Lecture 5

The principle of universality of free fall defines a path structure on spacetime. In affine Minkowski space the set of paths that realise force-free motion, also called inertial motion, are straight lines. The notion of "straight" is well defined in an affine space and is algebraically characterised by linear inhomogeneous equations of the form

$$x^\alpha(\lambda) = v^\alpha \lambda + a^\alpha \qquad (5.1)$$

where λ is an arbitrary parameter and v^α, a^α are the components of fixed vectors in V (associated to M^4).

The important thing to notice here is that (5.1) is valid only in special, so called affine bases, or affine charts.

We recall the definition of an affine chart and first that of affine space:
An affine space is a triple $(M, V, +)$ where M is a set, V a vector space, and "+" stands for a simply transitive action of V, considered as an abelian group,

$$V \times M \longrightarrow M$$

$$(v, m) \longmapsto m + v \qquad (5.2)$$

such that (action)

$$(m+v) + w = m + (v+w) \qquad (5.3)$$

action on M

addition in V

Simple transitivity means that for each pair of points $(p, q) \in M \times M$ there is a unique $v \in V$ such that

$$q = p + v$$

We then write

$$q - p = v \qquad (5.4)$$

i.e. "$q-p$" is that unique element of V which, when acting upon p, results in q.

An affine basis B is a pair

$$B = (O, b) \qquad (5.5)$$

where $O \in M$ and $b = \{e_1, \dots, e_n\}$ is a basis of V. Each affine basis B defines a bijection $\phi_B : M \to \mathbb{R}^n$ as follows, called an <u>affine chart</u>:

$$\left.\begin{aligned} \phi_B : \; & M \to \mathbb{R}^n \\ & p \mapsto (\theta^1(p-o), \dots, \theta^n(p-o)) \end{aligned}\right\} \quad (5.6)$$

Where $\{\theta^1, \dots, \theta^n\}$ is the dual basis to $\{e_1, \dots, e_n\}$ that is part of B.
The inverse map is

$$\left.\begin{aligned} \phi_B^{-1} : \; & \mathbb{R}^n \to M \\ (x^1 \dots, x^n) & \mapsto o + x^a e_a \end{aligned}\right\} \quad (5.7)$$

Given two affine bases B and B' and their corresponding affine charts ϕ_B and $\phi_{B'}$, then the transition functions

$$\phi_{BB'} := \phi_B \circ \phi_{B'}^{-1} : \mathbb{R}^n \to \mathbb{R}^n \quad (5.8)$$

are given by

$$\left.\begin{aligned} & x^a \mapsto R^a{}_b x^b + b^a \\ & \text{where} \quad e'_a = R^b{}_a e_b \\ & \text{and} \quad o' - o = b^a e_a \end{aligned}\right\} \quad (5.9)$$

The image of a straight line in M under an affine chart becomes a straight line in $\mathbb{R}^n$, and vice versa.

In practice we recognise straight lines on M by their straight images under a chart that we <u>assume</u> to be an affine chart. The task is to find an intrinsic characterisation that makes no reference to specially selected charts.

For this we note that straight-lines in Minkowski space are geodesics of the Minkowski metric η, where now we consider (M, η) as a Lorentzian Manifold

$$\eta = \eta_{\mu\nu} \, dx^\mu \otimes dx^\nu \qquad (5.10)$$

η is a non-degenerate, symmetric bilinear form of signature $(1,-1,-1,-1)$

Geodesics are stationary points of the "energy-functional" for curves

$$\left.\begin{aligned} \gamma : \mathbb{R} &\to M \\ \lambda &\mapsto \gamma(\lambda) \end{aligned}\right\} (5.11)$$

$$E[\gamma] = \frac{1}{2} \int \eta(\dot{\gamma}, \dot{\gamma})\, d\lambda \qquad (5.12)$$

between fixed end points.

Let $X^\mu : M \supset \mathcal{U} \to \mathbb{R}$ be the coordinate functions of some local chart the domain of which contains the path, i.e. the image of the map γ (the "curve"), then we write

$$Z^\alpha := X^\alpha \circ \gamma . \qquad (5.13)$$

then $E[\gamma]$ turns into a functional $E[Z(\lambda)]$ which reads

$$E[Z(\lambda)] = \frac{1}{2} \int d\lambda\ \eta_{\alpha\beta}(Z(\lambda))\, \dot{Z}^\alpha(\lambda)\, \dot{Z}^\beta(\lambda)$$

$$= \int d\lambda\ L\,(Z(\lambda), \dot{Z}(\lambda)) \qquad (5.14)$$

with Lagrange-function

$$L = \frac{1}{2}\ \eta_{\alpha\beta}(Z(\lambda))\, \dot{Z}^\alpha(\lambda)\, \dot{Z}^\beta(\lambda) \qquad (5.15)$$

Note that the koefficients $\eta_{\alpha\beta}$ of the Minkowski metric depend on the point in the chart image, i.e. the X^α (considered as $\in \mathcal{U} \subset \mathbb{R}^n$). Only in affine charts are these coefficients constant.

Indeed, if

$$\eta = \eta_{\alpha\beta}\, dx^\alpha \otimes dx^\beta = \bar{\eta}_{\mu\nu}\, d\bar{x}^\mu \otimes d\bar{x}^\nu \tag{5.16}$$

and $\bar{\eta}_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ because $\bar{x}^\mu$ are affine chart maps, then

$$\eta_{\alpha\beta}(x) = \bar{\eta}_{\mu\nu} \frac{\partial \bar{x}^\mu(x)}{\partial x^\alpha} \frac{\partial \bar{x}^\nu(x)}{\partial x^\beta} \tag{5.17}$$

where the Jacobi-matrices

$$J^\mu{}_\alpha(x) := \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \tag{5.18}$$

are just the derivatives of the transition maps

$$\bar{\phi} \circ \phi^{-1} : \mathbb{R}^n \supset \mathcal{U} \to \mathbb{R}^n \tag{5.19}$$

any chart

affine chart

which depend on x = point in $\mathcal{U} \subset \mathbb{R}^n$ ($\mathcal{U}$ = image of ϕ). It is not difficult to prove from (5.17) that $\eta_{\alpha\beta}$ is x-independent iff $\bar{x}$ is a linear inhomogeneous function of x.

Proof. Let $\frac{\partial}{\partial x^\gamma} f = f_{,\gamma}$, then

$$\eta_{\alpha\beta} = \bar{\eta}_{\mu\nu}\, \gamma^{\mu}_{\ \alpha}\, \gamma^{\nu}_{\ \beta} \qquad (5.20)$$

$$\left.\begin{aligned}
\eta_{\alpha\beta,\gamma} &= \bar{\eta}_{\mu\nu}\left(\gamma^{\mu}_{\ \alpha,\gamma}\, \gamma^{\nu}_{\ \beta} + \gamma^{\mu}_{\ \alpha}\, \gamma^{\nu}_{\ \beta,\gamma}\right)\\
\eta_{\beta\gamma,\alpha} &= \bar{\eta}_{\mu\nu}\left(\gamma^{\mu}_{\ \beta,\alpha}\, \gamma^{\nu}_{\ \gamma} + \gamma^{\mu}_{\ \beta}\, \gamma^{\nu}_{\ \gamma,\alpha}\right)\\
\eta_{\gamma\alpha,\beta} &= \bar{\eta}_{\mu\nu}\left(\gamma^{\mu}_{\ \gamma,\beta}\, \gamma^{\nu}_{\ \alpha} + \gamma^{\mu}_{\ \gamma}\, \gamma^{\nu}_{\ \alpha,\beta}\right)
\end{aligned}\right\} (5.21)$$

We add the first two lines and subtract the third. We see that

$$\left.\begin{aligned}
\gamma^{\mu}_{\ \alpha,\beta} &= \frac{\partial^2 \bar{X}^{\mu}}{\partial x^\alpha \partial x^\beta} = \frac{\partial^2 \bar{X}^{\mu}}{\partial x^\beta \partial x^\alpha}\\
&= \gamma^{\mu}_{\ \beta,\alpha}
\end{aligned}\right\} (5.22)$$

i.e. that the $\gamma^{\mu}_{\ \alpha,\beta}$ are symmetric in their lower indices. We also see that $\bar{\eta}_{\mu\nu} = \bar{\eta}_{\nu\mu}$. Then we see that the 2nd term on the right of the first line cancels the 1st of the third:

$$\eta_{\mu\nu}\left(\gamma^{\mu}_{\ \alpha}\, \gamma^{\nu}_{\ \beta,\gamma} - \gamma^{\mu}_{\ \gamma,\beta}\, \gamma^{\nu}_{\ \alpha}\right)$$

$$= \eta_{\mu\nu}\left(\gamma^{\mu}_{\ \alpha}\, \gamma^{\nu}_{\ \beta,\gamma} - \gamma^{\nu}_{\ \beta,\gamma}\, \gamma^{\mu}_{\ \alpha}\right) = 0 .$$

In the same way the 1st of the second cancels the 2nd of the third line. The 1st term in the first line equals the 2nd in the second line and they add. Hence

$$2\,\bar\eta_{\mu\nu}\,\gamma^{\mu}{}_{\alpha,\gamma}\,\gamma^{\nu}{}_{\beta}$$

$$= \left(\eta_{\alpha\beta,\gamma} + \eta_{\beta\gamma,\alpha} - \eta_{\gamma\alpha,\beta}\right) \qquad (5.23)$$

This can be solved for $\gamma^{\mu}{}_{\alpha,\gamma}$

$$\gamma^{\mu}{}_{\alpha,\gamma} = \frac{1}{2}\,\bar\eta^{\mu\nu}\,\bar\gamma^{\beta}{}_{\nu}\left(\eta_{\alpha\beta,\gamma} + \eta_{\beta\gamma,\alpha} - \eta_{\gamma\alpha,\beta}\right) \qquad (5.24)$$

where

$$\left.\begin{aligned} &\bar\eta^{\mu\nu}\,\bar\eta_{\nu\lambda} = \delta^{\mu}{}_{\lambda} \\ &\bar\gamma^{\beta}{}_{\nu}\,\gamma^{\nu}{}_{\alpha} = \delta^{\beta}{}_{\alpha} \\ \text{i.e.}\quad &\bar\gamma^{\beta}{}_{\nu} = \frac{\partial \bar X^{\beta}}{\partial X^{\nu}} \end{aligned}\right\} \qquad (5.25)$$

(5.24) shows that $\eta_{\alpha\beta,\gamma} = 0 \Rightarrow$

$$\gamma^{\mu}{}_{\alpha,\gamma} = 0 \iff X^{\mu} = R^{\mu}{}_{\nu}\,\bar X^{\nu} + b^{\mu} \qquad (5.26)$$

constants

Conversely, if (5.26) holds then (5.17) obviously implies $\eta_{\alpha\beta}$ = const. Hence we have shown that C^2-charts leave the coefficients constant iff they are affine charts.

How does the law of straight lines read in arbitrary charts? To deduce that, we have to calculate the Euler-Lagrange equation for L given in (5.15):

$$\frac{\partial L}{\partial z^\gamma} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{z}^\gamma}\right) = 0 \qquad (5.27)$$

$$\frac{\partial L}{\partial z^\gamma} = \frac{1}{2}\eta_{\alpha\beta,\gamma}(z(\lambda))\,\dot{z}^\alpha(\lambda)\,\dot{z}^\beta(\lambda) \qquad (5.28)$$

$$\frac{\partial L}{\partial \dot{z}^\gamma} = \eta_{\alpha\gamma}(z(\lambda))\,\dot{z}^\alpha(\lambda) \qquad (5.29)$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{z}^\gamma}\right) = \underbrace{\eta_{\alpha\gamma,\beta}\,\dot{z}^\alpha\dot{z}^\beta}_{\frac{1}{2}\left(\eta_{\alpha\gamma,\beta}+\eta_{\beta\gamma,\alpha}\right)\dot{z}^\alpha z^\beta} + \eta_{\alpha\gamma}\,\ddot{z}^\alpha \qquad (5.30)$$

symmetrised in (α,β) because of

Hence (5.28) is equivalent to

$$\frac{1}{2}\left(\eta_{\alpha\beta,\gamma} - \eta_{\alpha\gamma,\beta} - \eta_{\beta\gamma,\alpha}\right)\dot{Z}^{\alpha}\dot{Z}^{\beta} - \eta_{\alpha\gamma}\ddot{Z}^{\alpha} = 0$$

or

$$\boxed{\ddot{Z}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{Z}^{\beta}\dot{Z}^{\gamma} = 0} \qquad (5.31)$$

Geodesic Equation

Where

$$\boxed{\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\lambda}\left(-\eta_{\beta\gamma,\lambda} + \eta_{\lambda\beta,\gamma} + \eta_{\gamma\lambda,\beta}\right)} \qquad (5.32)$$

Christoffel Symbols

In this way, the equation of a "straight line" in affine Minkowski space can be written in <u>any</u> chart e.g. in spatial polar coordinates or in a rotating reference frame. The information on the internal structure entirely resides in η, not in the coordinates used.

The parameter λ in (5.31) was not specified. Suppose we introduce another one:

$$S = S(\lambda) \qquad (5.33)$$

Where $\quad \dot{s} = \dfrac{ds}{d\lambda} \neq 0 \qquad (5.34)$

Then

$$\left.\begin{aligned} \frac{d}{d\lambda} &= \dot{s}\,\frac{d}{ds} \\ \frac{d^2}{d\lambda^2} &= \ddot{s}\,\frac{d}{ds} + \dot{s}^2\,\frac{d^2}{ds^2} \end{aligned}\right\} \quad (5.35)$$

Writing $f' = df/ds$, (5.31) is equivalent to

$$\ddot{s}\, z'^{\alpha} + \dot{s}^2 z''^{\alpha} + \dot{s}^2\, \Gamma^{\alpha}_{\beta\gamma} z'^{\alpha} z'^{\beta} = 0$$

or

$$z''^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} z'^{\alpha} z'^{\beta} = -\frac{\ddot{s}}{\dot{s}^2}\, z'^{\alpha} \qquad (5.36)$$

The right-hand side vanishes iff $\ddot{s} = 0$. Hence the geodesic equation determines the parametrisation of its solution curve up to <u>affine reparametrisations</u> ($a, b \in \mathbb{R}$, $a \neq 0$)

$$\lambda \to s(\lambda) = a\lambda + b \,. \qquad (5.37)$$

Conversely, suppose we had started with a generalisation of the geodesic equation, when $h(\lambda)$ is any function:

$$\boxed{\ddot{z}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{z}^\beta \dot{z}^\gamma = h(\lambda)\dot{z}^\alpha} \qquad (5.38)$$

Equation of Autoparallel

If we rewrite this in terms of new parameter $s = s(\lambda)$, we get

$$\ddot{s} z'^\alpha + \dot{s}^2 z''^\alpha + \dot{s}^2 \Gamma^\alpha_{\beta\gamma} z'^\beta z'^\gamma = h(\lambda)\dot{s} z'^\alpha$$

$$\Leftrightarrow \quad z''^\alpha + \Gamma^\alpha_{\beta\gamma} z'^\beta z'^\gamma = \frac{z'^\alpha}{\dot{s}^2}\left(h(\lambda)\dot{s} - \ddot{s}\right) \qquad (5.39)$$

Hence the right hand side can be made to vanish for any h, provided we choose s appropriately, namely according to

$$\ddot{s} = \dot{s}\, h$$

$$\Leftrightarrow \quad [\ln(\dot{s})]^\cdot = h$$

$$\left.\begin{aligned} &\Leftrightarrow \quad s(\lambda) = a + b\int_0^\lambda d\lambda' \exp\left\{\int_0^{\lambda'} d\lambda'' h(\lambda'')\right\} \\ &\text{with } s(0) = a, \ \dot{s}(0) = b. \end{aligned}\right\} \quad (5.40)$$

This means that autoparallels are just "badly parametrized" geodesics. A "good parametrization" is one where the right hand side of (5.39) vanishes. It is called an <u>affine parametrisation</u>.

This distinguished class of parametrisations can be geometrically characterised

To see this we calculate

$$\left.\begin{aligned} &\frac{d}{d\lambda}\left[\frac{1}{2}\,\eta_{\alpha\beta}(z(\lambda))\,\dot z^\alpha(\lambda)\,\dot z^\beta(\lambda)\right] \\ &= \frac{1}{2}\,\eta_{\alpha\beta,\gamma}\,\dot z^\gamma \dot z^\alpha \dot z^\beta + \eta_{\alpha\beta}\,\ddot z^\alpha \dot z^\beta \end{aligned}\right\}(5.41)$$

If $z(\lambda)$ is a geodesic and satisfies (5.31), we may replace $\ddot z^\alpha$ by $-\Gamma^\alpha_{\beta\gamma}\dot z^\beta \dot z^\gamma$. Using (5.32) this gives

$$\frac{1}{2}\,\eta_{\alpha\beta,\gamma}\,\dot z^\alpha \dot z^\beta \dot z^\gamma - \frac{1}{2}\,\dot z^\beta\left(-\eta_{\alpha\gamma,\beta} + \eta_{\beta\alpha,\gamma} + \eta_{\gamma\beta,\alpha}\right)\cdot \dot z^\alpha \dot z^\gamma$$

$$= \frac{1}{2}\,\dot z^\alpha \dot z^\beta \dot z^\gamma\left(\eta_{\alpha\beta,\gamma} + \eta_{\alpha\gamma,\beta} - \eta_{\beta\alpha,\gamma} - \eta_{\gamma\beta,\alpha}\right)$$

$$= 0 \qquad (5.42)$$

because $\dot z^\alpha \dot z^\beta \dot z^\gamma$ enforces total symmetry in (α, β, γ)

Hence $\eta_{\alpha\beta}(z(\lambda))\dot{z}^{\alpha}(\lambda)\dot{z}^{\beta}(\lambda)$ is constant along a geodesic which means that

1.) The tangent vector along a geodesic is either always timelike ($\eta(\dot{\gamma},\dot{\gamma}) > 0$), lightlike ($\eta(\dot{\gamma},\dot{\gamma})=0$), or spacelike $\eta(\dot{\gamma},\dot{\gamma}) < 0$).

2.) If the geodesic is timelike or spacelike (i.e. non lightlike), its parametrisation is affinely equivalent to arc length

The second statement follows from the expression for arc length:

$$\left.\begin{aligned} S(\lambda) &= \int_{\lambda_0}^{\lambda} d\lambda \left[\underbrace{\left|\eta_{\alpha\beta}(z(\lambda))\dot{z}^{\alpha}(\lambda)\dot{z}^{\beta}(\lambda)\right|}_{=a=\text{const}}\right]^{1/2} \\ &= a\lambda + b \end{aligned}\right\} \tag{5.43}$$

with $b = -a\lambda_0$.

Finally we wish to show that the geodesic equation accounts for the "inertial forces" if the equations are written with respect to non affine charts.

Let $(O, \{e_0, \dots, e_3\})$ be an affine basis and let $(O, \{\bar{e}_0, \dots, \bar{e}_3\})$ be a time dependent basis the spatial part of which rotates against an affine one:

$$\left.\begin{aligned} \bar{e}_0 &= e_0 \\ \bar{e}_a &= \bar{e}_a(t) = R^b{}_a(t)\, e_b \end{aligned}\right\} \quad (5.44)$$

$$\rightsquigarrow \quad \dot{\bar{e}}_a = \dot{R}^b{}_a\, e_b = \dot{R}^b{}_a\, R^{-1\,c}{}_b\, \bar{e}_c$$

$$= \Omega^c{}_a(t)\, \bar{e}_c \qquad (5.45)$$

Where $\Omega^c{}_a := R^{-1\,c}{}_b\, \dot{R}^b{}_a$ (5.46)

is the angular-velocity map : $\mathbb{R}^3 \to \mathbb{R}^3$
It is antisymmetric,

$$\Omega_{ab} = -\Omega_{ba} \qquad (5.47)$$

Where $\Omega_{ab} = \delta_{ac}\, \Omega^c{}_b$ (5.48)

We have $\Omega^a{}_b X^b = (\vec{\omega} \times \vec{X})^a$ (5.49)

Where $\omega^a = -\frac{1}{2}\, \varepsilon^a{}_{bc}\, \Omega^{bc}$ (5.50)

From

$$X^a e_a = \bar{X}^b \bar{e}_b = \bar{X}^b R^a{}_b e_a$$

have

$$X^a = R^a{}_b \bar{X}^b \qquad (5.51)$$

We now rewrite the Minkowski metric

$$\eta = c^2 dt \otimes dt - \underbrace{d\vec{X} \dot{\otimes} d\vec{X}}_{\delta_{ab}\, dX^a \otimes dX^b} \qquad (5.52)$$

in rotating coordinates $\bar{X}^a$ we have from (5.51) in matrix notation

$$d\vec{X} = R\, d\vec{\bar{X}} + (\dot{R} \vec{\bar{X}})\, dt$$

$$= R\,(d\vec{\bar{X}} + \Omega \vec{\bar{X}}\, dt)$$

$$= R\,(d\vec{\bar{X}} + \vec{\omega} \times \vec{\bar{X}}\, dt) \qquad (5.53)$$

$$\Rightarrow \eta = c^2 dt \otimes dt$$

$$- R\,(d\vec{\bar{X}} + \vec{\omega} \times \vec{\bar{X}}\, dt) \dot{\otimes} R\,(d\vec{\bar{X}} + \vec{\omega} \times \vec{\bar{X}}\, dt)$$

cancel

$$= c^2 dt \otimes dt - (d\vec{\bar{X}} + \vec{\omega} \times \vec{\bar{X}}\, dt) \dot{\otimes} (d\vec{\bar{X}} + \vec{\omega} \times \vec{\bar{X}}\, dt)$$

$$(5.54)$$

The Lagrangian for the energy functional is

$$L = \frac{1}{2}\left[c^2 \dot{t}^2 - (\dot{\vec{Z}} + \vec{\omega} \times \vec{Z}\,\dot{t})^2\right] \quad (5.55)$$

where we now wrote $\vec{X} \circ \gamma = \vec{Z}$.

$$\frac{\partial L}{\partial t} = 0$$

$$\frac{\partial L}{\partial \dot{t}} = c^2 \dot{t} - (\dot{\vec{Z}} + \vec{\omega} \times \vec{Z}) \cdot \vec{\omega} \times \vec{Z}$$

$$\frac{\partial L}{\partial t} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} = 0 \iff \frac{\partial L}{\partial \dot{t}} = \text{const} = k$$

$$\iff \quad \dot{t} - \frac{1}{c^2}\left[(\vec{\omega} \times \vec{Z}) \cdot (\dot{\vec{Z}} + \vec{\omega} \times \vec{Z})\right] = k \quad (5.57)$$

$$\frac{\partial L}{\partial \vec{Z}} = -(\dot{\vec{Z}} + \vec{\omega} \times \vec{Z}) \times \vec{\omega}\; \dot{t}$$

$$\frac{\partial L}{\partial \dot{\vec{Z}}} = -(\dot{\vec{Z}} + \vec{\omega} \times \vec{Z}\,\dot{t})$$

$$\frac{\partial L}{\partial \vec{Z}} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\vec{Z}}}\right) = -(\dot{\vec{Z}} + \vec{\omega} \times \vec{Z}) \times \vec{\omega}\,\dot{t}$$

$$+ \ddot{\vec{Z}} + \dot{\vec{\omega}} \times \vec{Z}\,\dot{t} + \vec{\omega} \times \dot{\vec{Z}}\,\dot{t} + \vec{\omega} \times \vec{Z}\,\ddot{t}$$

$$\left.\begin{aligned} = \ddot{\vec{Z}} + 2(\vec{\omega} \times \dot{\vec{Z}})\,\dot{t} + \vec{\omega} \times (\vec{\omega} \times \vec{Z})\,\dot{t} + \dot{\vec{\omega}} \times \vec{Z}\,\dot{t} \\ + (\omega \times \vec{Z})\,\ddot{t} = 0 \end{aligned}\right\} \quad (5.58)$$

Equations (5.57) and (5.58) are four equations for four functions $t(\lambda)$ and $\vec{z}(\lambda)$. If we wish to compare them with the Newtonian equations in a rotating reference frame, we have to take the limit $c \to \infty$. Then (5.57) becomes

$$\dot{t} = k \qquad (5.58)$$

We choose $t = \lambda$ so that $\frac{d}{d\lambda} = \dot{} = \frac{d}{dt}$ = time-derivative w.r.t. Newtonian time. Then (5.58) becomes

$$\ddot{\vec{z}} + \underbrace{2(\vec{\omega} \times \dot{\vec{z}})}_{\text{Coriolis}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{z})}_{\text{centrifugal}} + \underbrace{\dot{\vec{\omega}} \times z}_{\text{Euler}} = 0 \qquad (5.59)$$

Note that $\Omega = R^{-1}\dot{R}$ and correspondingly $\vec{\omega}$ are the angular velocity components with respect to the rotating frame, (body-fixed frame).

This exemplifies that inertial forces are fully encoded in the spacetime metric η. Einstein's idea is now this: If gravity is locally like an inertial "force", all of gravity should also be encoded in a local metric field g, which equals η in the matter-free case