Lecture 4: On the possibility of Poincaré invariant generalisations of Newtonian gravity

Newtonian gravity is characterised by its field equation

$$\Delta \phi = 4\pi G \rho \tag{4.1}$$

and the equation of motion for a test mass in the potential ϕ:

$$\ddot{\vec{x}}(t) = -\vec{\nabla}\phi(t, \vec{x}(t)) \tag{4.2}$$

Is there an obvious way to generalise these to a form compatible with SR? The answer is: "yes".

We first consider the field equation. Recall that

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4.3}$$

in affine coordinates of 3-space ($\cong \mathbb{R}^3$). The obvious replacement is

$$-\Delta \longmapsto \Box = \left(\frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right) \tag{4.4}$$

Then the left-hand side of (4.1) turns into $-\Box\phi$

Note that

$$\Box = \eta^{\alpha\beta}\, \nabla_\alpha \nabla_\beta \tag{4.5}$$

This operator – called the "wave operator" or "d'Alembert-Operator" maps scalar functions to scalar functions. This means the following: Let $\phi : M^4 \to R$ a scalar function for the Poincaré group which acts on M^4:

$$\left.\begin{aligned} \text{Poin} \times M^4 &\to M^4 \\ ((a,\Lambda), X) &\mapsto (a,\Lambda)\cdot X \\ &:= \Lambda X + a \end{aligned}\right\} \tag{4.6}$$

Then Poin also acts on $C^\infty(M^4,\mathbb{R})$:

$$\left.\begin{aligned} \text{Poin} \times C^\infty(M^4,R) &\to C^\infty(M^4,R) \\ ((a,\Lambda),\phi) &\to \phi' = T_{(a,\Lambda)}\,\phi \\ &:= \phi \circ (a,\Lambda)^{-1} \end{aligned}\right\} \tag{4.7}$$

That is:

$$\phi'(X) = (T_{(a,\Lambda)}\,\phi)(X) := \phi(\Lambda^{-1}(X-a)) \tag{4.8}$$

This action is called the scalar representation of Poin on $C^\infty(M^4, \mathbb{R})$. A real-valued function transforming under the scalar representation is called a <u>scalar function</u>

Lemma : The operator $\Box$ commutes with the scalar representation, i.e.,

$$T_{(a,\Lambda)} \circ \Box = \Box \circ T_{(a,\Lambda)} \qquad (4.9)$$

or $$T_{(a,\Lambda)} \circ \Box \circ T_{(a,\Lambda)}^{-1} = \Box \qquad (4.10)$$

Proof. Let $\phi \in C^\infty(M^4, \mathbb{R})$, then

$$(\Box \circ T_{(a,\Lambda)})(\phi) = \Box(T_{(a,\Lambda)}\phi)$$

$$= \Box\left(\phi \circ (a,\Lambda)^{-1}\right) \qquad (4.11)$$

Evaluated at $x \in M^4$

$$\Box\,(\phi \circ (a,\Lambda)^{-1})(x) = \eta^{\alpha\beta}\nabla_\alpha\nabla_\beta\, \phi(\Lambda^{-1}(x-a))$$

Chain rule $\longrightarrow$ $$= \eta^{\alpha\beta}\nabla_\alpha\left[(\Lambda^{-1})^\nu{}_\beta\,\nabla_\nu\,\phi(\Lambda^{-1}(x-a))\right]$$

$$= \eta^{\alpha\beta}(\Lambda^{-1})^\mu{}_\alpha(\Lambda^{-1})^\nu{}_\beta\,\nabla_\mu\nabla_\nu\,\phi\,(\Lambda^{-1}(x-a))$$

(4.12)

But $\Lambda \in$ Lor which is equivalent to

$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\mu} \qquad (4.13)$$

$\Longleftrightarrow$ (matrix - notation)

$$\eta = \Lambda^{T} \eta \Lambda$$

$$\Longleftrightarrow \quad \eta^{-1} = \Lambda^{-1} \eta^{-1} (\Lambda^{T})^{-1}$$

$$\Longleftrightarrow \quad \eta^{\mu\nu} = \eta^{\alpha\beta} (\Lambda^{-1})^{\mu}{}_{\alpha} (\Lambda^{-1})^{\nu}{}_{\beta} \qquad (4.14)$$

Hence we have shown on precious page

$$\Box(\phi \circ (a,\Lambda)^{-1})(x) = (\Box \phi)(\Lambda^{-1}(x-a))$$

$$\Longleftrightarrow \quad \Box(T(a,\Lambda)\phi) = T(a,\Lambda)(\Box\phi) \qquad (4.15)$$

Since this is true for all $C^{\infty}(M^4, \mathbb{R})$, statement (4.9) follows. ■

For our proposed differential equation

$$-\Box\phi = 4\pi G \underbrace{X}_{\text{replacing } \rho \text{ as source for } \phi} \qquad (4.16)$$

The requirement of Poincaré invariance for the proposed equation (4.16) is now as follows:

For any solution ϕ of (4.16) with source-function X the Poincaré transformed field $T(a,\Lambda)\,\phi$ satisfies the same equation with Poincaré-transformed source function $T(a,\Lambda)$.

Therefore, for any ϕ solving

$$-\Box\,\phi = 4\pi G\, X \qquad (4.16)$$

we require $T(a,\Lambda) = \phi \circ (a,\Lambda)^{-1}$ to solve

$$-\Box\,(T(a,\Lambda)\,\phi) = 4\pi G\; T(a,\Lambda)\,X\,.$$

By the Lemma the left-hand side equals $T(a,\Lambda)\,(-\Box\,\phi) = (-\Box\,\phi)\circ(a,\Lambda)^{-1}$ and since ϕ satisfies (4.16) equals $4\pi G\; X\circ(a,\Lambda)^{-1}$. Hence we must have that

$$T(a,\Lambda)\,X = X\circ(a,\Lambda)^{-1} \qquad (4.17)$$

or in other words: The source X must be a scalar function.

Now the mass density

$$\rho = T^{00} / c^2 \qquad (4.18)$$

is not a scalar function (see transformation law for $T^{\alpha\beta}$ in Lecture 3). So what should we replace it with? The obvious scalar formed by means of $T^{\alpha\beta}$ is its η-trace:

$$T := \eta_{\alpha\beta} T^{\alpha\beta} = T^{\alpha}{}_{\alpha}$$

$$= (\text{Trace}(T_\downarrow)).$$

$$= T^{00} - T^{11} - T^{22} - T^{33} \qquad (4.19)$$

For materials where pressure $= T^{aa}$ $\ll$ energy density $=$ (mass-density) $\times c^2$ $T^{\alpha}{}_{\alpha} / c^2$ is at least approximately equal to ρ. Hence we propose as special relativistic scalar field equation

$$\boxed{\Box \phi = -\frac{4\pi G}{c^2} T^{\alpha}{}_{\alpha}} \qquad (4.20)$$

An immediate consequence of (4.20) is its solution by retarded Green function:

$$\phi(t,\vec{x}) = -\frac{G}{c^2}\int_{\mathbb{R}^3} \frac{T^2_{\;2}\left(t-\frac{\|\vec{x}-\vec{x}'\|}{c}, \vec{x}'\right)}{\|\vec{x}-\vec{x}'\|}\, d^3x' \qquad (4.21)$$

and the existence of grav. waves as solutions to homogeneous eq.

$\Box\phi = 0$, e.g. plane waves

$$\phi(t,\vec{x}) = A\sin(\omega t - \vec{k}\cdot\vec{x})$$

$$\text{where } \omega^2/c^2 = \vec{k}^2. \qquad (4.22)$$

Next we need to generalise the equations of motion (4.2)

$$\ddot{\vec{x}}(t) = d^2\vec{x}(t)/dt^2 = -\vec{\nabla}\phi(t,\vec{x}(t)).$$

An apparently obvious guess is to just generalise this to the four-acceleration and the four-gradient:

$$\frac{d^2x^\mu(\tau)}{d\tau^2} = \nabla^\mu\phi(x(\tau)) = \eta^{\mu\nu}\frac{\partial\phi}{\partial x^\nu}(x(\tau)) \qquad (4.23)$$

where τ = Eigentime along curve (world line).

We simply write

$$\ddot{X}^{\mu}(\tau) = \nabla^{\mu}\phi(X(\tau)) \qquad (4.23)$$

where

$$d\tau = ds/c$$
$$= \frac{1}{c}\left[\eta_{\mu\nu}\frac{dX^{\mu}}{d\lambda}\frac{dX^{\nu}}{d\lambda}\right]^{1/2} d\lambda \qquad (4.24)$$

λ = any parameter along curve

But – unfortunately – (4.23) is total nonsense. Or more precisely: It is totally overdetermined so as to admit almost no solution. This is because in the transition from (4.2) to (4.23) we added one equation (μ runs from 0 to 3, so (4.23) are four equations, (4.2) are just three). To see this more precisely recall that four-accelerations are always perpendicular to four-velocities. This follows from

$$\eta_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = c^2 \qquad \Big| \frac{d}{d\tau}$$

$$\rightarrow 2\eta_{\mu\nu}\dot{X}^{\mu}\ddot{X}^{\nu} = 0. \qquad (4.25)$$

Hence, multiplying (4.23) with $\dot{X}_\mu$ we get zero on left-hand side and therefore

$$\dot{X}^\mu \nabla_\mu \phi(X(\tau)) = 0 \qquad (4.26)$$

for any ϕ admitting a solution to (4.23). But this says that

$$\frac{d}{d\tau} \phi(X(\tau)) = 0 \qquad (4.27)$$

Hence (4.23) only admits solutions if ϕ is constant along $X(\tau)$, ie. precisely in those trivial cases we are <u>not</u> interested in.

How shall we proceed? Our method will be this: Rather than launching more or less random guesses, we employ the full aparatus of Lagrangian field theory to systematically guide us to the "right" equations of motion.
The first step in this procedure is to find the right action-functional for the field equation (4.20)

We wish to find a functional of ϕ and T^α_α that gives us (4.20) as Euler Lagrange Equation

$$S[\phi] = c \int_{\mathbb{R}^4} d^4x \, \underbrace{\mathcal{L}(\phi(x), T)}_{\text{Lagrange density}} \qquad (4.28)$$

$$\frac{\delta S}{\delta \phi(x)} = 0 \quad \text{(Variational eq.)} \qquad (4.29)$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left[\frac{\partial \mathcal{L}}{\partial (\nabla_\mu \phi)} \right] = 0} \qquad (4.30)$$

Euler - Lagrange - Eqns.

This equation determines $\mathcal{L}$, the Lagrange density only up to a multiplicative constant and up to an addition divergence $(\nabla_\alpha V^\alpha)$. Ignoring the latter, $\mathcal{L}$ is easy to guess

$$\mathcal{L} = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \kappa T^\alpha_\alpha \phi \qquad (4.31)$$

$$\text{where } \kappa = \frac{4\pi G}{c^2} \qquad (4.32)$$

Indeed:

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= -\kappa T^{\alpha}{}_{\alpha} \\ \frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} &= \nabla^\mu \phi \end{aligned}\right\} \qquad (4.33)$$

Hence

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial \nabla_\mu \phi} \right)$$

$$= -\kappa T^{\alpha}{}_{\alpha} - \Box \phi$$

$$= 0 \iff \Box \phi = -\kappa T^{\alpha}{}_{\alpha}$$

$$\iff (4.20). \qquad (4.34)$$

The multiplicative factor c in front of the integral (4.28) must be chosen such that 1) S has the physical dimension of an action (Energy × Time) and 2) that

$$c \int_{\mathbb{R}^3} \mathcal{L}\, d^3x = \text{kin. Energy} - \text{pot. Energy} \qquad (4.35)$$

so that

$$S = \int dt\, (E_{kin} - E_{pot}). \qquad (4.36)$$

We claim that this is satisfied if

$$C = \frac{1}{Kc^3} = \frac{1}{4\pi G c} \qquad (4.37)$$

1.) physical dimension

$$[\phi] = [\phi_{Newton}] = m^2 \cdot s^{-2}$$

$$[\nabla \phi] = m \cdot s^{-2}$$

$$[T] = Kg \cdot m^2 \cdot s^{-2} \cdot m^{-3} = Kg \cdot m^{-1} \cdot s^{-2}$$

$$[\phi T] = Kg \cdot m \cdot s^{-4}$$

$$[K] = \left[\frac{G}{c^2}\right] = m^3 \cdot Kg^{-1} \cdot s^{-2} \cdot m^{-2} \cdot s^2$$

$$= m \cdot Kg^{-1}$$

$$[K\phi T] = m^2 \cdot s^{-4}$$

$$[(\nabla \phi)^2] = m^2 \cdot s^{-4}$$

$$[d^4x] = m^4$$

$$[d^4x\,(\tfrac{1}{2}(\nabla\phi)^2 - K\phi T)] = m^6 \cdot s^{-4}$$

$$[1/Kc^3] = Kg \cdot m^{-1} \cdot m^{-3} \cdot s^3$$

$$= Kg \cdot m^{-4}\, s^3$$

$$\Rightarrow \left[\frac{d^4x\,(\frac{1}{2}(\nabla\phi)^2 - K\phi T)}{Kc^3}\right] = Kg \cdot m^2 \cdot s^{-1}$$

$$= J \cdot s \qquad (4.38)$$

2) Numerical prefactor and sign

$$\nabla_\mu \phi \, \nabla^\mu \phi = \eta^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \phi$$

$$= \frac{\partial^2 \phi}{\partial x^{0\,2}} - (\vec{\nabla}\phi)^2 = \frac{1}{c^2}\dot{\phi}^2 - (\vec{\nabla}\phi)^2 \qquad (4.39)$$

where $x^0 = ct$ and $\dot{\phi} := \partial\phi/\partial t$.

Hence

$$S = \frac{1}{2\kappa c^3} \int c\, dt \wedge d^3x \, \Big\{$$

$$\frac{1}{c^2}\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - 2\kappa\phi\,(T^{00} + T^a{}_a)\Big\} \qquad (4.40)$$

We evaluate this in the Newtonian case where

$$|T^a{}_a| \ll T^{00} = c^2 \rho \qquad (4.41)$$

Then S has the form

$$S = \int dt \;\; L \qquad (4.42)$$

with

$$L = \int d^3x \left\{ \frac{1}{2\kappa c^4}\dot{\phi}^2 - \frac{1}{2\kappa c^2}(\vec{\nabla}\phi)^2 - \phi\rho \right\}$$

$$= E_{kin} - E_{pot} \qquad (4.43)$$

$$E_{kin} = \frac{1}{2\kappa c^4} \int d^3x \, \dot{\phi}^2 \qquad (4.44)$$

$$E_{pot} = \int d^3x \left\{ \frac{1}{2\kappa c^2} (\vec{\nabla}\phi)^2 + \phi \varrho \right\} \qquad (4.45)$$

Note that because of Gauss' theorem and $\Delta \phi = 4\pi \varrho$ we have

$$\int d^3x \quad \phi \varrho = \frac{1}{4\pi G} \int d^3x \quad \phi \Delta \phi$$

$$= \frac{1}{4\pi G} \underbrace{\int_{S^2(\infty)}}_{\substack{\text{2-Sphere of} \\ \text{radius } \infty}} \phi \, (\vec{\nabla}\phi) \cdot \vec{n} \, d\Omega - \frac{1}{4\pi G} \int d^3x (\vec{\nabla}\phi)^2$$

$$= - \frac{1}{4\pi G} \int d^3x (\vec{\nabla}\phi)^2 \qquad (4.46)$$

because the surface-integral vanishes due to $\phi \, (r \to \infty) \sim \frac{1}{r}$, $\vec{\nabla}\phi \, (r\to\infty) \sim \frac{1}{r^2}$, hence $\phi \vec{\nabla}\phi \cdot \vec{n} \; (r \to \infty) \sim r^{-3}$.
Therefore, since $1/2\kappa c^2 = 1/8\pi G$, we have

$$E_{pot} = - \frac{1}{8\pi G} \int_{\mathbb{R}^3} d^3x \, \| \vec{\nabla}\phi \|^2$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} d^3x \quad \phi \, \varrho \qquad (4.47)$$

This is indeed the Newtonian pot. Energy!
Compare Exercise 4 on Sheet 1.

So we have found the right action for the scalar gravitational field:

$$S_g = \frac{1}{\kappa c^3} \int d^4x \left[\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \kappa T^\alpha{}_\alpha \phi \right] \quad \text{where } \kappa = \frac{4\pi G}{c^2} \tag{4.48}$$

In order to find the equation of motion for a point particle we have to couple it to the gravitational field. But how? Well, (4.48) already tells us how matter, here represented by its energy-momentum tensor $T^{\alpha\beta}$ couples to ϕ: Through the term $-\kappa T^\alpha{}_\alpha \phi$! This shall be our guiding principle: All matter, not just the one that sources the gravitational field, but also the one that just moves in an external gravitational field, couples to gravity in the very same way

For a point particle of rest-mass m, moving along a world-line $z(\tau)$, the energy momentum tensor is given by $T_p^{\alpha\beta}$ that reads as follows:

for "particle"

$$T_p^{\alpha\beta}(x) = mc \int \dot{z}^\alpha(\tau)\,\dot{z}^\beta(\tau)\,\delta^{(4)}(x - z(\tau))\,d\tau \tag{4.49}$$

Here $\delta^{(4)}$ is the 4-dim. Dirac-delta distribution. (We will learn more about this expression on Sheet 3 of the exercises). Note the right physical dimension:

$$[T_p^{\alpha\beta}(x)] = kg \cdot m \cdot s^{-1} \cdot m^2 \cdot s^{-2} \cdot m^{-4} \cdot s$$

$$= kg\, m^2 \cdot s^{-2} \cdot m^{-3} = J \cdot m^{-3}$$

i.e. that of an energy-density

Now, since $\eta_{\alpha\beta}\,\dot{z}^\alpha \dot{z}^\beta = c^2$, the trace is

$$T_{p\,\alpha}^{\alpha}(x) = mc^3 \int \delta(x - z(\tau))\,d\tau$$

and hence the coupling in the action

$$S_{coupl.} = \frac{1}{\kappa c^3} \int -\kappa\, T_{p\,\alpha}^{\alpha}(x)\,\phi(x)\,d^4x$$

$$= -m \iint \phi(x)\,\delta^{(4)}(x - z(\tau))\,d^4x\,d\tau$$

$$= -m \int \phi(z(\tau))\,d\tau \tag{4.50}$$

To that we have to add the action for the free (relativistic) particle:

$$S_p^{(free)} := -mc^2 \int d\tau \qquad (4.51)$$

Note the right sign: For any parameter λ have

$$d\tau = \frac{1}{c}\left(\eta_{\alpha\beta}\frac{dz^\alpha}{d\lambda}\frac{dz^\beta}{d\lambda}\right)^{1/2} d\lambda \qquad (4.52)$$

For $\lambda = t$ have

$$d\tau = \frac{1}{c}\left(c^2 - v^2\right)^{1/2} dt$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt$$

$$\cong \left[1 - \frac{1}{2}\frac{v^2}{c^2} + O\left(\frac{v^4}{c^4}\right)\right] dt \qquad (4.53)$$

so that in leading order approximation in (v/c) we just get the Newtonian kinetic energy, as it must be

$$\left.\begin{aligned} S_p^{free} &= \int \frac{1}{2} m v^2 \, dt + \text{const} \\ &\quad + \text{Terms} \int O\left(\frac{v^4}{c^4}\right) dt \end{aligned}\right\} \quad (4.54)$$

right sign!

The total action for the particle in the gravitational field is the sum of its free part and its coupling to ϕ:

$$S_p = S_p^{free} + S_{coupling}$$

$$= -mc^2 \int d\tau \left(1 + \phi(z(\tau))/c^2\right) \qquad (4.55)$$

This action determines the equation of motion for the particle. In order to apply the Euler-Lagrange equation, we have to make the dependence of the Eigentime-intervall $d\tau$ on the path of the particle explicit. This means that we have to rewrite the integral in the action in terms of a parameter λ that is independent of $z(\tau)$:

$$d\tau = \frac{1}{c} \left(\eta_{\alpha\beta}\, z'^{\alpha}(\lambda)\, z'^{\beta}(\lambda)\right)^{1/2} d\lambda \qquad (4.52)$$

$$\text{where } z'^{\alpha}(\lambda) := \frac{dz^{\alpha}(\lambda)}{d\lambda} \qquad (4.56)$$

$$S_p = -mc \int \left(z'^{\alpha}(\lambda)\, z'_{\alpha}(\lambda)\right)^{1/2} \left(1 + \frac{\phi(z(\lambda))}{c^2}\right) d\lambda \qquad (4.57)$$

or

$$S_P = \int L(z(\lambda), z'(\lambda))\, d\lambda \qquad (4.58)$$

with Lagrange-function

$$L(z(\lambda), z'(\lambda)) = -mc\left(z'^{\alpha}(\lambda)\, z'_{\alpha}(\lambda)\right)^{1/2} \times \left[1 + \frac{\phi(z(\lambda))}{c^2}\right] \qquad (4.59)$$

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial z^{\alpha}(\lambda)} - \frac{d}{d\lambda}\left[\frac{\partial L}{\partial z'^{\alpha}(\lambda)}\right] = 0 \qquad (4.60)$$

After we have performed the differentiations w.r.t. z'^{α} we may re-introduce the parameter τ for notational convenience (we just needed to use λ to display the z'-dependence of $d\tau$). In what follows we write

$$(\cdot\cdot)^{1/2} \text{ for } \left(z'^{\alpha} z'_{\alpha}\right)^{1/2} \qquad (4.61)$$

so that

$$\frac{d}{d\lambda} = \frac{1}{c}(\cdot\cdot)^{1/2}\frac{d}{d\tau} \qquad (4.62)$$

Now

$$\frac{\partial L}{\partial Z^\alpha} = -\frac{m}{c}(\dots)^{1/2}\,\nabla_\alpha\phi \qquad (4.63)$$

$$\frac{\partial L}{\partial Z'^\alpha} = -mc\left(1+\frac{\phi}{c^2}\right)\frac{Z'_\alpha}{(\dots)^{1/2}}$$

$$= -m\left(1+\frac{\phi}{c^2}\right)\dot{Z}_\alpha \qquad (4.64)$$

where $\dot{Z}_\alpha := dZ_\alpha/d\tau$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial Z'^\alpha}\right) = \frac{1}{c}(\dots)^{1/2}\,\frac{d}{d\tau}\left[-m\left(1+\frac{\phi}{c^2}\right)\dot{Z}_\alpha\right]$$

$$= -\frac{m}{c}(\dots)^{1/2}\left[\left(1+\frac{\phi}{c^2}\right)\ddot{Z}_\alpha + \frac{\dot{Z}_\alpha\dot{Z}^\beta}{c^2}\nabla_\beta\phi\right] \qquad (4.65)$$

Here we used $d\,\phi(Z(\tau))/d\tau = \nabla_\beta\phi\,\dot{Z}^\beta$.

Hence the full Euler-Lagrange equations are

$$\frac{\partial L}{\partial Z^\alpha} - \frac{d}{d\lambda}\left[\frac{\partial L}{\partial Z'^\alpha}\right] = \frac{m}{c}(\dots)^{1/2}\left(1+\frac{\phi}{c^2}\right)$$

$$\cdot\left\{\ddot{Z}_\alpha - \underbrace{\frac{\nabla_\beta\phi}{1+\phi/c^2}}_{c^2\nabla_\beta\left[\ln(1+\phi/c^2)\right]}\underbrace{\left(\delta^\beta_\alpha - \frac{\dot{Z}^\beta\dot{Z}_\alpha}{c^2}\right)}_{=:P^\beta_\alpha}\right\} = 0 \qquad (4.66)$$

Euler Lagrange equation:

$$\ddot{z}^\alpha = P^{\alpha\beta} \nabla_\beta \left(c^2 \ln(1+\phi/c^2)\right)$$

$$= P^{\alpha\beta} \nabla_\beta \psi \qquad (4.67)$$

where $\psi := c^2 \ln(1+\phi/c^2)$ (4.68)

and $P^{\alpha\beta} := \left(\eta^{\alpha\beta} - \frac{\dot{z}^\alpha \dot{z}^\beta}{c^2}\right)$ (4.69)

Note that

$$\left.\begin{aligned} &P^\alpha{}_\beta \dot{z}^\beta = 0 \\ &P^\alpha{}_\beta V^\beta = V^\alpha \quad \text{if} \quad V \perp \dot{z} \\ &P^\alpha{}_\gamma P^\gamma{}_\beta = P^\alpha{}_\beta \end{aligned}\right\} (4.70)$$

that is: $P^\alpha{}_\beta$ are the components of the η-orthogonal projection onto

$$\dot{z}^\perp := \{ v \in V : \eta(v, \dot{z}) = 0 \} \qquad (4.71)$$

Note that it is precisely this projector that renders (4.67) meaningful, in contrast to (4.23), which is over-determined. Here we do not have that problem from $\dot{z}_\alpha \ddot{z}_\alpha \equiv 0$ since also $\dot{z}_\alpha P^{\alpha\beta} \nabla_\beta \phi \equiv 0$.

We could have guessed that simple "projection - solution" just after (4.23) had been written down and the $\dot{x}$ - orthogonality of the lhs. had been noticed. But that would have led to

$$\ddot{z}^{\alpha} = P_{\alpha}^{\beta} \nabla_{\beta} \phi \tag{4.72}$$

whereas we have found

$$\ddot{z}^{\alpha} = P^{\alpha\beta} \nabla_{\beta} [c^2 \ln (1 + \phi / c^2)] \tag{4.73}$$

For $|\phi| \ll c^2$ this is approximately the same, but not for "strong" field strengths. Clearly, we could just redefine (i.e. rename) the scalar field

$$\psi := c^2 \ln (1 + \phi / c^2) \tag{4.74}$$

Then

$$\ddot{z}^{\alpha} = P^{\alpha\beta} \nabla_{\beta} \psi \tag{4.75}$$

But then the field equation for ϕ

$$\Box \phi = - \kappa T^{\alpha}{}_{\alpha}$$

with $\quad \phi = c^2 \left(\exp (\psi / c^2) - 1\right) \quad (4.76)$

and $\quad \nabla \phi = \nabla \psi \exp(\cdot\cdot)$

$$\Box \phi = [\Box \psi + (\nabla \psi)^2 / c^2] \exp(\psi / c^2)$$

reads

$$\Box \psi + \frac{1}{c^2} \nabla^\alpha \psi \nabla_\alpha \psi = -\kappa \exp(-\psi/c^2) T^\alpha{}_\alpha \qquad (4.77)$$

We conclude the existence of a reasonable possibility to generalize Newtonian gravity to a Special Relativistic – i.e. Poincaré invariant – theory which we can either present in the form

$$\boxed{\begin{aligned} \Box \phi &= -\kappa T^\alpha{}_\alpha \\ \ddot{z}^\alpha &= P^{\alpha\beta} \nabla_\beta \left[c^2 \ln(1 + \phi/c^2) \right] \end{aligned}} \qquad (4.78)$$

or in the form

$$\boxed{\begin{aligned} \Box \psi + \frac{1}{c^2} \nabla^\alpha \psi \nabla_\alpha \psi &= -\kappa \exp(-\psi/c^2) T^\alpha{}_\alpha \\ \ddot{z}^\alpha &= P^{\alpha\beta} \nabla_\beta \psi \end{aligned}} \qquad (4.79)$$

where $\kappa := \frac{4\pi G}{c^2}$

‖ But: Is that theory compatible with observations? ‖

A first general observation is this: Gravity couples to systems only via the <u>trace</u> of the energy-momentum tensor. This means that systems the energy-momentum tensor of which has vanishing trace do not at all couple to gravity: They do not produce any gravitational fields, nor is their dynamics affected by external grav. fields. Now, there is a general statement from (quantum-) field theory that fundamental fields whose mediating particle (Boson) is massless (i.e. long-ranging fields) has a traceless T. Examples are electromagnetism and Yang-Mills theory (describing strong interaction).

Thus special-relativistic scalar gravity will not describe gravitational redshift and not any deflection of light by gravitational fields.

What about compatibility with observed planetary motion? Can scalar gravity account for the full perihelion precession of Mercury?

We wish to calculate the periastron precession of this theory. The gravitational potential is that outside a static, spherically symmetric stress-energy distribution. Then outside the star, we have

$$\phi(r) = -\frac{GM}{r} \qquad (4.80)$$

where $M = \frac{1}{c^2}\int d^3x \, T^{\alpha}{}_{\alpha}$

$$= \frac{1}{4\pi G}\int_{S^2(\infty)} (\vec{\nabla}\phi)\cdot\vec{n}\, d\Omega . \qquad (4.81)$$

Hence

$$\psi = c^2 \ln(1+\phi/c^2)$$
$$= c^2 \ln\left(1-\frac{GM}{c^2 r}\right) \qquad (4.82)$$

This becomes singular for

$$r \leq \frac{GM}{c^2}$$

so we assume the star's radius R to be larger than that critical Radius

$$R > GM/c^2 \qquad (4.83)$$

For us ψ will be easier to work with than ϕ since the equations of motion for ψ are simpler:

$$\ddot{z}^{\alpha} = P^{\alpha\beta} \nabla_\beta \psi \tag{4.84}$$

We now want to evaluate them:

Staticity implies

$$\nabla_0 \psi = 0 \tag{4.85}$$

In components

$$\dot{z}^{\mu} = \frac{d}{d\tau} z^{\mu} = c\gamma\,(1, \vec{\beta}) \tag{4.86}$$

where
$$\vec{\beta} = \frac{1}{c}\frac{d\vec{z}}{dt} = \frac{\vec{v}}{c} \tag{4.87}$$

and
$$\gamma = (1-\vec{\beta}^2)^{-1/2} \tag{4.88}$$

Also
$$\frac{d}{d\tau} = \gamma \frac{d}{dt} \tag{4.89}$$

Hence
$$\frac{d\gamma}{dt} = \frac{d}{dt}(1-\vec{\beta}^2)^{-1/2} = \gamma^3 \frac{\vec{a}\cdot\vec{v}}{c^2} \tag{4.90}$$

where
$$\vec{a} := \frac{d\vec{v}}{dt} \tag{4.91}$$

Now

$$\ddot{z}^\alpha = \frac{d}{d\tau} \dot{z}^\alpha = \gamma \frac{d}{dt}\, c\gamma (1, \vec{\beta})$$

$$= \gamma c\, \gamma^3 \frac{\vec{a}\cdot\vec{v}}{c^2} (1, \vec{\beta})$$

$$+ \gamma^2 c\, (0, \vec{a}/c)$$

$$= \gamma^4 \left(\vec{a}\cdot\vec{\beta},\ (\vec{a}\cdot\vec{\beta})\vec{\beta} + \gamma^{-2}\vec{a} \right)$$

$$= \gamma^4 \left(\vec{a}\cdot\vec{\beta},\ \vec{a}_\parallel + \gamma^{-2}\vec{a}_\perp \right) \qquad (4.92)$$

where

$$\vec{a}_\parallel := \frac{(\vec{a}\cdot\vec{\beta})\vec{\beta}}{\beta^2} \qquad (4.93)$$

$$\vec{a}_\perp := \vec{a} - \vec{a}_\parallel \qquad (4.93)$$

so that $\vec{a} = \vec{a}_\parallel + \vec{a}_\perp$ ($\parallel$ and $\perp$ to $\vec{\beta} = \vec{v}/c$), and where we used $\gamma^{-2} = 1 - \beta^2$

Equation (4.83) gives the left-hand side of (4.75) in terms of 3-dimensional velocities and accelerations $\vec{v}, \vec{a}$.

Next we do the same for the right-hand side of (4.75):

$$\left.\begin{aligned} P^{\alpha\beta}\,\nabla_\beta\,\psi &= \left(\eta^{\alpha\beta} - \frac{\dot{z}^\alpha\,\dot{z}^\beta}{c^2}\right)\nabla_\beta\,\psi \\ \text{where } \dot{z}^\alpha &= c\gamma\,(1, \vec{\beta}) \end{aligned}\right\} \quad (4.94)$$

<u>Set $\alpha = 0$</u>

$$P^{0\beta}\,\nabla_\beta\,\psi = \left[\eta^{0\beta} - \frac{\dot{z}^0\,\dot{z}^\beta}{c^2}\right]\nabla_\beta\,\psi$$

$$= \left[\eta^{0n} - \frac{\dot{z}^0\,\dot{z}^n}{c^2}\right]\nabla_n\,\psi$$

$$(\text{since } \nabla_0\,\psi = 0)$$

$$= -\gamma^2\,(\vec{\beta}\cdot\vec{\nabla})\,\psi \qquad (4.95)$$

<u>Set $\alpha = a$</u>

$$P^{a\beta}\,\nabla_\beta\,\psi = P^{ab}\,\nabla_b\,\psi \qquad (\nabla_0\,\psi = 0)$$

$$= \left(\eta^{ab} - \frac{\dot{z}^a\,\dot{z}^b}{c^2}\right)\nabla_b\,\psi$$

$$= \nabla^a\,\psi - \gamma^2\,\beta^a\,(\vec{\beta}\cdot\vec{\nabla})\,\psi$$

$$= \left(-\vec{\nabla}\,\psi - \gamma^2\,\beta^2\,\vec{\nabla}_{\parallel}\,\psi\right)^a \qquad (\gamma^2\beta^2 = \gamma^2 - 1)$$

$$= \left[-\vec{\nabla}_{\perp}\,\psi - \gamma^2\,\vec{\nabla}_{\parallel}\,\psi\right]^a \qquad (4.96)$$

where

$$\vec{\nabla}_{\parallel} = \frac{\vec{\beta}\,(\vec{\beta}\cdot\vec{\nabla})}{\beta^2} \qquad (4.97)$$

$$\vec{\nabla}_{\perp} = \vec{\nabla} - \vec{\nabla}_{\parallel} \qquad (4.98)$$

so that $\vec{\nabla} = \vec{\nabla}_{\parallel} + \vec{\nabla}_{\perp}$ (4.99)

and where $\nabla^a = \eta^{ab}\nabla_b = -\delta^{ab}\nabla_b$
$= (-\vec{\nabla})^a$.

Hence

$$P^{\alpha\beta}\nabla_\beta\psi = \left(-\gamma^2(\vec{\beta}\cdot\vec{\nabla})\psi\,,\right.$$

$$\left. -\vec{\nabla}_{\perp}\psi - \gamma^2\vec{\nabla}_{\parallel}\psi\right) \qquad (4.100)$$

Equality to (4.92) is equivalent to

$$\left.\begin{aligned} \vec{a}\cdot\vec{\beta} &= -\gamma^{-2}(\vec{\beta}\cdot\vec{\nabla})\psi \\ \vec{a}_{\parallel} &= -\gamma^{-2}\vec{\nabla}_{\parallel}\psi \\ \vec{a}_{\perp} &= -\gamma^{-2}\nabla_{\perp}\psi \end{aligned}\right\} \qquad (4.101)$$

$$\Leftrightarrow \boxed{\vec{a} = \frac{d^2\vec{z}}{dt^2} = -\gamma^{-2}\vec{\nabla}\psi} \qquad (4.102)$$

The full content of (4.84) is therefore contained in (4.102), which looks like a Newtonian equation for the gravitational potential φ except for the occurence of γ^{-2}

$$\gamma^{-2} = 1 - \frac{1}{c^2}\left(\frac{d\vec{Z}}{dt}\right)^2 = 1 - \vec{Z}'^2/c^2 \qquad (4.103)$$

The integration of (4.102) works as in Newtonian theory via a conservation law

$$\frac{\vec{Z}''}{1-\vec{Z}'^2/c^2} = -\vec{\nabla}\varphi(\vec{Z}(t)) \qquad \Big|\cdot \vec{Z}'$$

$$\underbrace{\frac{\vec{Z}''\cdot\vec{Z}'}{1-\vec{Z}'^2/c^2}}_{\underbrace{-\frac{1}{2}c^2\left[\ln\left(1-\vec{Z}'/c^2\right)\right]'}_{c^2(\ln\gamma)'}} = -(\vec{Z}'\cdot\vec{\nabla}\varphi) = -\varphi'$$

$$\Leftrightarrow \quad \underbrace{\left(\ln(\gamma) + \varphi/c^2\right)}_{\text{is constant along solution curve}}{}' = 0 \qquad (4.104)$$

$$\Rightarrow \gamma(z(t)) = \gamma_0 \exp(-\psi(z(t))/c^2) \qquad (4.104)$$

where γ_0 is a constant

This we insert back into (4.102), i.e.

$$\gamma^{-2}(z(t)) = \gamma_0^{-2} \exp(2\psi(z(t))/c^2)$$

Hence

$$\vec{z}''(t) = -\gamma_0^{-2} \exp(2\psi(z(t))/c^2)\, \vec{\nabla}\psi(z(t))$$

$$= -\vec{\nabla}\Phi(z(t)) \qquad (4.105)$$

where

$$\Phi = \frac{c^2}{2}\gamma_0^{-2} \exp(2\psi/c^2) \qquad (4.106)$$

Recall (4.74): $\psi = c^2 \ln(1+\phi/c^2)$; hence

$$\Phi = \frac{c^2}{2}\gamma_0^{-2}(1+\phi/c^2)^2 \qquad (4.106)$$

So we arrive at our equation of motion

$$\left.\begin{aligned}\frac{d^2\vec{r}}{dt^2} &= -\vec{\nabla}\Phi \\ \Phi &= \frac{c^2}{2}\gamma_0^{-2}\left(1+\phi/c^2\right)^2 \\ &= \frac{c^2}{2}\gamma_0^{-2}\left(1-\frac{GM}{c^2 r}\right)^2 \\ &= k - \frac{\alpha}{r} + \frac{\delta_2}{r^2}\end{aligned}\right\} \quad (4.107)$$

where

$$\left.\begin{aligned} k &= \frac{c^2}{2}\gamma_0^{-2} \\ \alpha &= \gamma_0^{-2}\, GM \\ \delta_2 &= \gamma_0^{-2}\, GM\, \frac{GM}{2c^2} \\ &= \alpha\, \frac{GM}{2c^2}\end{aligned}\right\} \quad (4.108)$$

If we now apply the result from problem sheet 2, exercise 1, we get the following expression for the periastron shift per revolution:

$$\Delta\varphi = -2\pi \frac{\delta_2/\alpha}{a(1-\varepsilon^2)}$$

$$= -\pi \frac{GM/c^2}{a(1-\varepsilon^2)} \qquad (4.109)$$

retrograde !

The value predicted by GR is

$$(\Delta\varphi)_{GR} = 6\pi \frac{GM/c^2}{a(1-\varepsilon^2)} \qquad (4.110)$$

Hence

$$(\Delta\varphi)_{scalar} = -\frac{1}{6}(\Delta\varphi)_{GR} \qquad (4.111)$$