Lecture 20

Spherically symmetric perfect-fluid stars.
The Tolman-Oppenheimer-Volkov equation.
The interior Schwarzschild solution.
The Buchdahl Limit, binding energies.

In this lecture we seek static, spherically-symmetric solutions to Einstein's equations with $\Lambda = 0$ and an energy-momentum tensor of a perfect fluid. The latter depends on 5 functions:

$$\left.\begin{aligned} \rho &= \text{density of rest-mass} \\ p &= \text{pressure} \\ u &= \text{four velocity} \end{aligned}\right\} \quad (20.1)$$

Note that u satisfies $g(u,u) = c^2$, so only three out of the four components of u are independent. This gives us the $1 + 1 + 3 = 5$ free functions. The energy momentum tensor is (see Problem 1, Sheet 3):

$$\left.\begin{aligned} T^{\alpha\beta} &= \rho\, u^\alpha u^\beta + \left(\frac{u^\alpha u^\beta}{c^2} - g^{\alpha\beta}\right) p \\ &= (\rho + p/c^2)\, u^\alpha u^\beta - g^{\alpha\beta} p \end{aligned}\right\} \quad (20.2)$$

The metric will be assumed of the form

$$g = e^{2a(r)}\, c\,dt \otimes c\,dt - e^{2b(r)}\, dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi) \qquad (20.3)$$

Note that now – in contrast to the derivation of the exterior Schwarzschild solution – we explicitly assume the functions a and b to be independent of t. We cannot expect the gravitational field to be automatically static inside matter, irrespective of spherical symmetry, for matter could radially move inwards in a spherically symmetric way, thereby giving rise to time dependent metric coefficients at fixed r. We are seeking solutions for which this is not the case. This is ~~is~~ clearly only possible if the matter moves along the integral lines of the timelike Killing vector field

$$\left.\begin{aligned} u &= c\,\frac{k}{[g(k,k)]^{1/2}} \\ &= e^{-a}\,\frac{\partial}{\partial t} \end{aligned}\right\} \quad (20.4)$$

We shall evaluate the Einstein equation in the term

$$G_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta = \kappa\, T_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$$

$$\Rightarrow \left(G_{\alpha\beta} - \kappa T_{\alpha\beta}\right) \theta^\alpha \otimes \theta^\beta = 0 \qquad (20.5)$$

i.e. with covariant components referring to the orthonormal co-frame

$$\left.\begin{aligned} \theta^0 &= e^a\, dx^0 \\ \theta^1 &= e^b\, dr \\ \theta^2 &= r\, d\theta \\ \theta^3 &= r \sin(\theta)\, d\varphi \end{aligned}\right\} (20.6)$$

Then

$$\begin{aligned} T &= T_{\alpha\beta}\, \theta^\alpha \otimes \theta^\alpha \\ &= (\rho + p/c^2)\, u^{\downarrow} \otimes u^{\downarrow} - p\, g \\ &= \left((\rho + p/c^2)\, u_\alpha u_\beta - p\, g_{\alpha\beta}\right) \theta^\alpha \otimes \theta^\beta \end{aligned}$$

$$\left.\begin{aligned} &= \rho c^2\, \theta^0 \otimes \theta^0 \\ &\quad + p\left(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3\right) \end{aligned}\right\} (20.7)$$

and, from Lecture 17, equations (17.31a–g) for $\dot{b} = 0$:

$$G = G_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta \qquad (20.8)$$

with

$$G_{00} = \left(\frac{2b'}{r} - \frac{1}{r^2}\right) e^{-2b} + \frac{1}{r^2}$$
$$= \frac{1}{r^2}\left[1 - (r e^{-2b})'\right] \qquad (20.9a)$$

$$G_{11} = \left(\frac{2a'}{r} + \frac{1}{r^2}\right) e^{-2b} - \frac{1}{r^2} \qquad (20.9b)$$

$$G_{22} = e^{-2b}\left(a'' + a'^2 - a'b' + \frac{a' - b'}{r}\right)$$
$$G_{33} = G_{22} \qquad (20.9c)$$

and all other components vanish.

It will turn out useful to also use the integrability condition

$$\nabla_\alpha T^{\alpha\beta} = 0 \iff \mathrm{Trace}_{(12)}(\nabla T) = 0 \qquad (20.10)$$

where (using $g^{-1} = g^{\alpha\beta} e_\alpha \otimes e_\beta$)

$$\left.\begin{aligned} T &= T^{\alpha\beta} e_\alpha \otimes e_\beta \\ &= (\rho c^2 + p)\, e_0 \otimes e_0 - p\, g^{-1} \end{aligned}\right\} \qquad (20.11)$$

Hence using

$$\nabla g^{-1} = 0 \qquad (20.12)$$

and

$$\nabla e_\lambda = \omega^\mu{}_\lambda \otimes e_\mu \qquad (20.13)$$

where $\omega^\mu{}_\lambda$ are the connection 1-forms, we get

$$\left.\begin{aligned} \nabla T = {} & d(\varrho c^2 + p) \otimes e_0 \otimes e_0 - dp \otimes g^{-1} \\ & + (\varrho c^2 + p)\, \omega^\mu{}_0 \otimes e_\mu \otimes e_0 \\ & + (p c^2 + p)\, \omega^\mu{}_0 \otimes e_0 \otimes e_\mu \, . \end{aligned}\right\} (20.14)$$

Taking the trace over the first and second tensor factor means to apply the 1-form in the first tensor factor to the vector in the second.

$$\left.\begin{aligned} \mathrm{Trace}_{(12)}(\nabla T) = {} & e_0(\varrho c^2 + p)\, e_0 - (dp)^\uparrow \\ & + (\varrho c^2 + p)\left(\omega^\mu{}_0(e_\mu)\, e_0 + \omega^\mu{}_0(e_0)\, e_\mu \right) \end{aligned}\right\} (20.15)$$

The connection 1-forms $\omega^\alpha{}_\beta$ were calculated in Lecture 17, and listed in formulae (17.20 - 25). We immediately see that $\omega^\alpha{}_\beta(e_0)$ is non zero if and only if $\alpha = 0$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 0$, with value $a' e^{-b}$,

$$\left.\begin{aligned} \omega^{\mu}{}_{0}(e_0) &= 0 \quad \text{for } \mu \neq 1 \\ \omega^{1}{}_{0}(e_0) &= e^{-b} a' = e_1(a) \end{aligned}\right\} \quad (20.16)$$

since $e_1 = e^{-b} \partial r$ (20.17)

We also see that

$$\omega^{\mu}{}_{0}(e_\mu) = \omega^{1}{}_{0}(e_1) = e^{-a} \dot{b} = e_0(b) \quad (20.18)$$

Hence

$\mathrm{Trace}_{(12)}(\nabla T)$

$$\left.\begin{aligned} &= e_0 (\rho c^2 + p) e_0 - e_\mu(p) ((\theta^\mu))^\uparrow \\ &\quad + (\rho c^2 + p) (e_0(b) e_0 + e_1(a) e_1) \end{aligned}\right\} \quad (20.19)$$

Using $(\theta^\mu)^\uparrow = \eta^{\mu\lambda} e_\lambda$, this becomes

$$\left.\begin{aligned} &= e_0 [c^2 e_0(\rho) + (\rho c^2 + p) e_0(b)] \\ &\quad + e_1 [e_1(p) + (\rho c^2 + p) e_1(a)] \\ &\quad + e_2 e_2(p) + e_3 e_3(p) \end{aligned}\right\} \quad (20.20)$$

If this is to vanish we must have

$$e_2(p) = e_3(p) = 0 \quad (20.21a)$$

and

$$c^2\, e_0(\varrho) + (\varrho c^2 + p)\, e_0(b) = 0$$

$$\Leftrightarrow \quad c^2\, \dot{\varrho} + (\varrho c^2 + p)\, \dot{b} = 0 \qquad (20.21b)$$

with $\dot{} = \frac{\partial}{\partial t}$, since $e_0 = e^{-a}\, \partial/\partial t$.

And also

$$e_1(p) + (\varrho c^2 + p)\, e_1(a) = 0$$

$$\Leftrightarrow \quad p' + (p + c^2 \varrho)\, a' = 0 \qquad (20.21c)$$

Now, in our case $\dot{b} = 0 \Rightarrow \dot{\varrho} = 0$ and spherical symmetry is also taken to apply to ϱ and p, i.e. $e_2(\varrho) = e_3(\varrho) = e_2(p) = e_3(p) = 0$. In this case the integrability condition is equivalent to

$$\nabla_\alpha T^{\alpha\beta} = 0 \quad \Leftrightarrow \quad \mathrm{Trace}_{(12)}(\nabla T) = 0$$

$$\left.\begin{aligned} &\Leftrightarrow \quad p' + a'(\varrho c^2 + p) = 0 \\ &\Leftrightarrow \quad a' = -p'/(\varrho c^2 + p) \end{aligned}\right\} (20.22)$$

It is this equation, (20.22), that we shall use together with Einstein's equations.

Einstein's equations

$$G_{\alpha\beta} = \kappa T_{\alpha\beta}$$

Left hand side : (20.9)
Right hand side : (20.7)

Hence, listing all equations (plus (20.22)), we get:

$$\left(\frac{2b'}{r} - \frac{1}{r^2}\right) e^{-2b} + \frac{1}{r^2} = \frac{1}{r^2}\left[r\left(1-e^{-2b}\right)\right]' = \kappa \rho c^2 \qquad (20.23a)$$

$$\left(\frac{2a'}{r} + \frac{1}{r^2}\right) e^{-2b} - \frac{1}{r^2} = \kappa p \qquad (20.23b)$$

$$\left(a'' + a'^2 - a'b' + \frac{a'-b'}{r}\right) e^{-2b} = \kappa p \qquad (20.23c)$$

$$a' = -p' / (\rho c^2 + p) \qquad (20.23d)$$

These four equations are all information we get from Einsteins equations

The first observation is that since $b = b(r)$, and $\rho = \rho(r)$, (20.23a) is an ordinary differential equation that we can just integrate :

$$\left[r\left(1-e^{-2b}\right)\right]' = \kappa r^2 \rho c^2$$

$$\rightsquigarrow \quad r\left(1-e^{-2b}\right) = \kappa c^2 \int_0^r dr'\, r'^2 \rho(r') \qquad (20.24)$$

Define

$$M(r) := 4\pi \int_0^r dr'\, r'^2 \rho(r') \qquad (20.25)$$

then the (00)-equation is equivalent to

$$e^{-2b(r)} = 1 - \frac{2\,G\,M(r)}{c^2 r} \qquad (20.26)$$

Having solved (20.23a) we may solve (20.23a + 23b) instead of (20.23b):

$$G_{00} + G_{11} = \frac{2}{r}(a+b)'\, e^{-2b}$$

$$= \kappa\left(\rho c^2 + p\right) \qquad (20.27)$$

$$\Rightarrow \quad (a+b)' = \frac{r}{2}\kappa\, e^{2b}\left(\rho c^2 + p\right)$$

$$= \frac{4\pi G}{c^2} e^{2b}\, r\left(\rho + p/c^2\right) \qquad (20.28)$$

Integration with boundary conditions

$$a(r\to\infty) = b(r\to\infty) = 0 \qquad (20.29)$$

(asymptotic flatness)

gives

$$a(r) + b(r) = \frac{4\pi G}{c^2} \int_{\infty}^{r} dr' \, e^{2b(r')} \, r' \left(\rho(r') + p(r')/c^2 \right) \tag{20.30}$$

If $\rho(r)$ is known then $M(r)$ follows from (20.25) and $b(r)$ from (20.26).
If $p(r)$ is known, e.g. through some equation of state $p = p(\rho)$, then $a(r)$ can be calculated through (20.30).
This determines the metric completely.
This will be the way we proceed in the idealising case of incompressible matter, where $\rho = \rho_0 = $ const. inside the star.

In order to get an equation for $\rho(r)$ alone, once we have an equation of state, analogous to

$$-p'(r) = G \, \frac{M(r)\,\rho(r)}{r^2} \tag{20.31}$$

in the Newtonian case, we have to eliminate $a(r)$ from (20.30). For this we use (20.23d). We proceed as follows:
From (20.28) have

$$a'(r) = -b'(r) + \frac{4\pi G}{c^2} \, e^{2b(r)} \, r \left(\rho(r) + p(r)/c^2 \right) \tag{20.32}$$

From (20.26) we get by differentiation

$$-2b' e^{-2b} = -2\frac{G}{c^2}\left(\frac{M}{r}\right)'$$

$$b' = e^{2b}\frac{G}{c^2}\left(\frac{M}{r}\right)' \qquad (20.33)$$

Insert this into (20.32)

$$a' = e^{2b}\frac{G}{c^2}\left[4\pi r(\rho + p/c^2) - \left(\frac{M}{r}\right)'\right]$$

and since by (20.25)

$$M'(r) = 4\pi r^2 \rho(r) \qquad (20.34)$$

we get

$$a'(r) = e^{2b(r)}\frac{G}{c^2}\left[4\pi r\frac{p(r)}{c^2} + \frac{M(r)}{r^2}\right] \qquad (20.35)$$

Here the right-hand side is only a function of ρ and p (note that ρ appears in integrated form in $M(r)$.

Finally, we can eliminate $a'(r)$ on the left-hand side through (20.23d) to obtain an equation involving only ρ and p:

$$\boxed{-p'(r) = G\,\frac{M(r)+4\pi r^3 p(r)/c^2}{r^2\left[1-\frac{2GM(r)}{c^2 r}\right]}\left(\rho(r)+p(r)/c^2\right)} \quad (20.36)$$

TOV: Tolman - Oppenheimer - Volkov
1939

This equation replaces the Newtonian equation (20.31), though direct comparison should be done with care: In (20.31) and (20.36) the r - coordinate is the area-radius; in that sense they are comparable. But only in (20.31) is r also the metric radial distance; that is <u>not</u> true for (20.36). If we wish to interpret $-p'(r)$ as the increase in pressure per unit inward radial distance, we have to replace the $p' = \partial p/\partial r$ - derivative by the derivative with respect to the unit radial tangent vector

$$e_r = e^{-b}\,\frac{\partial}{\partial r} = \left[1-\frac{2GM}{c^2 r}\right]^{1/2}\frac{\partial}{\partial r} \qquad (20.37)$$

then:

$$\boxed{-e_r(p) = G\,\frac{M(r)+4\pi r^3 p(r)/c^2}{r^2\left[1-\frac{2GM(r)}{c^2 r}\right]^{1/2}}\left(\rho(r)+p(r)/c^2\right)} \quad (20.38)$$

The TOV-equation (20.36) – or (20.38) – replaces the Newtonian equation (20.31). Relative to the latter, the TOV-equation contains three characteristic corrections which are easily identifiable by the factor $1/c^2$ that each one of them carries:

1.) $M(r) \rightarrow M(r) + 4\pi r^3 p(r)/c^2$ (20.39)

pressure enhances active gravitational mass

2) $\rho(r) \rightarrow \rho(r) + p(r)/c^2$ (20.40)

pressure enhances passive gravitational mass

3.) $$\frac{1}{r^2} \rightarrow \frac{1}{r^2}\left[1 - \frac{2GM(r)}{c^2 r}\right]^{-n}$$ (20.41)

When $n=1$ or $n=1/2$, depending on whether we understand r as area-radius or radial proper distance, respectively. In both cases this term leads to an increase of the negative pressure gradient due to the curvature of space. [Compare (17.28 f, h, j) to see how e^{2b} produces spatial curvature.]

Important observation: All $\frac{1}{c^2}$ corrections of (20.36), or (20.38), relative to (20.31) point in the same direction, i.e. lead to an enhancement of the negative p-gradient $-p'(r)$. This means that GR corrections require the pressure inside stars to grow faster as one moves inward than predicted by Newtonian theory. This may eventually lead to instabilities and stellar collapse, because higher pressure leads to higher active and passive gravitational mass and then, in turn, to higher weight, which can only be stabilised by yet higher pressures. This may, eventually, lead to diverging pressures. This is in contrast to Newtonian theory, where any star can be stabilized by finite pressures, given that the equation of state allow the required pressure to be build up. In GR we face the possibility that stars become unstable, quite independent of their equation of state (which is only limited by energy-conditions).

Once we specify an equation of state,

$$p = p(\rho) \qquad (20.42)$$

We can determine $\rho(r)$ and $p(r)$ from the TOV equation (20.36), then $b(r)$ from (20.26) and finally $a(r)$ from (20.30).

Standard equations of state are polytropes

$$p = K \rho^{\frac{n+1}{n}} \qquad (20.43)$$

where n = polytropic index, e.g.

$$n = \begin{cases} 3: \text{ for main sequence stars (Sun)} \\ 1.5: \text{ white/brown dwarfs} \\ \quad\;\; \text{cores of red giants} \\ 0.5 - 1.0: \text{ neutron star} \\ 0: \text{ "rocky stars"} \end{cases} \qquad (20.44)$$

The limit $n \searrow 0$ corresponds to incompressible stars, where $\rho = \rho_0 =$ const. but varying $p = p(r)$. This is strictly speaking an unphysical idealisation since incompressibility implies an infinite speed of sound,

$$v_s = \left(\frac{dp}{d\rho}\right)^{1/2} = \left[K \, \frac{n+1}{n} \, \rho^{1/n}\right]^{1/2} \xrightarrow[n \to 0]{} \infty \qquad (20.45)$$

But exact analytical solutions can be obtained in that case, as we show next.

Before we do that we remark that we have not yet involved equation (20.23c), i.e. the (22) (and (33)) component of Einsteins equation. As in the vacuum case, it can be shown that (20.23c) is implied by (20.23a, b, d). We leave this as an exercise for the next sheet.

Incompressible stars: The interior Schwarzschild solution

$$\rho(r) = \begin{cases} \rho_0 > 0 & \text{for } r \le R \\ 0 & \text{for } r > R \end{cases} \tag{20.46}$$

$$\Rightarrow M(r) = \begin{cases} \frac{4\pi}{3}\rho_0 r^3 = M\frac{r^3}{R^3} & \text{for } r \le R \\ \frac{4\pi}{3}\rho_0 R^3 =: M & \text{for } r > R \end{cases} \tag{20.47}$$

Inserting this into the TOV-equation (20.36), we get

$$\frac{p'(r)}{(p(r)+\rho_0 c^2)(p(r)+\rho_0 c^2/3)} = -\frac{4\pi G r}{c^4}\left[1-\frac{8\pi G\rho_0 r^2}{3c^2}\right]^{-1}$$

(20.48)

Instead of r we use the variable

$$x := \left[\frac{8\pi G \rho_0}{3c^2}\right]^{1/2} r \qquad (20.49)$$

then

$$\frac{dp}{(p+\rho_0 c^2)(p+\rho_0 c^2/3)} = -\frac{3}{2\rho_0 c^2}\,\frac{x\,dx}{1-x^2} \qquad (20.50)$$

But

$$\frac{1}{(p_0+\rho_0 c^2)(p_0+\rho_0 c^2/3)}$$

$$= \frac{3}{2\rho_0 c^2}\left[\frac{-1}{p+\rho_0 c^2} + \frac{1}{p+\rho_0 c^2/3}\right] \qquad (20.51)$$

hence

$$\frac{dp}{p+\rho_0 c^2} - \frac{dp}{p+\rho_0 c^2/3} = \frac{x\,dx}{1-x^2} \qquad (20.52)$$

Integrating that from $r' = r$ to $r' = R$, where $r' = R$ is defined to be the star's radius, which in turn is given by $p = 0$, we get

$$\ln\left(\frac{\rho_0 c^2}{p(r)+\rho_0 c^2}\right) - \ln\left(\frac{\rho_0 c^2/3}{p(r)+\rho_0 c^2/3}\right)$$

$$= -\frac{1}{2}\ln(1-x^2)\Big|_{r}^{r=R}$$

$$= \ln\left[\frac{1-x^2(r)}{1-x^2(R)}\right]^{1/2} \qquad (20.53)$$

hence

$$\frac{3p(r)+\rho_0 c^2}{p(r)+\rho_0 c^2} = \left[\frac{1-8\pi G\rho_0 r^2/3c^2}{1-8\pi G\rho_0 R^2/3c^2}\right]^{1/2}$$

$$= \left[\frac{1-r^2 r_s/R^3}{1-r_s/R}\right]^{1/2} \qquad (20.54)$$

where $r_s = \frac{2GM}{c^2} = \frac{2G}{c^2}\frac{4\pi}{3}\rho_0 R^3$

$$= \frac{8\pi G\rho_0}{3c^2}R^3 \qquad (20.55)$$

This can be solved for $p(r)$

$3p + \rho_0 c^2 = [\cdot\cdot]^{1/2}(p+\rho_0 c^2)$

$\rightsquigarrow (3-[\cdot\cdot]^{1/2})\,p = \rho_0 c^2([\cdot\cdot]^{1/2}-1)$

$\rightsquigarrow \; p = \rho_0 c^2([\cdot\cdot]^{1/2}-1)/(3-[\cdot\cdot]^{1/2})$

(20.56)

This gives us the pressure profile $p(r)$:

$$\boxed{p(r) = \rho_0 c^2 \frac{\left(1-\frac{r^2 r_s}{R^3}\right)^{1/2} - \left(1-\frac{r_s}{R}\right)^{1/2}}{3\cdot\left(1-\frac{r_s}{R}\right)^{1/2} - \left(1-\frac{r^2 r_s}{R^3}\right)^{1/2}}} \quad (20.57)$$

This is the GR-generalization of the Newtonian expression, that follows from

$$-p' = \frac{GM(r)}{r^2}\rho(r) \quad (20.58)$$

and $\rho = \rho_0$, hence

$$M(r) = \frac{4\pi}{3}\rho_0 r^3 \quad (20.59)$$

and

$$-p' = \frac{1}{3} 4\pi G \rho_0^2 r \quad (20.60)$$

or

$$p(r) = \frac{2\pi G}{3}\rho_0^2 (R^2 - r^2)$$

$$= \frac{MG}{2R}\rho_0\left[1-\left(\frac{r}{R}\right)^2\right] \quad (20.61)$$

Indeed, the $1/c \to 0$ limit of (20.57) is just that:

$$p(r) = \rho_0 c^2 \frac{\frac{1}{2}\left(\frac{r_s}{R} - \frac{r^2 r_s}{R^3}\right)}{2}$$

$$= \frac{1}{4}\rho_0 c^2 \frac{r_s}{R}\left[1-\left(\frac{r}{R}\right)^2\right] = \frac{MG}{2R}\rho_0\left[1-\left(\frac{r}{R}\right)^2\right].$$

Both functions (20.61) and (20.57) for $p(r)$ are monotonically increasing inwards, from zero at $r = R$ to the central pressure $p(0)$:

Newtonian central pressure

$$p(0) = \frac{MG}{2R} \rho_0 \qquad (20.62)$$

GR - central pressure

$$p(0) = \rho_0 c^2 \frac{1 - \left(1 - r_s/R\right)^{1/2}}{3\left(1 - \frac{r_s}{R}\right)^{1/2} - 1} \qquad (20.62)$$

The GR pressure profile is always above the Newtonian pressure profile between $r = R$ and $r = 0$. But the crucial difference is that the GR - central pressure diverges for

$$3\left(1 - \frac{r_s}{R}\right)^{1/2} = 1$$

$$\Leftrightarrow \quad \frac{r_s}{R} = \frac{2GM}{c^2 R} = \frac{8}{9} \qquad (20.63)$$

$$\boxed{p(0) \nearrow \infty \text{ for } R \searrow \frac{9}{8} r_s} \qquad (20.64)$$

Buchdahl - Limit

This means that according to GR stars cannot be stabilized for radii smaller than 9/8 the Schwarzschild radius, at least not incompressible, spherically-symmetric ones. Buchdahl proved in 1959 that this result is more general, in fact it holds for all stars with equations of state that imply $\rho'(r) < 0$ within the star.

In fact, the Buchdahl-Limit can be strengthened by imposing the condition of energy dominance

$$p \leq \rho_0 c^2 \tag{20.65}$$

(compare Problem 5 of Sheet 3, in particular equation 7c). This leads via (20.57) to

$$1 - \left(1 - \frac{r_s}{R}\right)^{1/2} \leq 3\left(1 - \frac{r_s}{R}\right)^{1/2} - 1$$

or $$\left(1 - \frac{r_s}{R}\right)^{1/2} \geq \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{r_s}{R} \leq \frac{3}{4} \tag{20.66}$$

or $$R \geq \frac{4}{3} r_s \tag{20.67}$$

Having determined $p(r)$, we also need to determine the metric, i.e. functions $e^{2a(r)}$ and $e^{2b(r)}$. The latter follows immediately from (20.26) and (20.47)

$$e^{2b(r)} = \begin{cases} \left(1 - \frac{r_s r^2}{R^3}\right)^{-1} & \text{for } r \leq R \\ \left(1 - \frac{r_s}{r}\right)^{-1} & \text{for } r > R \end{cases} \tag{20.68}$$

Hence the Riemannian metric of the slice $t = \text{const.}$, e.g. $t = 0$, is given by the metric inside the star ($r \leq R$):

$$h = -g\big|_{t=0}$$

$$= \frac{dr \otimes dr}{\left(1 - r^2 \frac{r_s}{R^3}\right)} \tag{20.69}$$

$$+ r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi \right)$$

This is a metric of constant (sectional) curvature. In fact, it is locally isometric to the metric of a (round) 3-sphere of radius K,

$$S_K := \{ X \in \mathbb{R}^4 : \|X\| = K \} \tag{20.70}$$

where $K = \left(R^3 / r_s\right)^{1/2}$

Indeed, a metric of the form

$$\frac{dr^2}{1-k^2r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{20.71}$$

transforms via

$$k r = \sin\psi$$

into

$$\frac{1}{k^2}\left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right] \tag{20.72}$$

which is the metric on a 3-sphere of radius $K = 1/k$. Applied to our case $k^2 = r_S/R^3$, so that (20.70) follows.

But as the range of r is limited by R, which in turn is limited by the Buchdahl limit, we have

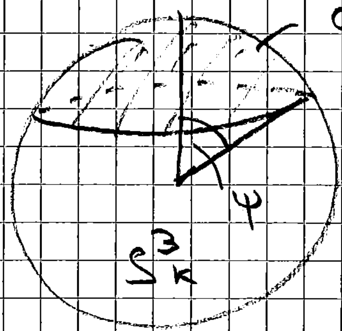
$$\sin\psi = r\cdot\left(\frac{r_S}{R^3}\right)^{1/2} \leq R\left(\frac{r_S}{R^3}\right)^{1/2}$$

$$\leq \left(\frac{r_S}{R}\right)^{1/2} \leq \left(\frac{8}{9}\right)^{1/2} = \sqrt{2}\,\frac{2}{3}$$

Hence

$$\psi \leq \sin^{-1}\left(\sqrt{2}\,\frac{2}{3}\right) = 70.53^\circ \tag{20.73}$$

$\Rightarrow$ The interior of a star makes up less than a 70.53° - cap of a 3-sphere.



(20.74)

Sphere of radius $K = R\left(\frac{R}{r_s}\right)^{1/2}$.

Finally, we calculate $a(r)$. From (20.30)

$$a(r) = -b(r) + \frac{4\pi G}{c^2}\int_\infty^r dr'\, e^{2b(r')}\, r'\left(\rho(r') + p(r')/c^2\right)$$

We get for $r \geq R$, where $p(r) = \rho(r) = 0$ that $a(r) = -b(r)$ and hence

$$e^{2a(r)} = e^{-2b(r)} = \left(1 - \frac{r_s}{r}\right) \quad \text{for } r \geq R$$

For $r \leq R$ we have the integral

$$\int_\infty^r dr'\, r' \;\frac{\rho_0 + p(r')/c^2}{1 - \frac{r'^2 r_s}{R^3}} \qquad (20.75)$$

Now, from the expression (20.57) for $p(r)$ we infer

$$\rho_0 + p(r)/c^2 = 2\rho_0\, \frac{\left(1 - \frac{r_s}{R}\right)^{1/2}}{3\left(1 - \frac{r_s}{R}\right)^{1/2} - \left(1 - r^2\frac{r_s}{R^3}\right)^{1/2}}$$

(20.76)

Hence

$$\frac{4\pi G}{c^2}\int_\infty^r \cdots = \frac{8\pi G}{c^2}\,\varrho_0\left(1-\frac{r_s}{R}\right)^{1/2}\times$$

$$\int_R^r dr'\, r'\left(1-\frac{r'^2 r_s}{R^3}\right)^{-1}\left[3\left(1-\frac{r_s}{R}\right)^{\frac{1}{2}}-\left(1-r'^2\frac{r_s}{R}\right)^{1/2}\right]^{-1} \quad (20.77)$$

First we set

$$y := 1-\frac{r_s r^2}{R^3} \quad (20.78)$$

$$\Rightarrow y(r=R) =: y_R = 1-\frac{r_s}{R}$$

and $dy = -2\frac{r_s}{R^3}\, r\, dr$

Then $\frac{4\pi G}{c^2}\int_\infty^r \cdots =$

$$-\frac{R^3}{2r_s}\cdot\frac{8\pi G}{c^2}\,\varrho_0\, y_R^{1/2}\int_{y_R}^{y_r}\frac{dy}{y\left[3y_R^{1/2}-y^{1/2}\right]} \quad (20.79)$$

Now set $z := y/(9y_R)$; then (20.80)

$$= -\underbrace{\frac{1}{r_s}\cdot\frac{4\pi}{3}R^3\varrho_0\cdot\frac{G}{c^2}}_{1/2}\;\underbrace{\int_{1/9}^{z_r}\frac{dz}{z(1-\sqrt{z})}}_{\ln(z)-2\ln(1-\sqrt{z})} \quad (20.81)$$

$$= \ln\left(\frac{z}{(1-\sqrt{z})^2}\right)\Bigg|_{\frac{1}{9}}^{z_r}$$

$$a(r) = -b(r) + \frac{4\pi G}{c^2}\int_\infty^r \dots$$

$$= -\left\{ \ln y_r^{-1/2} + \ln\left[\frac{2\, y_r^{1/2}}{(3 y_R^{1/2} - y_r^{1/2})}\right]\right\}$$

$$= -\ln\left(2 / (3 y^{1/2}_R - y^{1/2}_r)\right)$$

$$= \ln\left(\frac{1}{2}\left(3 y_R^{1/2} - y^{1/2}_r\right)\right)$$

$$= \ln \frac{1}{2}\left[3\left(1 - \frac{r_s}{R}\right)^{1/2} - \left(1 - r^2 \frac{r_s}{R^3}\right)\right] \qquad (20.82)$$

Finally:

$$e^{2a(r)} = \begin{cases} \frac{1}{4}\left[3\left(1 - \frac{r_s}{R}\right)^{1/2} - \left(1 - r^2\frac{r_s}{R^3}\right)^{1/2}\right]^2, & r \le R \\ \left(1 - \frac{r_s}{r}\right), & r > R \end{cases}$$

(20.83)

and all metric coefficients are determined.

Note that $e^{2a(r)} = g_{00}(r) \to 1$ for $r \to \infty$. $g_{00}(r)$ is smallest for $r = R$, i.e. on the surface of the star, and that is smallest for the smallest possible R if r_s is given, i.e. the mass of the star is given. But R has the Buchdahl lower bound. This means that g_{00} is bounded below by

$$e^{2a(r)}\Big|_{\substack{r=R \\ R=\frac{9}{8}r_s}} = 1 - \frac{r_s}{\frac{9}{8}r_s}$$

$$= 1 - \frac{8}{9} = \frac{1}{9} \qquad (20.84)$$

We recall the redshift formula from Lecture 16, (16.24)

$$z = \frac{\omega_e - \omega_r}{\omega_r} = \left[\frac{g_{00}(x_r)}{g_{00}(x_e)}\right]^{1/2} - 1 \qquad (20.85)$$

Here the point of emission is the star's surface, the point of reception an observer "at infinity". Hence $g_{00}(\vec{x}_r) = 1$, and $g_{00}(x_e)$ is smallest for the value 1/9 calculated above. Hence

$$z \le z_{max} = 3 - 1 = 2 \qquad (20.86)$$

⇒ Light from the surface of a spherical star obeying the Buchdahl lower bound for its radius cannot show a redshift larger than two. For the limit (20.67) we obtained from energy dominance we get an even lower upper limit for redshift:

Energy-dominance lower limit on g_{00}

$$e^{2a(r)}\Big|_{\substack{r=R\\ R=\frac{4}{3}r_s}} = 1 - \frac{r_s}{\frac{4}{3}r_s}$$

$$= 1 - \frac{3}{4} = \frac{1}{4}$$

$$\rightsquigarrow \quad z_{max} = \left(\frac{1}{\frac{1}{4}}\right)^{1/2} - 1 = 1. \qquad (20.87)$$

$\Rightarrow$ Light from the surface of a spherical star obeying the lower bound on its radius from energy dominance cannot show a redshift larger than one.

Binding energy

For the homogeneous (incompressible) spherically symmetric star the total mass was given by

$$M := \frac{4\pi}{3} R^3 \rho_0 \qquad (20.88)$$

This is the mass that appears in the exterior Schwarzschild metric

$$g = \left(1 - \frac{r_s}{r}\right) c\,dt \otimes c\,dt - \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi\right) \qquad (20.89)$$

$$r_s = \frac{2GM}{c^2} \qquad (20.90)$$

to which our interior metric continuously connects at $r = R$

$$g = \frac{1}{4}\left[3\left(1 - \frac{r_s}{r}\right)^{1/2} - \left(1 - r^2 \frac{r_s}{R^3}\right)^{1/2}\right]^2 c\,dt \otimes c\,dt - \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1} dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi\right) \qquad (20.91)$$

where $M(r) = 4\pi \int_0^r dr'\, r'^2 \rho(r')$ (20.92)

The parameter M that enters (20.89) through (20.90) was shown in Lecture 18, equation (18.49) and (18.51), to correspond to the active gravitational mass, in the sense that it is the parameter that appears in the $1/r^2$ - law for the acceleration of a stationary body held at constant r (leading-order term in m/r-expansion).

In contrast, the rest mass of matter put into the star is

$$\tilde{M} = \int_{B_R} \rho_0 \underbrace{\theta^1 \wedge \theta^2 \wedge \theta^3}_{dV} \qquad (20.93)$$

where B_R is the interior of $\{(r,\theta,\varphi) : r \leq R\}$ and dV is the spatial volume element on a slice $t = \text{const.}$

$$dV = \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1/2} r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi \qquad (20.94)$$

Hence

$$\tilde{M} = \rho_0 4\pi \int_0^R dr' \, r'^2 \left(1 - \frac{2GM(r')}{c^2 r'}\right)^{-1/2}$$

$$> \rho_0 4\pi \int_0^R dr' \, r'^2 = \frac{4\pi}{3} R^3 \rho_0 = M \qquad (20.95)$$

Note that the expression for the gravitational mass looks deceptively like the volume integral of ϱ, but it is not because the geometric volume element contains a factor

$$e^{b(r)} = \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1/2} > 1 \qquad (20.96)$$

in addition to $4\pi r^2 dr$, making the geometric volume bigger than the $4\pi R^3/3$ appearing in (20.25).

The binding energy is defined to be the mass-defect $\Delta M := M - \tilde{M}$ times c^2:

$$\Delta E_{Bind} := (M - \tilde{M})c^2$$

$$= 4\pi c^2 \int_0^R dr' r'^2 \varrho(r') \left[1 - \left(1 - \frac{2GM(r')}{c^2 r'}\right)^{-1/2}\right] \qquad (20.97)$$

This formula is valid for all spherically symmetric, static stars, not just the incompressible ones. To leading order in a $(1/c)$-expansion we get

$$\left.\begin{aligned} \Delta E_{Bind} &= -4\pi \int_0^R dr' r'^2 \varrho(r') \frac{GM(r')}{r'} \\ &= -\frac{1}{2} \int d^3x' d^3x' \frac{\varrho(\vec{x})\,\varrho(\vec{x}')}{\|\vec{x} - \vec{x}'\|} \end{aligned}\right\} \qquad (20.98)$$

which is precisely the Newtonian binding energy.

In the incompressible case we have

$$M(r) = M \frac{r^3}{R^3} \qquad (20.99)$$

hence

$$\Delta E_{Bind} = -4\pi \rho_0 \int_0^R dr' r'^2 \frac{G}{r'} M \frac{r'^3}{R^3}$$

$$= -4\pi \rho_0 \frac{G}{R^3} M \frac{1}{5} R^5$$

$$= -\frac{3}{5} \frac{GM^2}{R} \qquad (20.100)$$

which is just the Newtonian gravitational binding energy of a homogeneous ball of total mass M and radius R.

Stars are sometimes termed "relativistic" if the magnitude of the binding energy is of the order of magnitude of Mc^2.

Note that we can also calculate (20.97) exactly for the homogeneous case (20.99):

$$\Delta E_{Bind} = 4\pi c^2 \rho_0 \int_0^R dr\, r^2 \left[1 - \left(1 - r^2 \frac{r_s}{R^3}\right)^{-1/2} \right]$$

$$= 4\pi c^2 \rho_0 \left[\frac{R^3}{3} - \int_0^R dr\, r^2 \left(1 - r^2 \frac{r_s}{R^3}\right)^{-1/2} \right]$$

(20.101)

The second integral is

$$\int_0^R dr\, r^2 \left(1 - \frac{r^2 r_s}{R^3}\right)^{-1/2} = \left(\frac{R^3}{r_s}\right)^{3/2} \int_0^{\left(\frac{r_s}{R}\right)^{1/2}} \frac{y^2\, dy}{(1-y^2)^{1/2}}$$

where $y = r \cdot \left(\frac{r_s}{R}\right)^{1/2}$

But $$\int \frac{dy\, y^2}{(1-y^2)^{1/2}} = \frac{1}{2}\left(\sin^{-1}(y) - y(1-y^2)^{1/2}\right)$$

(20.102)

Hence

$$\Delta E_{Bind} = 4\pi c^2 \rho \left\{ \frac{R^3}{3} - \frac{1}{2}\left(\frac{R^3}{r_s}\right)^{3/2} \left[\sin^{-1}\left(\frac{r_s}{R}\right)^{1/2} - \left(\frac{r_s}{R}\right)^{1/2}\left(1 - \frac{r_s}{R}\right)^{1/2} \right] \right\}$$

$$= Mc^2 \left\{ 1 - \frac{3}{2}\left(\frac{R}{r_s}\right)^{3/2} \left[\sin^{-1}\left(\frac{r_s}{R}\right)^{1/2} - \left(\frac{r_s}{R}\right)^{1/2}\left(1 - \frac{r_s}{R}\right)^{1/2} \right] \right\}$$

(20.103)

(exact formula for binding energy)

Alternatively, also expanding in $\left(\frac{r_s}{R}\right)$,

$$\boxed{\begin{aligned} \frac{-\Delta E_{Bind}}{Mc^2} &= \frac{3}{2}\left(\frac{R}{r_s}\right)^{3/2}\left[\sin^{-1}\left(\frac{r_s}{R}\right)^{1/2} - \left(\frac{r_s}{R}\right)^{1/2}\left(1-\frac{r_s}{R}\right)^{1/2}\right] - 1 \\ &= \frac{3}{10}\,\frac{r_s}{R} + \frac{9}{56}\left(\frac{r_s}{R}\right)^2 + \frac{5}{48}\left(\frac{r_s}{R}\right)^3 + \dots \end{aligned}} \quad (20.104)$$

(Binding energy: Exact and expansion)

In leading order we get again

$$\Delta E_{Bind} \overset{(1)}{=} -Mc^2\,\frac{3}{10}\,\frac{r_s}{R}$$

$$= -\frac{3}{5}\,GM^2/R \qquad (20.105)$$

For the Buchdahl Limit $R = \frac{9}{8} r_s$ we get from the exact expression

$$-\frac{\Delta E_{Bind}}{Mc^2} = \frac{3}{2}\left(\frac{3}{2\cdot\sqrt{2}}\right)^3\left(\sin^{-1}\left(\frac{2\cdot\sqrt{2}}{3}\right) - \frac{2\cdot\sqrt{2}}{3}\,\frac{1}{3}\right)$$

$$= \underline{0.64} \qquad (20.106)$$

and for the limit $R = \frac{4}{3} r_s$ set by energy dom.:

$$-\frac{\Delta E_{Bind}}{Mc^2} = \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right)^3\left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\,\frac{1}{2}\right]$$

$$= \underline{0.42} \qquad (20.107)$$

Had we just used the leading-order term we would have obtained

$$\left(\frac{-\Delta E_{Bind}}{Mc^2}\right)_{Buchdahl} = 0.64 \overset{(1)}{=} \frac{3}{10}\,\frac{8}{9} = \frac{4}{3\cdot 5} = 0.2\bar{6} \qquad (20.107)$$

$$\left(\frac{-\Delta E_{Bind}}{Mc^2}\right)_{\substack{Energy\\dom.}} = 0.42 \overset{(1)}{=} \frac{3}{10}\,\frac{3}{4} = \frac{9}{40} = 0.225 \qquad (20.108)$$

That means that a Newtonian calculation of the binding energy would have underestimated the exact value considerably. In the Buchdahl case the exact value is well over twice as much, in the energy-dominance case still almost twice as much.

Compare this to nuclear binding energies: They go up to about 9 MeV / nucleon for iron. But for a nucleon $Mc^2 = 938$ MeV. So nuclear binding is at most 10% of the rest mass. A neutron star of $R = 10$ km and $M = 1.5\, M_\odot \Rightarrow r_s/R = 0.3$

$$\left(\frac{-\Delta E_{Bind}}{Mc^2}\right)_{\substack{Neutron\\Star}} \cong 0.11 \qquad (20.109)$$