^2 r_s}\right)^{1/2}\left\{ \frac{\pi}{4} + \frac{1}{2}\sin^{-1}\left(1-\frac{2r}{R}\right) + \left(\frac{r}{R}\right)^{1/2}\left(1-\frac{r}{R}\right)^{1/2} \right\} \quad (19.52)$$

For $r = r_s$ get

$$\tau(R,r_s) = \frac{R}{c}\left(1-\frac{r_s}{R}\right)^{1/2} + \frac{R}{c}\left(\frac{R}{r_s}\right)^{1/2}\left\{ \frac{\pi}{4} + \frac{1}{2}\sin^{-1}\left(1-\frac{2r}{R}\right)\right\} \quad (19.53)$$

and for $r = 0$

$$\tau(R,0) = \frac{R}{c}\left(\frac{R}{r_s}\right)^{1/2}\frac{\pi}{2} \quad (19.54)$$

Note: timelike geodesics are length maximising. Hence any other timelike curve from $r = R$ to $r = 0$ has eigentime <u>less</u> than (19.54) !

Equation (19.54) even applies for $R \leq r_s$. Hence the eigentime duration after passing the horizon $r = r_s$ until hitting the singularity at $r = 0$ is bounded above by

$$\tilde{\tau} \leq \frac{r_s}{c} \frac{\pi}{2} \qquad (19.55)$$

You live longest along the geodesic. That means that any action that lets you deviate from the geodesic, like firing your rocket boosters in an attempt to escape, will shorten your lifetime.

The ellapsed coordinate time $t(R,r)$ for a radial fall from $R > r_s$ to $R > r > r_s$ can also be calculated. From (19.46) have

$$d\tilde{\tau} = -\frac{1}{c}\left(\frac{r_s}{r} - \frac{r_s}{R}\right)^{-1/2} dr \qquad (19.56)$$

and from (19.8) and (19.44)

$$\dot{t} = \frac{\tilde{E}}{c^2}\left(1 - \frac{r_s}{r}\right)^{-1} = \frac{(1 - r_s/R)^{1/2}}{(1 - r_s/r)} \qquad (19.57)$$

Hence

$$dt = \dot{t}\, d\tilde{\tau} = -\frac{(1 - r_s/R)^{1/2}}{c} \cdot \frac{dr}{\left(1 - \frac{r_s}{r}\right)\left(\frac{r_s}{r} - \frac{r_s}{R}\right)^{1/2}} \qquad (19.58)$$

Hence

$$t(R,r) = \frac{1}{c}\left(1-\frac{r_s}{R}\right)^{1/2}$$

$$\times \underbrace{\int_r^R \frac{dr'}{\left(1-\frac{r_s}{r'}\right)\left(\frac{r_s}{r'}-\frac{r_s}{R}\right)^{1/2}}}$$

$$r_s \int_{r/r_s}^{R/r_s} \frac{dx}{\left(1-\frac{1}{x}\right)\left(\frac{1}{x}-\frac{r_s}{R}\right)^{1/2}} \qquad (19.59)$$

where $x := r'/r_s$. For $r' \in [r, R]$ have $x \in [r/r_s, R/r_s]$ and

$$\left(\frac{1}{x}-\frac{r_s}{R}\right)^{1/2} \in \left[\left(\frac{r_s}{r}-\frac{r_s}{R}\right)^{1/2}, 0\right] \qquad (19.60)$$

so that

$$\int_{r/r_s}^{R/r_s} \frac{dx}{\left(1-\frac{1}{x}\right)\left(\frac{1}{x}-\frac{r_s}{R}\right)^{1/2}} \geq \left(\frac{r_s}{r}-\frac{r_s}{R}\right)^{-1/2} \int_{r/r_s}^{R/r_s} \frac{dx}{\left(1-\frac{1}{x}\right)}$$

$$= \left(\frac{r_s}{r}-\frac{r_s}{R}\right)^{-1/2} \Big[(x-1)+\ln(x-1)\Big]_{r/r_s}^{R/r_s}$$

$$= \left(\frac{r_s}{r}-\frac{r_s}{R}\right)^{-1/2} \cdot \frac{R-r}{r_s} \cdot \ln\left(\frac{\frac{R}{r_s}-1}{\frac{r}{r_s}-1}\right) \qquad (19.61)$$

For $r \searrow r_s$ this diverges like $-\ln\left(\frac{r}{r_s}-1\right)$ i.e. logarithmically to $+\infty$.

This means that the radial geodesic reaches $r = r_s$ at no finite value of coordinate time t. On the other hand, a stationary observer at $r > r_s$ has a constant relation between his/her eigentime and coordinate time

$$d\tau_{obs}(r) = [g_{00}(\vec{x})]^{1/2}\, dt$$

$$= \left(1 - \frac{r_s}{r}\right)^{1/2} dt \qquad 19.62$$

Hence a stationary observer will not say that the radial infall through $r = r_s$ of the particle has "happened" at any finite eigentime of his/her.