# Lecture 3: Energy-Momentum Tensors

The energy-momentum tensor of a physical system describes its local distribution (i.e. densities) of energy and momentum. In Special Relativity, where affine Minkowski space has an associated Vektor space $V$ ($\cong \mathbb{R}^4$), the energy momentum tensor $T$ can be considered as element of either

$$\left.\begin{array}{lll} T \in V \otimes V & & \text{(contravariant)} \\ T_{\downarrow} \in V \otimes V^* & & \text{(mixed)} \\ T_{\downarrow\downarrow} \in V^* \otimes V^* & & \text{(covariant)} \end{array}\right\} \quad (3.1)$$

These are related by the Isomorphism

$$\left.\begin{array}{l} \eta_{\downarrow} : V \to V^*, \quad v \mapsto \eta(v, \cdot) \\ \eta_{\uparrow} := (\eta_{\downarrow})^{-1} : V^* \to V \end{array}\right\} \quad (3.2)$$

that is defined by the Minkowski metric $\eta \in V^* \otimes V^*$, which is a symmetric non-degenerate bilinear form on $V$ of signature $(+,-,-,-)$.

Let $\{e_\alpha : \alpha = 0,1,2,3\}$ be a basis of V. It is called <u>orthonormal</u> (with respect to $\eta$) if and only if

$$\eta_{\alpha\beta} := \eta(e_\alpha, e_\beta) = \mathrm{diag}(1,-1,-1,-1)$$

Let $\{\theta^\alpha : \alpha = 0,1,2,3\}$ be the corresponding dual Basis, i.e.,

$$\theta^\alpha(e_\beta) = \delta^\alpha{}_\beta \qquad (3.3)$$

then

$$\eta = \eta_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta \qquad (3.4)$$

and either

$$\left.\begin{aligned} T &= T^{\alpha\beta}\, e_\alpha \otimes e_\beta, \quad \text{or} \\ T_\downarrow &= T^\alpha{}_\beta\, e_\alpha \otimes \theta^\beta, \quad \text{or} \\ T_{\downarrow\downarrow} &= T_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta, \end{aligned}\right\} \quad (3.5)$$

Where

$$\left.\begin{aligned} T^\alpha{}_\beta &:= T^{\alpha\gamma}\, \eta_{\gamma\beta} \\ T_{\alpha\beta} &:= T^{\gamma\delta}\, \eta_{\gamma\alpha}\, \eta_{\delta\beta} \end{aligned}\right\} \quad (3.6)$$

In what follows we assume that $T$ is symmetric, i.e.

$$\left.\begin{aligned} &\eta(T_\downarrow(v), w) = \eta(v, T_\downarrow(w)) \\ &\forall\, v, w \in V. \end{aligned}\right\} \quad (3.7)$$

That is: $T_\downarrow \in V \otimes V^* \cong \mathrm{End}(V)$ is a symmetric map with respect to the Minkowski-metric $\eta$.

In components this means (i.e. for $v = e_\alpha$, $w = e_\beta$)

$$\eta(T_\downarrow(e_\alpha), e_\beta) = T^\gamma{}_\alpha\, \eta_{\gamma\beta}$$

$$= \eta(e_\alpha, T_\downarrow(e_\beta)) = T^\gamma{}_\beta\, \eta_{\alpha\gamma}$$

$$\left.\begin{aligned} \Leftrightarrow \quad & T_{\alpha\beta} = T_{\beta\alpha} \\ \Leftrightarrow \quad & T^{\alpha\beta} = T^{\beta\alpha} \end{aligned}\right\} \quad (3.8)$$

Whether one regards $T$, $T_\downarrow$ or $T_{\downarrow\downarrow}$ as natural representation of the physical energy-momentum distribution depends somewhat on taste. We will freely change between these and refer to all of them as EMT.

Let $u \in V$ be the four-velocity of an inertial observer; then $\eta(u, u) = c^2$. We set

$$e_0 := u / c \tag{3.9}$$

and complete $e_0$ by $e_1, e_2, e_3$ to an orthonormal basis. With respect to that basis $T$ has the following representation

$$T = T^{\alpha\beta} e_\alpha \otimes e_\beta \tag{3.10}$$

$$T^{\alpha\beta} = \left[\begin{array}{c|ccc} W & cG^1 & cG^2 & cG^3 \\ \hline \frac{1}{c}S^1 & \Sigma^{11} & \Sigma^{12} & \Sigma^{13} \\ \frac{1}{c}S^2 & \Sigma^{21} & \Sigma^{22} & \Sigma^{23} \\ \frac{1}{c}S^3 & \Sigma^{31} & \Sigma^{32} & \Sigma^{33} \end{array}\right] \tag{3.11}$$

$$= \left(\begin{array}{c|c} W & c\vec{G}^T \\ \hline \frac{1}{c}\vec{S} & \overleftrightarrow{\Sigma} \end{array}\right) \tag{3.12}$$

where

$$\left.\begin{aligned} W &= \text{energy density} \\ \vec{S} &= \text{energy current density} \\ \vec{G} &= \text{momentum density} \\ \overleftrightarrow{\Sigma} &= \text{momentum current density} \end{aligned}\right\} \tag{3.13}$$

In Special Relativity, local conservation of energy and momentum is expressed by ( $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$):

$$\nabla_\alpha T^{\alpha\beta} := \frac{\partial T^{\alpha\beta}}{\partial x^\alpha} = 0 \qquad (3.14)$$

$$\Leftrightarrow \quad \left. \begin{aligned} \dot{W} + \nabla_a S^a &= 0 , \\ \dot{G}^b + \nabla_a \Sigma^{ab} &= 0 . \end{aligned} \right\} (3.15)$$

Note that we regard the components $T^{\alpha\beta}$ as smooth functions on spacetime, so that $T^{\alpha\beta}(x)$ denotes the energy-momentum distribution at spacetime-point $x$.

The Poincaré group is the group of inhomogeneous Lorentz transformations:

$$\mathrm{Poin} := \{ (a, \Lambda) \in V \times \mathrm{Lor} \} \qquad (3.16)$$

$$\mathrm{Lor} := \{ \Lambda \in GL(V) : \eta(\Lambda v, \Lambda w) = \eta(v, w) \ \forall v, w \in V \} \qquad (3.17)$$

We have (semi-direct product)

$$\mathrm{Poin} \cong V \rtimes \mathrm{Lor} \qquad (3.18)$$

which here means the following rule of multiplication:

$$(a_1, \Lambda_1) \cdot (a_2, \Lambda_2)$$

$$= (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2). \qquad (3.19)$$

This implies

$$(a, \Lambda)^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}). \qquad (3.20)$$

The action of Poin on spacetime points (Minkowski space), in components with respect to orthonormal basis, is

$$x^\alpha \mapsto x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha \qquad (3.21)$$

with inverse

$$x^\alpha \mapsto x'^\alpha = (\Lambda^{-1})^\alpha{}_\beta (x^\beta - a^\beta)$$

$$(3.22)$$

We shall simply write (supressing indices)

$$\left.\begin{aligned} x &\mapsto x' = \Lambda x + a \\ x &\mapsto x' = \Lambda^{-1}(x - a) \end{aligned}\right\} \qquad (3.23)$$

Note that $\Lambda \in \mathrm{Lor}$

$$\Leftrightarrow \quad \eta(\Lambda v, \Lambda w) = \eta(v, w) \quad \forall\, v, w \in V \qquad (3.24)$$

$$\Leftrightarrow \quad \eta(\Lambda e_\alpha, \Lambda e_\beta) = \eta(e_\alpha, e_\beta) \quad \forall\, \alpha, \beta \in \{0, 1, 2, 3\}$$

$$\Leftrightarrow \quad \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \quad \Big|\, \eta^{\alpha\lambda}$$

$$\Leftrightarrow \quad (\eta_{\mu\nu} \Lambda^\mu{}_\alpha \eta^{\alpha\lambda}) \Lambda^\nu{}_\beta = \delta^\lambda{}_\beta$$

$$\Leftrightarrow \quad (\Lambda^{-1})^\lambda{}_\nu = \eta_{\mu\nu} \Lambda^\mu{}_\alpha \eta^{\alpha\lambda}. \qquad (3.24)$$

That means that the components of the inverse $\Lambda^{-1}$ are obtained from the components of $\Lambda$ by pulling down the upper index and raising the lower index using $\eta$.

Under a Poincaré transformation the functions $x \mapsto T(x) \in V \otimes V$ obeys the following (active!) transformation law:

$$((a, \Lambda), T) \to \bar{T}$$

$$= (\Lambda \otimes \Lambda)\left(T \circ (a, \Lambda)^{-1}\right) \qquad (3.25)$$

transform each upper index with $\Lambda$

the argument of the function is transformed with inverse group-element!

This is a <u>left action</u> on the space of tensor fields:

$$((a_1, \Lambda_1), ((a_2, \Lambda_2), T))$$

$$= (\underbrace{(a_1, \Lambda_1) \cdot (a_2, \Lambda_2)}_{\text{compare } (3.19)}, T) \qquad (3.26)$$

In components (3.25) reads

$$\bar{T}^{\alpha\beta}(x) = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu T^{\mu\nu}(\underline{x}) \qquad (3.27)$$

where $\underline{x} := \Lambda^{-1}(x-a)$

The impact of pure translations ($\Lambda = \mathrm{id}$), or pure spatial rotations ($a = 0$, $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$) is rather obvious. More interesting is how the energy-momentum distribution changes under pure boosts in, say, $e_1$-direction, where $a = 0$ and

$$(\Lambda^{\alpha}{}_{\beta}) = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.28}$$

This gives

$$\underline{x} = \Lambda^{-1} x = \left( \gamma (x^0 - \beta x^1), \gamma (x^1 - \beta x^0), x^2, x^3 \right), \tag{3.29}$$

$$\bar{T}^{00}(x) = \gamma^2 \left[ T^{00}(\underline{x}) + \beta^2 T^{11}(\underline{x}) + 2\beta T^{01}(\underline{x}) \right] \tag{3.30a}$$

$$\bar{T}^{11}(x) = \gamma^2 \left[ T^{11}(\underline{x}) + \beta^2 T^{00}(\underline{x}) + 2\beta T^{01}(\underline{x}) \right] \tag{3.30b}$$

$$\bar{T}^{01}(x) = \gamma^2 \left[ (1+\beta^2) T^{01}(\underline{x}) + \beta \left( T^{00}(\underline{x}) + T^{11}(\underline{x}) \right) \right] \tag{3.30c}$$

$$\bar{T}^{0n}(x) = \gamma \left[ T^{0n}(\underline{x}) + \beta T^{1n}(\underline{x}) \right] \tag{3.30d}$$

$$\bar{T}^{1n}(x) = \gamma \left[ T^{1n}(\underline{x}) + \beta T^{0n}(\underline{x}) \right] \tag{3.30e}$$

$$\bar{T}^{nm}(x) = T^{nm}(\underline{x}) \tag{3.30f}$$

Here $n$ and $m$ may assume the values 2 or 3.

Note that tensions, here $T^{11}$, i.e. in the direction of boost, contribute to the energy density and energy-current density / momentum-density of the boosted system. This means: A body under tension / pressure in motion has a larger energy / mass than the same body (same number of baryons) without tension / pressure.

We will learn more about EMTs on problem sheet 3.