Lecture 21

Generalisations of exterior Schwarzschild maintaining spherical symmetry

We look for solutions of the form

$$g = e^{2a}\, c\,dt \otimes c\,dt$$
$$- e^{2b}\, dr \otimes dr$$
$$- r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi \right)$$
$$= \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \qquad (21.1)$$

where

$$\left.\begin{aligned} \theta^0 &= e^a\, dt \quad , \quad \theta^1 = e^b\, dr \\ \theta^2 &= r\, d\theta \quad , \quad \theta^3 = r \sin(\theta)\, d\varphi \end{aligned}\right\} (21.2)$$

Also $a = a(r)$, $b = b(r)$. (21.3)

We wish to consider solutions with a spherically symmetric electric field and also include a non-zero cosmological constant

Electromagnetic field

$$F = F_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$$
$$= \frac{1}{2} F_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \qquad (21.4)$$

Note $\quad \theta^\alpha \wedge \theta^\beta = \theta^\alpha \otimes \theta^\beta - \theta^\beta \otimes \theta^\alpha \qquad (21.5)$

Maxwell's equations without sources are

$$\left.\begin{aligned} dF &= 0 \\ d * F &= 0 \end{aligned}\right\} (21.6)$$

Note $\quad *(\theta^\alpha \wedge \theta^\beta) = \frac{1}{2} \varepsilon^{\alpha\beta}{}_{\mu\nu}\, \theta^\mu \wedge \theta^\nu \qquad (21.7)$

Hence

$$*F = \frac{1}{2} (*F)_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \qquad (21.8)$$

where

$$(*F)_{\alpha\beta} = \frac{1}{2} \varepsilon^{\mu\nu}{}_{\alpha\beta}\, F_{\mu\nu}$$
$$= \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu}\, F^{\mu\nu} \qquad (21.9)$$

In orthonormal components

$$\varepsilon_{\alpha\beta\mu\nu} = \operatorname{sign} \begin{pmatrix} 0 & 1 & 2 & 3 \\ \alpha & \beta & \mu & \nu \end{pmatrix} \qquad (21.10)$$

Note : $\quad * \circ * = - id \quad$ (on 2-forms) $\qquad (21.11)$

We solve

$$dF = 0 \quad \text{by} \quad F = dA \qquad (21.12)$$

or

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \qquad (21.13)$$

where (compare Lecture 9, eqn. (9.55))

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \qquad (21.14)$$

We seek solutions of electrostatic form

$$A = \phi(r) \; c\,dt = A_0 \, \theta^0 \qquad (21.15)$$

where

$$A_0 = e^{-a} \phi \, . \qquad (21.16)$$

Then

$$F = dA = \phi' \, dr \wedge c\,dt$$

$$= -\phi' \, e^{-(a+b)} \, \theta^0 \wedge \theta^1 \qquad (21.17)$$

and

$$* F = -\phi' \, e^{-(a+b)} \, \theta^2 \wedge \theta^3$$

$$= -\phi' \, r^2 \sin\theta \, e^{-(a+b)} \, d\theta \wedge d\varphi \qquad (21.18)$$

The other half of Maxwell's equations then demand:

$$\left.\begin{aligned} & d * F = 0 \iff \\ & 0 = d\left(-\phi' r^2 \sin\theta\, e^{-(a+b)}\right) d\theta \wedge d\varphi \\ & \quad = \left(-\phi' r^2 \sin\theta\, e^{-(a+b)}\right)' dr \wedge d\theta \wedge d\varphi \end{aligned}\right\} \quad (21.19)$$

$$\iff \; -\phi' r^2 e^{-(a+b)} = k = \text{const}$$

$$\iff \; -\phi' e^{-(a+b)} = k / r^2$$

$$\iff \; F = (k/r^2)\; \theta^0 \wedge \theta^1 \qquad (21.20)$$

Hence

$$\left.\begin{aligned} *F &= \frac{k}{r^2}\; \theta^2 \wedge \theta^3 \\ &= k \sin\theta\, d\theta \wedge d\varphi \end{aligned}\right\} \quad (21.21)$$

$$\leadsto \quad \int_{S^2(r)} *F = 4\pi k = \int \frac{1}{c} (\vec{E} \cdot \vec{n})\, do$$

$$= \frac{1}{c\varepsilon_0}\, Q$$

$$\Rightarrow \quad k = \frac{1}{4\pi\varepsilon_0} \frac{Q}{c} \qquad (21.22)$$

(MKSA-units)

The energy-momentum tensor of the electromagnetic field is

$$T = T_{\alpha\beta}\ \theta^\alpha \otimes \theta^\beta$$

$$= \frac{1}{\mu_0}\left(-F_{\alpha\lambda} F_\beta{}^\lambda + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}\right) \theta^\alpha \otimes \theta^\beta$$

$$= \frac{1}{\mu_0}\left(F^2 - \frac{1}{4}\ g\ \mathrm{Tr}(F^2)\right) \qquad (21.23)$$

where we regard $F \in T^*M \otimes T^*M$ and multiplication via g^{-1}, i.e.

$$(\theta^\alpha \otimes \theta^\beta) \cdot (\theta^\gamma \otimes \theta^\delta)$$

$$= \theta^\alpha \otimes \theta^\delta\ g^{-1}(\theta^\beta, \theta^\gamma)$$

$$= g^{\beta\gamma}\ \theta^\alpha \otimes \theta^\delta \qquad (21.24)$$

With $$F = \left(\frac{k}{r^2}\right)\ \theta^0 \wedge \theta^1$$

$$= \left(\frac{k}{r^2}\right)\left(\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0\right) \qquad (21.25)$$

$$\curvearrowright \quad F^2 = \left(\frac{k}{r^2}\right)^2 \left(\theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1\right) \qquad (21.26)$$

$$\mathrm{Trace}(F^2) = 2\left(\frac{k}{r^2}\right)^2 \qquad (21.27)$$

$$T = \frac{1}{\mu_0}\left(\frac{k}{r^2}\right)^2 \Big[\theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1$$

$$- \frac{1}{4}\, 2 \Big(\theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a\Big)\Big]$$

$$= \frac{1}{2\mu_0}\left(\frac{k}{r^2}\right)^2 \left[\theta^0 \otimes \theta^0 - \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3\right] \tag{21.28}$$

Check: $\mathrm{Trace}(T) = \frac{\mu_0}{2}\left(\frac{k}{r^2}\right)^2 (1+1-1-1) = 0.$

$\checkmark$ OK.

Using (21.22) we get for the components:

$$\left.\begin{aligned} T_{00} &= -T_{11} = T_{22} = T_{33} \\ &= \frac{1}{2\mu_0}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{Q^2}{c^2 r^4} = \frac{\varepsilon_0}{2}\cdot\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{Q^2}{r^4} \\ &= \frac{Q^2}{32\pi^2\varepsilon_0}\,\frac{1}{r^4} \end{aligned}\right\} \tag{21.29}$$

Note: It is due to our use of the area-radius r in the metric (21.1-2) that F in (21.20) and $*F$ in (21.21) had no explicit dependence on a and b. The Coulomb-field is determined by the flux of $\vec{D} = \varepsilon_0 \vec{E}$ that must equal Q, hence the simple $1/r^2$-dependence, since Area $= 4\pi r^2$.

Einstein's equation with cosmological constant are

$$G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

or

$$G_{\alpha\beta} = \kappa \left(T_{\alpha\beta} + \frac{\Lambda}{\kappa} g_{\alpha\beta} \right) \qquad (21.30)$$

We set

$$q^2 := Q^2 \cdot \kappa \; \frac{\varepsilon_0}{2} \left(\frac{1}{4\pi\varepsilon_0} \right)^2$$

$$= Q^2 \cdot \frac{8\pi G}{c^4} \; \frac{1}{32\pi^2 \varepsilon_0}$$

$$= Q^2 \; \frac{G}{4\pi\varepsilon_0 c^4} \qquad (21.31)$$

so that q has the physical dimension of length. Then the right-hand side of (21.30) becomes

$$\frac{q^2}{r^4} \left[\Theta^0 \otimes \Theta^0 - \Theta^1 \otimes \Theta^1 + \Theta^2 \otimes \Theta^2 + \Theta^3 \otimes \Theta^3 \right]$$

$$+ \Lambda \left[\Theta^0 \otimes \Theta^0 - \Theta^1 \otimes \Theta^1 - \Theta^2 \otimes \Theta^2 - \Theta^3 \otimes \Theta^3 \right]$$

$$\left.\begin{aligned} = \left(\frac{q^2}{r^4} + \Lambda \right) \Theta^0 \otimes \Theta^0 - \left(\frac{q^2}{r^4} + \Lambda \right) \Theta^1 \otimes \Theta^1 \\ + \left(\frac{q^2}{r^4} - \Lambda \right) \Theta^2 \otimes \Theta^2 - \left(\frac{q^2}{r^4} - \Lambda \right) \Theta^3 \otimes \Theta^3 \end{aligned}\right\} (21.32)$$

Using once more the formulae from Lecture 17 for the components of the Einstein Tensor in the spherically-symmetric metric and $\dot{b}=0$, only diagonal components are non-zero and Einstein's equations become

$$G_{00} = \frac{1}{r^2}\left[r\left(1-e^{-2b}\right)\right]'$$

$$= \frac{1}{r^2}\left(1-e^{-2b}\right) + \frac{2b'}{r}e^{-2b}$$

$$= \frac{q^2}{r^4} + \Lambda \qquad (21.33a)$$

$$G_{11} = -\frac{1}{r^2}\left(1-e^{-2b}\right) + \frac{2a'}{r}e^{-2b}$$

$$= -\left(\frac{q^2}{r^4} + \Lambda\right) \qquad (21.33b)$$

$$G_{22} = e^{-2b}\left(a'' + a'^2 - a'b' + \frac{a'-b'}{r}\right)$$

$$= \frac{q^2}{r^4} - \Lambda \qquad (21.33c)$$

and identically for G_{33}.

Addition of (00) and (11) components yield as in the vacuum case

$$(a+b)' = 0 \Rightarrow a(r) = -b(r) + f(t)$$

Again we can absorb $f(t)$ via redefinition of time coordinate (cf. (18.18))

$$t \to t' = \int^t e^{f(\tilde{t})} d\tilde{t} .$$

Hence we can set

$$a(r) = -b(r) . \qquad (21.34)$$

The function $b(r)$ follows from the (00) - component alone

$$\left.\begin{aligned} \left[r\left(1-e^{-2b}\right)\right]' &= \frac{q^2}{r^2} + r^2 \Lambda \\ r\left(1-e^{-2b}\right) &= -\frac{q^2}{r} + \frac{1}{3}\Lambda r^3 + 2m \end{aligned}\right\} \qquad (21.35)$$

where $2m$ is the integration constant

$$\rightsquigarrow \quad e^{-2b} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{1}{3}\Lambda r^2 \qquad (21.36)$$

Note that for $r \to \infty$ the metric is not asymptotically flat but de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$).
Hence from (21.34):

$$e^{2a} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{1}{3}\Lambda r^2 \qquad (21.37)$$

So, as in the vacuum case, the (00) and (11) components alone determine the metric. Hence we must check that the (22) component is also satisfied by this solution: For $a' = -b'$ (21.33c) becomes (cf. (18.8), (18.10))

$$G_{22} = e^{-2b}\left(-b'' + 2b'^2 - \frac{2}{r}b'\right)$$

$$= e^{-2b}\left[\frac{1}{2}e^{2b}\left(e^{-2b}\right)'' - \frac{2}{r}b'\right]$$

$$= \frac{1}{2}\left(e^{-2b}\right)'' + \frac{1}{r}\left(e^{-2b}\right)'$$

$$= \frac{1}{2}\;\frac{1}{r}\left(r e^{-2b}\right)''$$

$$= \frac{1}{2}\;\frac{1}{r}\left(r - 2m + \frac{q^2}{r} - \frac{1}{3}\Lambda r^3\right)''$$

$$= \frac{1}{2}\;\frac{1}{r}\left(2\frac{q^2}{r^3} - 2\Lambda r\right)$$

$$= \frac{q^2}{r^4} - \Lambda \qquad (21.38)$$

and is hence identically satisfied.

In total we have:

- Metric:

$$g = f(r)\, c\,dt \otimes c\,dt - f^{-1}(r)\, dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi \right) \quad (21.39)$$

- Orthonormal oneforms

$$\left.\begin{aligned} \theta^0 &= |f|^{1/2}\, c\,dt , & \theta^1 &= |f|^{-1/2}\, dr \\ \theta^2 &= r\, d\theta , & \theta^3 &= r \sin(\theta)\, d\varphi \end{aligned}\right\} \quad (21.40)$$

- Electromagnetic field

$$\left.\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0 c} \frac{Q}{r^2}\, \theta^0 \wedge \theta^1 \\ &= \frac{c}{\sqrt{4\pi\varepsilon_0 G}} \frac{q}{r^2}\, \theta^0 \wedge \theta^1 \end{aligned}\right\} \quad (21.41)$$

- Satisfying the Einstein-Maxwell-equations

$$\left.\begin{aligned} & G_{\alpha\beta} - g_{\alpha\beta}\Lambda = \kappa T_{\alpha\beta} \\ & T_{\alpha\beta} = \frac{1}{\mu_0}\left(-F_{\alpha\lambda} F_\beta{}^\lambda + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right) \\ & dF = d * F = 0 \end{aligned}\right\} \quad (21.42)$$

Special cases

1.) $q = 0$

$$f = \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2\right) \quad (21.43)$$

$\Rightarrow$ Schwarzschild De Sitter $(\Lambda > 0)$
" anti " $(\Lambda < 0)$.

2.) $\Lambda = 0$

$$f = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \quad (21.44)$$

$\Rightarrow$ Reissner - Nordström

Let's look at the latter.

It is a spherically symmetric, static solution of a "charged star". The static killing vector field is

$$K = \frac{\partial}{\partial t} \quad (21.45)$$

with norm

$$g(K, K) = c^2 f(r) \quad (21.46)$$

The zeros of f are

$$r^2 - 2m + q^2 = 0$$

$$\Leftrightarrow \quad r_{1,2} = m \pm (m^2 - q^2)^{1/2} \tag{21.47}$$

As calculated in Lecture 18, the acceleration of

$$u = c \; k \, / \, |g(k,k)|^{1/2} \tag{21.48}$$

is (cf. equation (18.43))

$$a^\flat = -\frac{c^2}{2} \; d \ln\left(g(k,k)/c^2 \right)$$

$$= -\frac{c^2}{2} \; d \ln(f) \tag{21.49}$$

here with f as above. So the zeros of f correspond to the <u>limits of stationarity</u>, which we here also call (Killing-) Horizons. Hence we have 3 cases:

1.) $q^2 < m^2 \Rightarrow$ two horizons, (21.50a)

2.) $q^2 > m^2 \Rightarrow$ no horizon, (21.50b)

3.) $q^2 = m^2 \Rightarrow$ one horizon. (21.50c)

The acceleration is

$$a^\flat = -\frac{c^2}{2}\left(\frac{2m}{r^2} - \frac{2q^2}{r^3}\right) dr$$

$$= -c^2 |f|^{1/2}\left(\frac{m}{r^2} - \frac{q^2}{r^3}\right)\theta^r \qquad (21.51)$$

hence

$$a = |f|^{1/2}\left(\frac{mc^2}{r^2} - \frac{(qc)^2}{r^3}\right) e_r \qquad (21.52)$$

which for $r \to \infty$ approximates GM/r^2 if

$$m = \frac{GM}{c^2} \qquad (21.53)$$

as before in the vacuum case. Hence we identify m with the mass of the central object according to (21.53) and assume

$$m \geqslant 0 \qquad (21.54)$$

The critical charge modulus, above which f has no zeros and no horizons exist, is $|q| = m$, or in SI-units:

$$\boxed{\begin{aligned} |Q_{crit}| &= \left(\frac{4\pi\varepsilon_0}{G}\right)^{1/2} mc^2 \\ &= (4\pi\varepsilon_0 G)^{1/2} M \end{aligned}} \qquad (21.55)$$

Remark: Formula (21.55) has the following heuristic interpretation: Consider in standard Electrodynamics a spherical shell of radius R and homogeneously distributed mass and charge M and Q, respectively.

Its electrostatic and gravitational (newtonian) energies are

$$E_{\text{electric}} = \frac{Q^2}{8\pi\varepsilon_0 R} \quad (21.56)$$

$$E_{\text{grav}} = -\frac{GM^2}{2R} \quad (21.57)$$

The total energy is negative, i.e. the system is in a bound stable state, if

$$E_{\text{electric}} + E_{\text{grav}} < 0$$

$$\Rightarrow \quad GM^2 > Q^2 / 4\pi\varepsilon_0$$

or $$|Q| < (4\pi\varepsilon_0 G)^{1/2} M \quad (21.58)$$

Curvature components were calculated in Lecture 17

$$R_{0101} = -e^{-2b}\left(-b'' + 2b'^2\right)$$

$$= -\frac{1}{2}\left(e^{-2b}\right)'' \qquad (21.56)$$

$$R_{0202} = R_{0303} = -R_{1212} = -R_{1313}$$

$$= \frac{b'}{r} e^{-2b} = -\frac{1}{2r}\left(e^{-2b}\right)' \qquad (21.57)$$

$$R_{2323} = -\frac{1}{r^2}\left(1 - e^{-2b}\right) \qquad (21.58)$$

Using

$$e^{-2b} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \qquad (21.59)$$

$$\Rightarrow R_{0101} = -\frac{1}{2}\left(-\frac{4m}{r^3} + 6\frac{q^2}{r^4}\right)$$

$$= \frac{2m}{r^3} - \frac{3q^2}{r^4} \qquad (21.60)$$

$$R_{0202} = -\frac{1}{2r}\left(\frac{2m}{r^2} - \frac{2q^2}{r^3}\right)$$

$$= -\frac{m}{r^3} + \frac{q^2}{r^4} \qquad (21.61)$$

$$R_{2323} = -\frac{1}{r^2}\left(\frac{2m}{r} - \frac{q^2}{r^2}\right) = \frac{-2m}{r^3} + \frac{q^2}{r^4}$$

$$(21.62)$$

And as in (18.32)

$$K := R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$$

$$= 4 \left(R_{0101}^2 + R_{0202}^2 + R_{0303}^2 + R_{1212}^2 + R_{1313}^2 + R_{2323}^2 \right)$$

(21.63)

$$= 4 \left(R_{0101}^2 + 2 R_{0202}^2 + 2 R_{1212}^2 + R_{2323}^2 \right)$$

$$= 4 \left(R_{0101}^2 + 4 R_{0202}^2 + R_{2323}^2 \right)$$

$$= 4 \left\{ \left(\frac{2m}{r^3} - \frac{3q^2}{r^4} \right)^2 + 4 \left(\frac{m}{r^3} - \frac{q^2}{r^4} \right)^2 + \left(\frac{2m}{r^3} - \frac{q^2}{r^4} \right)^2 \right\}$$

$$= 4 \left\{ 12 \left(\frac{m}{r^3} \right)^2 - 24 \frac{mq^2}{r^7} + 14 \left(\frac{q^2}{r^4} \right)^2 \right\}$$

$$= 48 \left\{ \left(\frac{m}{r^3} \right)^2 - 2 \frac{mq^2}{r^7} + \frac{7}{6} \left(\frac{q^2}{r^4} \right)^2 \right\}$$

$$= 48 \left(\frac{m}{r^3} \right)^2 \left\{ 1 - 2 \frac{q}{m} \frac{q}{r} + \frac{7}{6} \left(\frac{q}{m} \right)^2 \frac{q^2}{r^2} \right\} \quad (21.63)$$