Lecture 22

The Kerr-Newman solution

- Stationary axissymetric solution to Einstein-Maxwell equation
- Two commuting Killing vector fields

$$K_1 = \frac{\partial}{\partial t}, \qquad K_2 = \frac{\partial}{\partial \varphi} \qquad (22.1)$$

- Describes a stationary rotating charged black hole
- Astrophysically most interesting solution in the uncharged case
- In Boyer-Lindquist coordinates (t, r, θ, φ)

$$\left.\begin{aligned} g = {} & \frac{\Delta}{\varrho^2}\left[c\,dt - a\sin^2(\theta)\,d\varphi\right]^{\otimes 2} \\ & - \frac{\sin^2\theta}{\varrho^2}\left[(r^2+a^2)\,d\varphi - a\,c\,dt\right]^{\otimes 2} \\ & - \frac{\varrho^2}{\Delta}\,dr \otimes dr - \varrho^2\,d\theta \otimes d\theta \end{aligned}\right\} \qquad (22.2)$$

$$\left.\begin{aligned} \Delta &:= r^2 - 2mr + a^2 + q^2 = (r-m)^2 - (m^2 - a^2 - q^2) \\ \varrho^2 &:= r^2 + a^2\cos^2\theta \end{aligned}\right\} \qquad (22.3)$$

Parameters and their physical meaning

$$m := \frac{GM}{c^2}, \quad M = \text{mass} \tag{22.4}$$

$$ma := \frac{GJ}{c^3}, \quad J = \text{angular-momentum} \tag{22.5}$$

$$\Rightarrow \quad a = \frac{J}{Mc}$$

$$q^2 = \frac{1}{4\pi\varepsilon_0} \frac{GQ^2}{c^4}, \quad Q = \text{charge} \tag{22.6}$$

Kretschmann Scalar

$$K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$$

$$= 8 / (r^2 + a^2\cos^2\theta)^6 \times$$

$$\times \Big\{ 6m^2 (r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)$$
$$- 12mq^2 r (r^4 - 10a^2r^2\cos^2\theta + 5a^4\cos^4\theta)$$
$$+ q^4 (7r^4 - 34a^2r^2\cos^2\theta + 7a^4\cos^4\theta) \Big\} \tag{22.7}$$

(see Richard Conn Henry, Astrop. J. 535, 2000)

$\Rightarrow$ Singular for $r^2 + a^2\cos^2\theta = 0$, i.e.

$r = 0$ and $\theta = \pi/2$ (this is a "ring")

Grouping (22.2) into the individual terms $dx^\alpha \otimes dx^\beta$ we get for the coefficients

$c\,dt \otimes c\,dt$:

$$g_{00} = \frac{\Delta}{\varsigma^2} - \frac{\sin^2\theta}{\varsigma^2} a^2$$

$$= \varsigma^{-2} \left[r^2 - 2mr + q^2 + a^2 \cos^2\theta \right]$$

$$= 1 - \frac{2mr}{\varsigma^2} + \frac{q^2}{\varsigma^2} \qquad (22.8)$$

$d\varphi \otimes d\varphi$:

$$g_{33} = \frac{\Delta}{\varsigma^2} a^2 \sin^4\theta - \frac{\sin^2\theta}{\varsigma^2} (r^2 + a^2)^2$$

$$= \frac{\sin^2\theta}{\varsigma^2} \left[a^2 \Delta \sin^2\theta - (r^2 + a^2)^2 \right]$$

$$= \frac{\sin^2\theta}{\varsigma^2} \left[a^2 (r^2 + a^2) \sin^2\theta + a^2 (-2mr + q^2) \sin^2\theta - (r^2 + a^2)^2 \right]$$

$$= \frac{\sin^2\theta}{\varsigma^2} (r^2 + a^2) \left[- r^2 - a^2 \cos^2\theta \right] - a^2 (2mr - q^2) \frac{\sin^4\theta}{\varsigma^2}$$

$$= - (r^2 + a^2) \sin^2\theta - (2mr - q^2) a^2 \frac{\sin^4\theta}{\varsigma^2}$$

$$(22.9)$$

$dct \otimes d\varphi + d\varphi \otimes cdt$:

$$g_{03} = -a\frac{\Delta}{\varrho^2}\sin^2\theta + a(r^2+a^2)\frac{\sin^2\theta}{\varrho^2}$$

$$= a\frac{\sin^2\theta}{\varrho^2}\left[r^2+a^2-r^2-a^2-q^2+2mr\right]$$

$$= a(2mr-q^2)\frac{\sin^2\theta}{\varrho^2} \qquad (22.10)$$

$dr \otimes dr$:

$$g_{11} = -\frac{\varrho^2}{\Delta} = -\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2+q^2}$$

$$= -\frac{r^2+a^2\cos^2\theta}{r^2+a^2}$$

$$+\left(\frac{r^2+a^2\cos^2\theta}{r^2+a^2} - \frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2+q^2}\right)$$

$$= -\frac{r^2+a^2\cos^2\theta}{r^2+a^2} - \frac{(r^2+a^2\cos^2\theta)(2mr-q^2)}{(r^2+a^2)(r^2+a^2-(2mr-q^2))}$$

(22.11)

$d\theta \otimes d\theta$:

$$g_{22} = -\varrho^2 = -r^2-a^2\cos^2\theta \qquad (22.12)$$

Hence the metric can be written as

$$g = \left.\begin{aligned} & c\,dt \otimes c\,dt \\ & - \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\, dr \otimes dr \\ & - (r^2 + a^2\cos^2\theta)\, d\theta \otimes d\theta \\ & - (r^2+a^2)\sin^2\theta\, d\varphi \otimes d\varphi \end{aligned}\right\} =: \eta$$

$$+ \left(- \frac{2mr}{\rho^2} + \frac{q^2}{\rho^2} \right) c\,dt \otimes c\,dt$$

$$- \frac{\rho^2\,(2mr - q^2)}{\Delta\,(r^2+a^2)}\, dr \otimes dr$$

$$- (2mr - q^2)\, a^2 \frac{\sin^4\theta}{\rho^2}\, d\varphi \otimes d\varphi$$

$$+ (2mr - q^2)\, a \frac{\sin^2\theta}{\rho^2} (dct \otimes d\varphi + d\varphi \otimes dct)$$

$$= \eta - (2mr - q^2) \Big[\rho^{-2}\, c\,dt \otimes c\,dt$$

$$+ \rho^{-2} a^2 \sin^4\theta\, d\varphi \otimes d\varphi$$

$$- \rho^{-2} a \sin^2\theta\, (dct \otimes d\varphi + d\varphi \otimes dct)$$

$$+ \rho^2 / (\Delta (r^2+a^2))\, dr \otimes dr \Big]$$

Hence

$$g = \eta + h \qquad (22.13)$$

where

$$\eta = c\,dt \otimes c\,dt - \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\, dr \otimes dr - (r^2 + a^2\cos^2\theta)\, d\theta \otimes d\theta - (r^2+a^2)\sin^2\theta\, d\varphi \otimes d\varphi \qquad (22.14)$$

is the flat Minkowski metric in bi-focal elliptic coordinates (which we discuss below), and

$$h = -\frac{2mr - q^2}{\rho^2}\left[(c\,dt - a\sin^2\theta\, d\varphi)^{\otimes 2} + \frac{\rho^4}{(r^2+a^2)\Delta}\, dr \otimes dr \right] \qquad (22.15)$$

Note that (22.2) has the form

$$g = \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \qquad (22.16)$$

with

$$\theta^0 = \frac{\sqrt{\Delta}}{\varrho} \left(c\,dt - a \sin^2(\theta)\, d\varphi \right) \qquad (22.17a)$$

$$\theta^1 = \frac{\varrho}{\sqrt{\Delta}}\, dr \qquad (22.17b)$$

$$\theta^2 = \varrho\, d\theta \qquad (22.17c)$$

$$\theta^3 = \frac{\sin\theta}{\varrho} \left((r^2 + a^2)\, d\varphi - a\, c\,dt \right) \qquad (22.17d)$$

Hence

$$h = -\frac{2mr - q^2}{\varrho^2} \left[\frac{\varrho^2}{\Delta}\, \theta^0 \otimes \theta^0 + \frac{\varrho^2}{(r^2+a^2)}\, \theta^1 \otimes \theta^1 \right]$$

$$= \frac{\Delta - (r^2+a^2)}{\Delta} \left[\theta^0 \otimes \theta^0 + \frac{\Delta}{r^2+a^2}\, \theta^1 \otimes \theta^1 \right]$$

(22.18)

That (22.14) is the flat metric is seen as soon as we understand the (r, θ, φ) coordinates, which here are <u>not</u> the usual spherical polar ones.

<u>Intermezzo: Confocal elliptic coordinates.</u>

We are in $\mathbb{R}^3$ with its standard euclidean metric δ. In affine, rectangular coordinates x, y, z have

$$\delta = dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{22.19}$$

We set

$$\left.\begin{aligned} x &= f(r) \sin\theta \cos\varphi \\ y &= f(r) \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned}\right\} \tag{22.20}$$

We wish to determine $f(r)$ so that the 3 surfaces of constant r, θ, and φ are pairwise orthogonal at each point of intersection. This is true if and only if the curves of constant r and θ, respectively, are mutually orthogonal in any (and hence all) planes of constant φ; ie. in the $y = 0$ plane where $\varphi = 0$.

In that plane

$$\left.\begin{aligned} x &= f(r)\sin\theta \\ z &= r\cos\theta \end{aligned}\right\} \quad (22.21)$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}$$

$$= f'(r)\sin\theta\,\frac{\partial}{\partial x} + \cos\theta\,\frac{\partial}{\partial z} \quad (22.22)$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z}$$

$$= f(r)\cos\theta\,\frac{\partial}{\partial x} - r\sin\theta\,\frac{\partial}{\partial z} \quad (22.23)$$

$$\Rightarrow \delta\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = (f f' - r)\sin\theta\cos\theta \quad (22.24)$$

This vanishes for all θ and r iff

$$f f' = r \iff f^2(r) = r^2 + c. \quad (22.25)$$

We choose $c = \varepsilon^2 \geq 0$ so also take f to be positive; then

$$x = (r^2+\varepsilon^2)^{1/2}\sin\theta\cos\varphi\,, \quad (22.26a)$$

$$y = (r^2+\varepsilon^2)^{1/2}\sin\theta\sin\varphi\,, \quad (22.26b)$$

$$z = r\cos\theta\,. \quad (22.26c)$$

$$\frac{x^2}{\tau^2+\varepsilon^2} + \frac{y^2}{\tau^2+\varepsilon^2} + \frac{z^2}{\tau^2} = 1 \qquad (22.27)$$

$$\left(\frac{x}{\sin\theta}\right)^2 + \left(\frac{y}{\sin\theta}\right)^2 - \left(\frac{z}{\cos\theta}\right)^2 = \varepsilon^2 \qquad (22.28)$$

Equation (22.27) describes an oblate ellipsoid with semi-minor axis (in z-direction) b and semi-major axis orthogonal to that in the xy-plane a, where

$$a = (\tau^2+\varepsilon^2)^{1/2} \qquad (22.29)$$

$$b = \tau \qquad (22.30)$$

so that the eccentricity e

$$e = \left(1 - \frac{b^2}{a^2}\right)^{1/2} = \frac{\varepsilon}{\sqrt{\tau^2+\varepsilon^2}} \qquad (22.31)$$

and the distance c between the mid-point of the ellipse in the $y=0$ plane and a focal point, i.e. the radius of the focal ring in the $z=0$ plane in $\mathbb{R}^3$, is

$$c = e\,a = \varepsilon. \qquad (20.32)$$

independent of τ, hence con-focal.

Equation (22.28) describes one-sheeted hyperboloids intersecting the $z=0$ plane in the circle of radius

$$r = (x^2+y^2)^{1/2} = |\sin\theta|\, \varepsilon \qquad (20.33)$$

and asymptotic to the cone

$$\frac{z}{\sqrt{x^2+y^2}} = \cot\theta \qquad (20.34)$$

The surfaces of constant φ are, as in the standard spherical polar coordinates, half planes containing the z-axis as boundary and intersecting the $z=0$ plane in the line $x/y = \cot\varphi$.

For $r=0$ the ellipsoid degenerates to a disc of radius ε in $z=0$ plane

$$\left.\begin{aligned} x &= \varepsilon \sin\theta\cos\varphi \\ y &= \varepsilon \sin\theta \sin\varphi \\ z &= 0 \end{aligned}\right\} \qquad (20.35)$$

For $\theta = 0$ this is a ring of radius ε, i.e. the <u>focal ring</u>. This is a major difference to spherical polar coordinates, where $r=0$ is a single point

The Jacobi-matrix for (22.20) is

$$\{X^a{}_{,b}\} = \begin{pmatrix} X_{,r} & X_{,\theta} & X_{,\varphi} \\ y_{,r} & y_{,\theta} & y_{,\varphi} \\ z_{,r} & z_{,\theta} & z_{,\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} f' \sin\theta\cos\varphi, & f\cos\theta\cos\varphi, & -f\sin\theta\sin\varphi \\ f'\sin\theta\sin\varphi, & f\cos\theta\sin\varphi, & f\sin\theta\cos\varphi \\ \cos\theta & , -r\sin\theta & , 0 \end{pmatrix} \qquad (20.36)$$

The determinant of that is (develop last row)

$$\det\{X^a{}_{,b}\} = \cos\theta\, f^2 \sin\theta\cos\theta + r\sin\theta\, f f' \sin^2\theta$$

$$= \sin\theta\,(f^2\cos^2\theta + r f f' \sin^2\theta) \qquad (20.37)$$

with $f f' = r$, $f^2 = r^2 + \varepsilon^2$

$$\det\{X^a{}_{,b}\} = \sin\theta\,(r^2 + \varepsilon^2\cos^2\theta) \qquad (20.38)$$

The metric coefficients in (r,θ,φ)-coord. are

$$\delta = \delta_{ab}\, dX^a \otimes dX^b$$

$$= \delta_{ab}\, X^a{}_{,c} X^b{}_{,d}\, d\tilde{x}^c \otimes d\tilde{x}^d \qquad (20.39)$$

where $\tilde{x}^a = (r,\theta,\varphi)$

Hence, by orthogonality of $\tilde{x}^a = \text{const}$ surfaces:

$$\delta = [(X_{,r})^2 + (Y_{,r})^2 + (Z_{,r})^2]\, dr \otimes dr$$

$$+ [(X_{,\theta})^2 + (Y_{,\theta})^2 + (Z_{,\theta})^2]\, d\theta \otimes d\theta$$

$$+ [(X_{,\varphi})^2 + (Y_{,\varphi})^2 + (Z_{,\varphi})^2]\, d\varphi \otimes d\varphi \qquad (20.40)$$

$$= (f'^2 \sin^2\theta + \cos^2\theta)\, dr \otimes dr$$

$$+ (f^2 \cos^2\theta + r^2 \sin^2\theta)\, d\theta \otimes d\theta$$

$$+ f^2 \sin^2\theta \; d\varphi \otimes d\varphi \qquad (20.41)$$

Using $f^2 = r^2 + \varepsilon^2$, $f'^2 = \dfrac{r^2}{(r^2+\varepsilon^2)}$

this becomes the flat metric in con-focal elliptic coordinates:

$$\delta = \frac{r^2 + \varepsilon^2 \cos^2\theta}{r^2 + \varepsilon^2}\, dr \otimes dr$$

$$+ (r^2 + \varepsilon^2 \cos^2\theta)\, d\theta \otimes d\theta$$

$$+ (r^2 + \varepsilon^2) \sin^2\theta \; d\varphi \otimes d\varphi \qquad (20.42)$$

This is just minus the spatial part of (22.14) for $\varepsilon = a$, showing that η in (22.14) is just the Minkowski metric.

It also shows that Kerr-Newman metric is flat for $m = q = 0$ but $a \neq 0$. Interestingly it is not flat for just $m = 0$ but $q \neq 0$, as seen from the Kretschmann scalar (22.7). The singularity in the latter for

$$r = 0 \quad \text{and} \quad \theta = \pi/2 \tag{22.43}$$

is now seen by comparison with (20.33) to be a ring in the $z = 0$ plane of "radius" $r = \varepsilon = a$, which is just the focal ring of our confocal coord.

The electromagnetic field of the Kerr-Newman solution is given by the four potential

$$A = \frac{Q}{4\pi\varepsilon_0 c} \frac{r}{\rho^2} \left(c\,dt - a \sin^2\theta \, d\varphi \right) \qquad (22.44)$$

We recall that for the Reissner-Nordström solution we had

$$A = \phi(r)\, c\,dt \qquad (21.15)$$

with $\phi' = -(Q/4\pi\varepsilon_0 c)\, r^{-2}$

(follows from (21.20), (21.34) i.e. $a = -b$, and (21.22)), hence

$$A = \frac{Q}{4\pi\varepsilon_0 c} \frac{1}{r}\, c\,dt \qquad (22.45)$$

into which (22.44) turns for $a = 0$.

The Faraday tensor $F = dA$ then follows from

$$d\left[\frac{r}{\rho^2} \left(c\,dt - a \sin^2\theta \, d\varphi \right) \right]$$

$$= \rho^{-2}\, dr \wedge (c\,dt - a \sin^2\theta\, d\varphi)$$

$$-2r\rho^{-4} (r\,dr - a^2 \cos\theta \sin\theta \, d\theta) \wedge (c\,dt - a\sin^2\theta\, d\varphi)$$

$$-2r\rho^{-2} a \sin\theta \cos\theta \, d\theta \wedge d\varphi$$

$$= \frac{\varrho^2 - 2r^2}{\varrho^4} \, dr \wedge (c\,dt - a\sin^2\theta \, d\varphi)$$

$$+ \frac{2ra^2}{\varrho^4} \cos(\theta)\sin(\theta) \, d\theta \wedge (c\,dt - a\sin^2\theta \, d\varphi)$$

$$- \frac{2ra}{\varrho^2} (r^2 + a^2\cos^2\theta) \sin\theta\cos\theta \, d\theta \wedge d\varphi$$

$$= - \frac{r^2 - a^2\cos^2\theta}{\varrho^4} \, dr \wedge (c\,dt - a\sin^2\theta \, d\varphi)$$

$$+ \frac{2ra}{\varrho^4} \sin\theta\cos\theta \, d\theta \wedge [a\,c\,dt - (r^2 + a^2) d\varphi]$$

$$= \frac{r^2 - a^2\cos^2\theta}{\varrho^4} \, \theta^0 \wedge \theta^1$$

$$- \frac{2ra}{\varrho^4} \cos(\theta) \, \theta^2 \wedge \theta^3 \qquad (22.46)$$

Hence

$$F = dA = \left(\frac{Q}{4\pi\varepsilon_0 c}\right) \left\{ \frac{r^2 - a^2\cos^2\theta}{\varrho^4} \, \theta^0 \wedge \theta^1 - \frac{2ra}{\varrho^4} \cos\theta \, \theta^2 \wedge \theta^3 \right\} \qquad (22.47)$$

$$= \left(\frac{Q}{4\pi\varepsilon_0 c}\right) \left\{ \frac{r^2 - a^2\cos^2\theta}{\varrho^4} (c\,dt - a\sin^2\theta \, d\varphi) \wedge dr - \frac{2ra}{\varrho^4} \sin\theta\cos\theta \, d\theta \wedge [(r^2 + a^2) d\varphi - a\,c\,dt] \right\} \qquad (22.48)$$

Again, for $a=0$ this just coincides with (21.20), i.e. the F - field of Reissner - Nordström.

From F the components of the electric and magnetic field strengths follow. We refer them to asymptotically normalized coordinate basis ($\theta^{\hat{r}} = dr$, $\theta^{\hat{\theta}} = r\, d\theta$, $\theta^{\hat{\varphi}} = r \sin\varphi\, d\varphi$). So if the indices on F now refer to dx^μ - basis, we get

$$E_{\hat{r}} = c F_{01} = \frac{Q}{4\pi\varepsilon_0} \frac{r^2 - a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)^2}$$

$$= \left(\frac{Q}{4\pi\varepsilon_0}\right) \frac{1}{r^2} + O(r^{-4}) \qquad (22.49a)$$

$$E_{\hat{\theta}} = \frac{c}{r} F_{01} = \left(\frac{Q}{4\pi\varepsilon_0}\right) \frac{2a^2}{\rho^4} \cos\theta \sin\theta$$

$$= O(r^{-4}) \qquad (22.49b)$$

$$E_{\hat{\varphi}} = \frac{c}{r\sin\theta} F_{03} = 0 \qquad (22.49c)$$

$$B_{\hat{r}} = -\frac{1}{r^2 \sin\theta} F_{23}$$

$$= \frac{Q}{4\pi\varepsilon_0 c} \frac{2a}{r^3} \cos\theta + O(r^{-4}) \qquad (22.50a)$$

$$B_{\hat{\theta}} = \frac{1}{r\sin\theta} F_{13}$$

$$= \frac{Q}{4\pi\varepsilon_0 c} \frac{a}{r^3} \sin\theta + O(r^{-4}) \qquad (22.50b)$$

$$B_{\hat{\varphi}} = \frac{1}{r} F_{21} = 0 \qquad (22.50c)$$

Now, (22.49) corresponds to an asymptotic electric Coulomb field and (22.50) to an asymptotic magnetic dipole field. To see the latter, we recall that an magnetic dipole field of magnetic moment $\vec{m}$ has the form

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \left(-\vec{m}\cdot\vec{\nabla}\right) \frac{\vec{x}}{r^3}$$

$$= \frac{\mu_0}{4\pi} \frac{3\vec{n}(\vec{n}\cdot\vec{m}) - \vec{m}}{r^3} \qquad (22.51)$$

For $\vec{m} = \|\vec{m}\| \vec{e}_z$ this is

$$\vec{B}\cdot\vec{n} = B_{\hat{r}} = \frac{\mu_0}{4\pi} \frac{1}{r^3} 2(\vec{m}\cdot\vec{n})$$

$$= \frac{\mu_0}{4\pi} \frac{2\|\vec{m}\|}{r^3} \cos\theta \qquad (22.52a)$$

$$B_{\hat{\theta}} = - \frac{\mu_0}{4\pi} \frac{1}{r^3} \left(\vec{m}\cdot\vec{e}_{\hat{\theta}}\right)$$

$$= \frac{\mu_0}{4\pi} \frac{\|\vec{m}\|}{r^3} \sin\theta \qquad (22.52b)$$

Comparison of (22.52a) and (22.52b) with (22.50a) and (22.50b), respectively, shows that the asymptotic magnetic field of the Kerr-Newman solution corresponds to that of a magnetic dipole of magnetic moment $\|\vec{m}\|$ satisfying

$$\frac{a\,Q}{4\pi\varepsilon_0 c} = \frac{\mu_0}{4\pi}\,\|\vec{m}\| \qquad (22.53)$$

or

$$\|\vec{m}\| = \frac{Q}{\varepsilon_0\mu_0}\,\frac{a}{c} = Q\cdot c\cdot\frac{J}{Mc}$$

$$= \frac{Q\cdot J}{M} \qquad (22.54)$$

We recall the definition of the gyro-magnetic ratio g,

$$\vec{m} = g\left(\frac{Q}{2M}\right)\vec{J} \qquad (22.55)$$

Hence the "object" described by the Kerr-Newman solution has a universal (i.e. independent of mass, charge and angular momentum) gyromagnetic ratio

$$g = 2 \qquad (22.56)$$

just like the electron (without QED-corrections).

The interpretation of the parameter a expressed in (22.5) can be derived from the $r \to \infty$ form of the coefficient g_{03} in (22.10) and comparison with the weak-field metric (11.26)

$$g = \left(1 + \frac{2\phi}{c^2}\right) c\,dt \otimes c\,dt$$
$$- \left(1 - \frac{2\phi}{c^2}\right) d\vec{r} \otimes d\vec{r}$$
$$+ \vec{h}(\vec{x}) \left(c\,dt \otimes d\vec{x} + d\vec{x} \otimes c\,dt \right) \qquad (11.26)$$

with

$$\vec{h}(\vec{x}) = 2 \frac{G}{c^3} \vec{J} \times \frac{\vec{x}}{r^3} \qquad (11.62)$$

as derived in (11.62). For large distances r the expression for g_{03} in (22.5) must (for $q = 0$) be of the form of the corresponding term in (11.26) with (11.62). For that we must express (11.26) in the same coordinates. We only need to know the last - non diagonal - term in (11.26). Sine

$$\vec{J} = J\, \vec{e}_z \qquad (22.57)$$

have

$$(\vec{e}_z \times \vec{x}) \cdot d\vec{x} = \vec{e}_z (\vec{x} \times d\vec{x}) = x\,dy - y\,dx\,. \qquad (22.58)$$

Using

$$\left.\begin{aligned} x &= f(r)\sin\theta\cos\varphi \\ y &= f(r)\sin\theta\sin\varphi \end{aligned}\right\} \quad (22.59)$$

$$\left.\begin{aligned} dx &= \frac{f'}{f}x\,dr + f\cos\theta\cos\varphi\,d\theta \\ &\qquad - f\sin\theta\sin\varphi\,d\varphi \\ dy &= \frac{f'}{f}y\,dr + f\cos\theta\sin\varphi\,d\theta \\ &\qquad + f\sin\theta\cos\varphi\,d\varphi \end{aligned}\right\} \quad (22.60)$$

$x\,dy - y\,dx$

$$= f\sin\theta\cos\varphi\,(f\cos\theta\sin\varphi\,d\theta + f\sin\theta\cos\varphi\,d\varphi)$$

$$- f\sin\theta\sin\varphi\,(f\cos\theta\cos\varphi\,d\theta - f\sin\theta\sin\varphi\,d\varphi)$$

$$= f^2\sin^2\theta\,d\varphi$$

$$= (r^2+a^2)\sin^2\theta\,d\varphi \qquad (22.61)$$

Hence

$$\vec{h}\cdot(c\,dt\otimes d\vec{x} + d\vec{x}\otimes c\,dt)$$

$$= 2\,\frac{G}{c^3}\,\frac{J}{r^3}\sin^2\theta\,(r^2+a^2)\,(c\,dt\otimes d\varphi + d\varphi\otimes c\,dt)$$

$$= g_{0\varphi}\,(c\,dt\otimes d\varphi + d\varphi\otimes c\,dt) \qquad (22.62)$$

So for $r \to \infty$ the dominant ($\sim 1/r$) term is

$$g_{03} = 2 \frac{G}{c^3} \frac{J}{r} \sin^2\theta + O(1/r^3). \qquad (22.63)$$

Now, from (22.10) we get

$$g_{03} = a(2mr - q^2) \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}$$
$$= 2 \frac{am}{r} \sin^2\theta + O(1/r^3). \qquad (22.64)$$

Hence the asymptotic forms coincide iff

$$GJ/c^3 = am \qquad (22.65)$$

and with $m = \frac{GM}{c^2}$

$$a = \frac{GJ}{mc^3} = \frac{J}{Mc} \qquad (22.66)$$

which is just (22.5).

We now turn to a discussion of the horizon-structure of the Kerr-Newman geometry.

For that we remark that

$$K_1 = \frac{1}{c}\frac{\partial}{\partial t} \qquad (22.67)$$

is the unique (up to $\mathbb{R}$-multiples) Killing field that is timelike for $r \to \infty$.

Further,

$$K_2 = \frac{\partial}{\partial \varphi} \qquad (22.68)$$

is the unique (up to $\mathbb{R}$-multiples) Killing field whose orbits are closed

As a result, the following three $\mathbb{R}$-valued functions are geometrically defined, i.e. independent of the coordinates used:

$$g(K_1, K_1) = g_{00} \qquad (22.69a)$$

$$g(K_1, K_2) = g_{03} \qquad (22.69b)$$

$$g(K_2, K_2) = g_{33} \qquad (22.69c)$$

$$g_{00} = \rho^{-2}\,(\Delta - a^2 \sin^2\theta) \qquad (22.70a)$$

$$g_{03} = \rho^{-2}\, a \sin^2\theta\,(r^2 + a^2 - \Delta) \qquad (22.70b)$$

$$g_{33} = -\rho^{-2} \sin^2\theta \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right] \qquad (22.70c)$$

Note that

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ g_{30} & 0 & 0 & g_{33} \end{pmatrix} \quad (22.71)$$

or reodered by an <u>even</u> permutation
$(1 2 3) \to (3 1 2)$

$$= \left(\begin{array}{cc|cc} g_{00} & g_{03} & 0 & 0 \\ g_{30} & g_{33} & 0 & 0 \\ \hline 0 & 0 & g_{11} & 0 \\ 0 & 0 & 0 & g_{22} \end{array}\right) \quad (22.71)$$

Hence

$$\det\{g_{\alpha\beta}\} = g_{11} g_{22} (g_{00} g_{33} - g_{03}^2) \quad (22.72)$$

The inverse matrix is obtained by inverting the (2×2)-blocks

$$g^{\alpha\beta} = \left(\begin{array}{cc|cc} g^{00} & g^{03} & 0 & 0 \\ g^{03} & g^{33} & 0 & 0 \\ \hline 0 & 0 & g^{11} & 0 \\ 0 & 0 & 0 & g^{22} \end{array}\right) \quad (22.73)$$

$$g^{00} = \frac{g_{33}}{g_{00}\,g_{33} - g_{03}^2} \qquad (22.74a)$$

$$g^{03} = -\frac{g_{03}}{g_{00}\,g_{33} - g_{03}^2} = g^{30} \qquad (22.73b)$$

$$g^{33} = \frac{g_{00}}{g_{00}\,g_{33} - g_{03}^2} \qquad (22.73c)$$

$$g^{11} = g_{11}^{-1} \qquad (22.73d)$$

$$g^{22} = g_{22}^{-1} \qquad (22.73e)$$

Horizons and limits of stationarity

The Killing-field of stationarity, K_1, is timelike if (compare (22.8)):

$$g_{00} > 0$$

$$\Leftrightarrow \quad \Delta - a^2 \sin^2\theta > 0$$

$$\Leftrightarrow \quad r^2 - 2mr + q^2 + a^2\cos^2\theta > 0 \qquad (22.74)$$

Zeros

$$\tilde{r}_{1,2} = m \pm \left(m^2 - q^2 - a^2\cos^2\theta\right)^{1/2} \qquad (22.75)$$

If

$$m^2 - q^2 - a^2 \cos\theta < 0 \qquad (22.76)$$

there are no real roots and K_1 is timelike (for these θ).

If

$$m^2 - q^2 - a^2 \cos\theta \geq 0 \qquad (22.77)$$

there are real roots and K_1 is

$$\text{timelike for } r > \tilde{r}_+ \text{ or } r < \tilde{r}_- \qquad (22.78a)$$
$$\text{Spacelike for } r < \tilde{r}_+ \text{ or } r > \tilde{r}_- \qquad (22.78b)$$
$$\text{lightlike for } r = \tilde{r}_+ \text{ or } r = \tilde{r}_- \qquad (22.78c)$$

If (22.77) holds there are still stationary observers in the region $r < \tilde{r}_+$, but they cannot move along the integral lines of K_1. They must move along the integral lines of

$$K = K_1 + \Omega K_2 = \frac{\partial}{\partial ct} + \Omega \frac{\partial}{\partial \varphi} \qquad (22.79)$$

which is again a Killing field if $\Omega =$ const. Here Ω is the "angular velocity" measured in "time" ct, with which the observer as to follow ("rotate") $\frac{\partial}{\partial\varphi}$.

Note

$$g(k,k) = g_{00} + 2\Omega g_{03} + \Omega^2 g_{33}$$

For k to be timelike (at the space-time point where we evaluate the $g_{\alpha\beta}$) must have

$$\Omega^2 g_{33} + 2\Omega g_{03} + g_{00} > 0 \qquad (22.80)$$

Dividing by g_{33}, which according to (22.9) is negative unless

$$2mr - q^2 < 0$$

$$\Longleftrightarrow \quad r/|q| < \frac{|q|}{2m} < \frac{1}{2} \qquad (22.81)$$

(since $m \geq |q|$ from (22.77)).

or

$$r < \frac{1}{2}|q| < \frac{m}{2} \qquad (22.82)$$

which we exclude here (this is justified, as we will see below), we get

$$\Omega^2 + 2\Omega \frac{g_{03}}{g_{33}} + \frac{g_{00}}{g_{33}} < 0 \qquad (22.83)$$

($>$ changes to $<$ because we divided by something negative).

The zeros of the Ω-polynomial (22.83) are

$$\Omega_\pm = -\frac{g_{03}}{g_{33}} \pm \left(\frac{g_{03}^2 - g_{00}g_{33}}{g_{33}^2}\right)^{1/2}$$

$$= \omega \pm \left(\omega^2 - \frac{g_{00}}{g_{33}}\right)^{1/2} \qquad (22.84)$$

where

$$\omega := -\frac{g_{03}}{g_{33}} \qquad (22.85)$$

K is timelike if

$$\Omega_- < \Omega < \Omega_+ \qquad (22.86)$$

For real roots to exist must have

$$\omega^2 \geqslant \frac{g_{00}}{g_{33}}$$

$$\Leftrightarrow \left(\frac{g_{03}}{g_{33}}\right)^2 \geqslant \frac{g_{00}}{g_{33}}$$

$$\Leftrightarrow \quad g_{03}^2 - g_{00}g_{33} \geqslant 0 \qquad (22.87)$$

The existence of stationary observers is equivalent to the existence of real roots for Ω, i.e. to (22.87). Note that this is just the condition for $\det\begin{pmatrix} g_{00} & g_{04} \\ g_{40} & g_{33}\end{pmatrix} \geqslant 0$.

Lets evaluate this combination:

$$g_{03}^2 - g_{00}\, g_{33}$$

$$= \rho^{-4}\, a^2 \sin^4\theta\, (r^2 + a^2 - \Delta)^2$$

$$+ \rho^{-4} \sin^2\theta\, (\Delta - a^2 \sin^2\theta) \left[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta \right]$$

The two terms without Δ cancel each other. The two terms involving Δ^2 also cancel each other. Hence only the terms involving Δ linearly survive and combine to

$$\Delta \rho^{-4} \sin^2\theta \left[-2 \sin^2\theta\, a^2 (a^2+r^2) + (a^2+r^2)^2 \right.$$
$$\left. + a^4 \sin^4\theta \right]$$

$$= \Delta \rho^{-4} \sin^2\theta \left[(a^2+r^2) - a^2 \sin^2\theta \right]^2$$

$$= \Delta \rho^{-4} \sin^2\theta \left[r^2 + a^2 \cos^2\theta \right]^2$$

$$= \Delta \sin^2\theta \qquad (22.88)$$

A very simple result indeed!

In passing we note that this simple expression allows to simplify (22.73)

$$g^{00} = \varrho^{-2} \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Delta} \qquad (22.89a)$$

$$g^{03} = \varrho^{-2} a \frac{r^2+a^2-\Delta}{\Delta} \qquad (22.89b)$$

$$g^{33} = -\varrho^{-2} \frac{\Delta - a^2 \sin^2\theta}{\Delta \sin^2\theta} \qquad (22.89c)$$

Back to our stationarity problem, we see that the existence of stationary observers (22.87) is equivalent to

$$\Delta > 0$$

$$\Leftrightarrow \quad r^2 - 2mr + a^2 + q^2 > 0 \qquad (22.90)$$

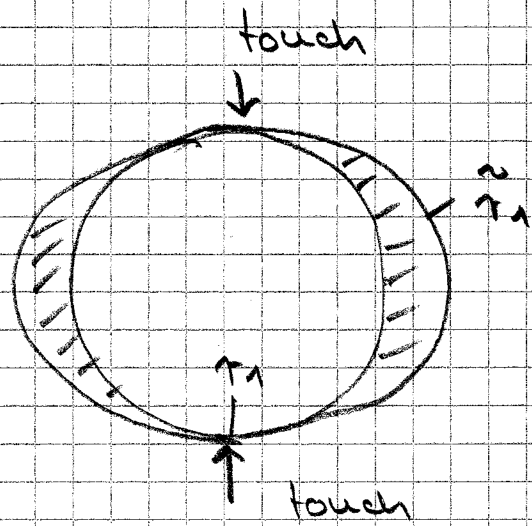
Note that the roots of this polynomial are

$$r_{1,2} = m \pm (m^2 - a^2 - q^2)^{1/2} \qquad (22.91)$$

These are the boundaries of stationary observers, that is – in our terminology – horizons. Compare (22.91) with (22.75). We have

$$\tilde{r}_1 \geq r_1 \quad \text{and} \quad \tilde{r}_1 = r_2 \Leftrightarrow \theta = 0, \frac{\pi}{2}$$

$$(22.92)$$



The surface $r = \tilde{\hat{r}}_1$ lies outside the surface $r = r_1$ and touches it at the poles $\theta = 0$ and $\theta = \pi/2$. The region

$$r_1 < r < \tilde{\hat{r}}_1 \tag{22.93}$$

is called the "Ergoregion". In that region stationary observers exist but $K_1 = \partial/\partial t$ is spacelike. This leads to the possibility to extract energy from the black hole by reducing its angular momentum via the so called "Penrose Process".

Note that horizons only exist for

$$m \geqslant (a^2 + q^2)^{1/2} \tag{22.94}$$

In which case $r_+ > m$. In any case, (22.82) was justified.

For the rotation velocity (wrt. ct) we had in (22.84)

$$\Omega_\pm = \omega \pm \left(\omega^2 - g_{00}/g_{33}\right)^{1/2}$$

$$= -\frac{g_{03}}{g_{33}} \pm \left(\frac{g_{03}^2 - g_{00}\,g_{33}}{g_{33}^2}\right)^{1/2}$$

$$= -\frac{g_{03}}{g_{33}} \pm \sqrt{\Delta}\,\left|\frac{\sin\theta}{g_{33}}\right| \qquad (22.95)$$

using (22.88). On the horizon $\Delta = 0$ have $\Omega = \Omega_+ = \Omega_- = \Omega_H$, where

$$\Omega\Big|_{\Delta=0} = \Omega_H := -\frac{g_{03}}{g_{33}}\Big|_{\Delta=0} \qquad (22.96)$$

Using (22.70b,c) this is

$$\Omega_H = \left[\frac{\rho^{-2}\,a\sin^2\theta\,(r^2+a^2-\Delta)}{\rho^{-2}\sin^2\theta\,[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]}\right]_{\substack{\Delta=0\\ r=r_+}}$$

$$= \frac{a}{r_+^2+a^2} = \frac{a}{2mr_+ - q^2} \qquad (22.97)$$

(since $r_+^2 = 2mr - a^2 - q^2$ by (22.90))

The true (i.e. wrt. time t rather than ct) angular velocity of the horizon $r = r_+$ is

$$\left.\begin{aligned} c\,\Omega_+ &= \frac{a\,c}{2m r_+ - q^2} \\ \text{with } r_+ &= m + (m^2 - a^2 - q^2)^{1/2} \end{aligned}\right\} \quad (22.98)$$

Note that the horizon rotates "rigidly", i.e. with an angular velocity independent of θ. This is connected to a deeper mathematical result called the "rigidity theorem".

The horizon $r = r_+$ has an interesting geometry. Its intersection with $t = \text{const}$ hypersurface is a spacelike embedded mfd.

$$\Sigma := \{(t, r, \theta, \varphi) : t = 0,\ r = r_+\} \quad (22.99)$$

which is topologically a 2-sphere but with a metric that is not round; it depends on θ. Setting $dt = dr = 0$ in (22.2) we get $(A = 0)$

$$g_\Sigma = \rho^2\, d\theta^2 + \rho^{-2} \sin^2\theta\, (r^2 + a^2)^2\, d\varphi^2$$

(22.100)

or $$g_\Sigma = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 \qquad (22.101a)$$

with $$\theta^1 = \rho \, d\theta$$
$$= (r_+^2 + a^2 \cos^2\theta)^{1/2} \, d\theta \qquad (22.101b)$$

$$\theta^2 = \frac{(r_+^2 + a^2)\sin\theta}{(r_+^2 + a^2\cos^2\theta)^{1/2}} \, d\varphi \qquad (22.101c)$$

The surface area is

$$A = \int \theta^1 \wedge \theta^2 = 4\pi \, (r_+^2 + a^2)$$
$$= 4\pi \, [2 m r_+ - q^2]$$

$$= 4\pi \, [2m^2 + 2\,(m^4 - (ma)^2 - m^2 q^2)^{1/2} - q^2]$$
$$= 4\pi \, [2m^2 + 2\,(m^4 - \ell^2 - m^2 q^2)^{1/2} - q^2] \qquad (22.102)$$

where $$\ell := ma = \frac{G J}{c^3} \qquad (22.103)$$

(angular momentum in geometric units.)

The relation $A(m, \ell, q)$ can be solved for m (mass) as function of (A, ℓ, q): Squaring (22.102) gives

$$\left(\frac{A}{4\pi} + q^2 - 2m^2\right)^2 = 4\left(m^4 - \ell^2 - m^2 q^2\right)$$

$$\left(\frac{A}{4\pi}\right)^2 + q^4 + \cancel{4m^4} + 2q^2\frac{A}{4\pi} - 4m^2\left(\frac{A}{4\pi}\right) - \cancel{4m^2}q^2$$

$$= \cancel{4m^4} - 4\ell^2 - \cancel{4m^2}q^2 \qquad (22.104)$$

Division by $4\left(\frac{A}{4\pi}\right)$ gives

$$\frac{1}{4}\left(\frac{A}{4\pi}\right) + \frac{q^2}{2} + \frac{q^4}{4}\left(\frac{4\pi}{A}\right) - m^2$$

$$= -\ell^2\left(\frac{4\pi}{A}\right) \qquad (22.105)$$

or

$$m = \left[\frac{A}{16\pi} + \frac{4\pi}{A}\ell^2 + \frac{q^2}{2} + \frac{\pi q^4}{A}\right]^{1/2}$$

$$= \left\{\left[\left(\frac{A}{16\pi}\right)^{1/2} + \frac{q^2}{4}\left(\frac{16\pi}{A}\right)^{1/2}\right]^2 + \frac{16\pi}{A}\frac{\ell^2}{4}\right\}^{1/2}$$

$$= m(A, \ell, q^2) \qquad (22.106)$$

We observe

$$m(\lambda A, \lambda \ell, \lambda q^2) = \lambda^{1/2}\, m(A, \ell, q^2) \qquad (22.107)$$

Applying to this $\left.\frac{d}{d\lambda}\right|_{\lambda=1}$ we get

$$\frac{1}{2} m = A \frac{\partial m}{\partial A} + l \frac{\partial m}{\partial l} + \underbrace{q^2 \frac{\partial m}{\partial q^2}}_{\frac{q}{2}\frac{\partial m}{\partial q}} \qquad (22.108)$$

We call

$$\left.\begin{aligned} \frac{\partial m}{\partial A} &= T = \text{surface tension} \\ \frac{\partial m}{\partial l} &= \Omega = \text{angular velocity} \\ \frac{\partial m}{\partial q} &= \Phi = \text{electric potential} \end{aligned}\right\} (22.109)$$

(all quantities in geometric units)

then

$$m = 2AT + 2l\Omega + q\Phi \qquad (22.110)$$

This is called the "Smarr-Formula". (Larry Smarr 1972)

One can show that $\Omega = \Omega_H$.

The Smarr-formula acquires a deeper meaning through the "thermodynamic analogy" in black-hole physics.

Geometry of $r = r_+$ horizon:

Curvature tensor via Cartan structure equations.

$$\theta^1 = \rho \, d\theta = (r_+^2 + a^2\cos^2\theta)^{1/2} d\theta$$

$$\theta^2 = \frac{(r_+^2 + a^2)\sin\theta}{(r_+^2 + a^2\cos^2\theta)^{1/2}} d\varphi$$

$$d\theta^1 = 0 = -\omega^1{}_2 \wedge \theta^2$$

$$\Rightarrow \quad \omega^1{}_2 = -\omega^2{}_1 \sim \theta^2 \qquad (22.111)$$

$$d\theta^2 = (r_+^2 + a^2) \frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{(r_+^2 + a^2\cos^2\theta)^{1/2}}\right] d\theta \wedge d\varphi$$

$$= (r_+^2 + a^2)\left[\frac{\cos\theta}{(\dots)^{1/2}} + \frac{1}{2}\frac{\sin\theta\,(a^2 2\cos\theta\sin\theta)}{(\dots)^{3/2}}\right]$$

$$\times \; d\theta \wedge d\varphi$$

$$= (r_+^2 + a^2)\left[\frac{\cos\theta(r_+^2 + a^2\cos^2\theta) + a^2\sin^2\theta\cos\theta}{(r_+^2 + a^2\cos^2\theta)^{3/2}}\right] d\theta \wedge d\varphi$$

$$= (r_+^2 + a^2) \frac{\cos\theta\,(r_+^2 + a^2)}{(r_+^2 + a^2\cos^2\theta)^{3/2}} d\theta \wedge d\varphi$$

$$= \frac{(r_+^2 + a^2)}{(r_+^2 + a^2\cos^2\theta)^{3/2}} \cot\theta \;\; \theta^1 \wedge \theta^2$$

using also (22.111),

$$= -\omega^2{}_1 \wedge \theta^1 = \omega^1{}_2 \wedge \theta^1$$

$$\Rightarrow \omega^1{}_2 = -\omega^2{}_1 = -\frac{\cot\theta\,(r_+^2+a^2)}{(r_+^2+a^2\cos^2\theta)^{3/2}}\,\theta^2 \qquad (22.112)$$

$$= -\frac{(r_+^2+a^2)^2\cos\theta}{(r_+^2+a^2\cos^2\theta)^2}\,d\varphi \qquad (22.113)$$

From the connection coefficients, the only independent curvature component follows

$$\Omega^1{}_2 = d\omega^1{}_2$$

$$= -(r_+^2+a^2)^2 \frac{\partial}{\partial\theta}\left[\frac{\cos\theta}{(r_+^2+a^2\cos^2\theta)^2}\right] d\theta\wedge d\varphi$$

$$= -(r_+^2+a^2)^2\left[\frac{-\sin\theta}{(\cdot\cdot)^2} + 2\,\frac{\cos\theta\,(a^2\,2\cos\theta\sin\theta)}{(\cdot\cdot)^3}\right]$$

$$\times\ d\theta\wedge d\varphi$$

$$= -(r_+^2+a^2)^2\left[\frac{-\sin\theta\,(r_+^2+a^2\cos^2\theta)+4a^2\cos^2\theta\sin\theta}{(\cdot\cdot)^3}\right]$$

$$\times\ d\theta\wedge d\varphi$$

$$= \frac{(r_+^2+a^2)^2\sin\theta\,(r_+^2-3a^2\cos^2\theta)}{(r_+^2+a^2\cos^2\theta)^3}\,d\theta\wedge d\varphi$$

Hence

$$\Omega^1{}_2 = \frac{(r_+^2 + a^2)(r_+^2 - 3a^2\cos^2\theta)}{(r_+^2 + a^2\cos^2\theta)^3}\,\theta^1 \wedge \theta^2 \qquad (22.114)$$

$$= \tfrac{1}{2} R^1{}_{2ab}\,\theta^a \wedge \theta^b$$

$$= R^1{}_{212}\,\theta^1 \wedge \theta^2$$

Hence the Gaussian curvature is
$K = R^1{}_{212} = R_{1212}$

$$\boxed{K = \frac{(r_+^2 + a^2)(r_+^2 - 3a^2\cos^2\theta)}{(r_+^2 + a^2\cos^2\theta)^3}} \qquad (22.115)$$

Gaussian curvature of outer event horizon $r = r_+$

The interesting property of that expression is that it may assume negative values for $\theta = 0, \pi$, i.e at the poles and in the neighbourhood. So $K < 0$ at $\theta = 0$ iff, for $q = 0$,

$$r_+^2 - 3a^2 < 0 \qquad (22.116)$$

$$\Leftrightarrow \quad (m + (m^2 - a^2)^{1/2})^2 < 3a^2$$

$$\Leftrightarrow \quad m^2 + (m^2 - a^2) + 2m(m^2 - a^2)^{1/2} < 3a^2$$

$$\Longleftrightarrow \quad 2m\left(m^2-a^2\right)^{1/2} < 4a^2 - 2m^2$$

$$\Longleftrightarrow \quad 4m^2\left(m^2-a^2\right) < 16a^4 - 16a^2m^2 + 4m^4$$

$$\Longleftrightarrow \quad 16a^4 - 12a^2m^2 > 0$$

$$\Longleftrightarrow \quad |a| > \frac{\sqrt{3}}{2}\cdot m \qquad (22.117)$$

Hence for $|a|$ in the intervall

$$\frac{\sqrt{3}}{2} < |a| < m \qquad (22.118)$$

the Gaussian curvature is negative in a neighbourhood of the poles.

The set of θ-values for which K is negative is given by

$$r_+^2 - 3a^2\cos^2\theta < 0$$

$$\Longleftrightarrow \quad |\cos\theta| > \frac{r_+}{\sqrt{3}|a|} = \frac{1}{\sqrt{3}}\left\{\frac{m}{|a|} + \left[\left(\frac{m}{a}\right)^2 - 1\right]^{1/2}\right\} \qquad (22.119)$$

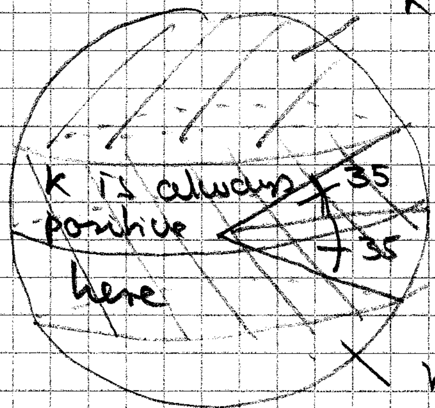
The right-hand side is smallest, and hence the negative-K θ-intervall largest, for $|a| = m$. Then

$$|\cos\theta| > \frac{1}{\sqrt{3}} \qquad (22.120)$$

$$\left.\begin{array}{ll} \Leftrightarrow & \theta < 54{,}7^\circ \\ \text{or} & \theta > 125{,}3^\circ \end{array}\right\} \qquad (22.121)$$

The strip extending $35{,}3^\circ$ north and south to the equator, i.e. of total width $\approx 70^\circ$, has always positive K.

K may be negative here



(22.122)

K may be negative here

There is no rotationally invariant isometric embedding of the horizon into $\mathbb{R}^3$, for that would necessarily have $K \geq 0$ at the poles (where the rotation axis intersects surface). Prove this!

Finally, lets integrate the Gaussian curvature over the whole surface. Using

$$\theta^1 \wedge \theta^2 = (r_+^2 + a^2) \sin\theta \, d\theta \wedge d\varphi \qquad (22.123)$$

We get

$$\int K \, \theta^1 \wedge \theta^2 = \int \frac{(r_+^2+a^2)^2 (r_+^2 - 3a^2\cos^2\theta)}{(r_+^2 + a^2\cos^2\theta)^3} \sin\theta \, d\theta \, d\varphi$$

$$= 2\pi (r_+^2+a^2)^2 \int_{-1}^{+1} dz \, \frac{(r_+^2 - 3a^2 z^2)}{(r_+^2 + a^2 z^2)^3}$$

$$= 2\pi (r_+^2+a^2)^2 \left. \frac{z}{(r_+^2 + a^2 z^2)^2} \right|_{-1}^{1}$$

$$= 4\pi \qquad (22.124)$$

independent of all parameters!

Why is that? Because of the Gauss-Bonnet Theorem, according to which

$$\int_\Sigma K = 2\pi \, \chi(\Sigma) = 4\pi (1-g) \qquad (22.125)$$

where $\chi(\Sigma)$ = Euler characteristic of Σ and g = genus of Σ. Hence our Σ is S^2!