Lecture 6

Before we get to GR proper, we wish to present one last and fairly general argument why gravity cannot be described by a Poincaré-invariant theory, at least if we believe that gravitational redshift should be accounted for as a result of the equivalence principle. Hence, what we are going to argue for is that no Poincaré-invariant theory, may it be based on scalar fields or more general constructions, will predict gravitational redshift

We recall that a fundamental hypothesis of SR is that proper time ($= \frac{1}{c} \times$ proper length) is measured by clocks, e.g. atomic clocks. That means the same atomic period of two atoms of the same kind (and held under the same conditions) cover the same amount of proper time if transported along two different worldlines (at least if the accelerations are small compared to the inner-atomic accelerations of the electrons, which are huge).

We assume a static gravitational field in Minkowski space, which means that the time translation VF

$$K = \frac{\partial}{\partial t} \qquad (6.1)$$

which is Killing :

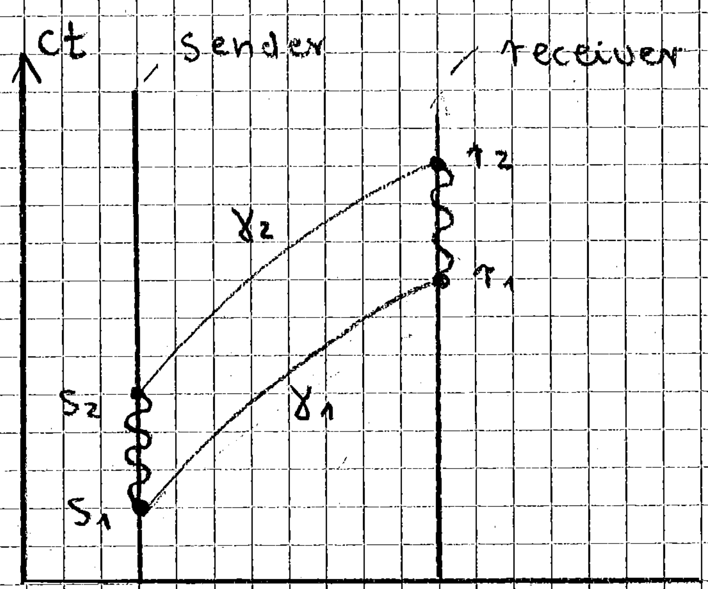
$$L_K \eta = 0 \qquad (6.2)$$

also leaves invariant the gravitational field

$$L_K \phi = 0 \qquad (6.3)$$

where ϕ stands collectively for all the gravitational fields, may they be scalars, vectors, tensors, or mixtures thereof

We consider two static observers, i.e. observers 1 and 2 whose world lines are integral curves of K. Observer 1, the "sender" transmits a number n of atomic periods, measured in his rest frame, by electromagnetic signals to observer 2, the "receiver". The signals are sent between events s_1 and s_2 on the "senders" world line and received between events r_1 and r_2 on the receivers world line; see figure.



The light rays γ_1 and γ_2 need not be straight in Minkowski space, as they may be affected by the gravitational field in a way we do not know but due to (6.2) and (6.3), i.e. time independence of the fields influencing the transmitting field (electromagnetism), we know that γ_2 is a time translation of γ_1. Hence s_2 is a time translation of s_1 and r_2 of r_1 by the same amount of flow time that is defined by K, i.e. Killing time. <u>But both worldlines of the observers are Killing orbits along which Killing-time = proper time.</u> Hence the proper time difference between s_2 and s_1 on the sender's worldline is the same as the proper time between r_2 and r_1 on the receiver's worldline. Hence the same number of atomic periods sent from 1 and received by 2 take the same proper

Here we use Mink. Geom.

time for sending and reception

$\Rightarrow$ no redshift

Einstein Equations

Gravity and inertia are essentially due to the same structure: The Lorentzian geometry of space time

$$g \in ST^0_2 M \tag{6.4}$$

In local chart

$$g = g_{\alpha\beta}(x)\, dx^\alpha \otimes dx^\beta \tag{6.5}$$

A test particle moves on a geodesic, if not acted upon by "forces"

$$\ddot{z}^\alpha(\lambda) + \Gamma^\alpha_{\beta\gamma}(z(\lambda))\, \dot{z}^\alpha(\lambda)\, \dot{z}^\beta(\lambda) = 0 \tag{6.6}$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma}(-g_{\beta\gamma,\sigma} + g_{\sigma\beta,\gamma} + g_{\gamma\sigma,\beta}) \tag{6.7}$$

Gravity is not a force, i.e. gravity causes no deviation from inertial motion defined through (6.6-7).

But how is g determined?

⇒ Einstein's field equation of November 25th 1915

$$\boxed{R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}} \qquad (6.7)$$

- 10 non-linear but quasi-linear [highest derivative appears only linearly] coupled partial differential equations for 10 functions $g_{\alpha\beta}$.

- $\kappa = \frac{8\pi G}{c^4}$, G = Newton's constant (6.8)

- $T_{\alpha\beta}$ = energy momentum tensor of all matter present

- $R_{\alpha\beta}$ = Ricci-tensor

  $= R^{\mu}{}_{\alpha\mu\beta}$ (6.9)

  where $R^{\mu}{}_{\alpha\nu\beta}$ = Riemann tensor with 20 indep. comp.

- $R = g^{\alpha\beta} R_{\alpha\beta}$ = Ricci-scalar or "scalar curvature". (6.10)

A little closer: The Riemann tensor

Its components are

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \partial_\nu \Gamma_\mu{}^\alpha{}_\beta + \Gamma_\mu{}^\alpha{}_\gamma \Gamma_\nu{}^\gamma{}_\beta - \Gamma_\nu{}^\alpha{}_\gamma \Gamma_\mu{}^\gamma{}_\beta \qquad (6.11)$$

Structurally

$$\Gamma = g^{-1}(-\partial g + \partial g + \partial g) \qquad (6.12)$$

$$\left.\begin{aligned} \text{Riem} &= g^{-1}(\sim \partial^2 g) + \\ & g^{-1} g^{-1} (\sim (\partial g)^2) \end{aligned}\right\} (6.13)$$

likewise Ric and scalar curv.

Look at (6.11)! Then

$$\left.\begin{aligned} \Omega^\alpha{}_\beta &= \tfrac{1}{2} R^\alpha{}_{\beta\mu\nu}\, dx^\mu \wedge dx^\nu \\ &= d\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta \end{aligned}\right\} (6.14)$$

= endomorphism-valued 2-form

$$\Gamma^\alpha{}_\beta = \Gamma_\lambda{}^\alpha{}_\beta\, dx^\lambda \qquad (6.15)$$

= endomorphism-valued 1-form.

Note that the left-hand side of Einsteins equations are a very particular combination of Curvature components

$$G_{\alpha\beta} = \text{Einstein Tensor}$$
$$= R^{\mu}{}_{\alpha\mu\beta} - \frac{1}{2} g_{\alpha\beta} R^{\mu\nu}{}_{\mu\nu} \qquad (6.16)$$

What makes this combination special analytically?

<u>Theorem</u> (David Lovelock, 1972):

Let (M, g) be a 4-dimensional Lorentzian manifold with Levi-Civita covariant derivative ∇.
Let $A \in ST^0_2 M$, so that

a) $A(p) = A(g(p), \partial g(p), \partial^2 g(p))$

b) $\nabla \cdot A = 0 \quad (\nabla^\alpha A_{\alpha\beta} = 0)$

then

$$A = a\, G + b\, g \quad , \; a, b \in \mathbb{R} \qquad (6.17)$$

G = Einstein Tensor

Note: Lovelock did not require that A was symmetric nor that it depends linearly on $\partial^2 g$, i.e. that it defined a quasi-linear differential operator. These properties are implied by his hypotheses.

Proof in: David Lovelock:
"The four-dimensionality of Space and the Einstein Tensor".
Journal of Mathematical Physics, Vol. 13, pages 874-6, year 1972.

- The hypothesis $\nabla \cdot A = 0$ was set because

$$\nabla T = 0 \Leftrightarrow \nabla_\alpha T^{\alpha\beta} = 0 \qquad (6.18)$$

Is interpreted as energy-momentum conservation of matter in the grav. field (careful!). This is required on physical grounds to be a consequence of Einstein's equations.

- Lovelock's theorem would have admitted an additional term on the left-hand side $\sim g_{\alpha\beta}$.

In modern terms ($b = -\Lambda$)

$$\boxed{R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}} \qquad (6.19)$$

Einstein 1917

Λ = cosmological constant

$[\Lambda] = (\text{length})^{-2}$

An alternative form of these equations is obtained by first taking the trace of (6.19), i.e. multiplying (6.19) with $g^{\alpha\beta}$, where $g^{\alpha\beta} g_{\alpha\beta} = 4$,

$$R - 2R - 4\Lambda = \kappa T$$

$$\Rightarrow \quad R = -4\Lambda - \kappa T \qquad (6.20)$$

and inserting this into (6.19) for R

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (-4\Lambda - \kappa T) - \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

$$\Leftrightarrow \quad \boxed{R_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa \left(T_{\alpha\beta} + \frac{1}{2} g_{\alpha\beta} T \right)} \qquad (6.21)$$

alternative form of (6.19)

In this lecture, we will mostly work with $\Lambda = 0$.

The current value of Λ, determind by cosmological observations (Supernovae of type 1A, Planck-Sat.),

$$\Lambda \cong 1.1056 \times 10^{-52}\ m^{-2} \qquad (6.22)$$

corresponding to a curvature radius of $\sim 10^{26}$ m $\cong 10^{10}$ lightyears

One can show that the Einstein tensor has the following geometric interpretation: Let n be a timelike unit vector, then

$$G(n,n) = G_{\alpha\beta} n^\alpha n^\beta$$

$$= \sum_{a<b=1}^{3} \mathrm{Sec}(a,b) \qquad (6.23)$$

= Sum of Sectional curvatures of any 3 mutually perpendicular planes in the orthogonal complement to n

= "mean spatial curvature of space $\perp$ to n."

On the other hand

$T(n,n)$ = Energy density in rest frame of n.

Hence (setting $\Lambda = 0$)

$$G(n,n) = \kappa\, T(n,n)$$
$$= \kappa c^2 \,(\text{Mass density}) \qquad (6.24)$$

$$\Rightarrow \kappa c^2 = \frac{\text{Spatial curvature}}{\text{mass density}} = \frac{8\pi G}{c^2} \qquad (6.25)$$

Note

$$[\kappa] = \frac{\text{Curvature}}{\text{Energy density}} = \frac{m^{-2}}{J \cdot m^{-3}}$$
$$= \frac{1}{N} \qquad (6.26)$$

$$G = 6.67408(31) \cdot 10^{-11}\ m^3 \cdot kg^{-1}\, s^{-2}$$

$$c = 299.792.458\ m \cdot s^{-1} \text{ (exactly!)}$$

[Remark: Accuracy of $G \sim 10^{-5}$
Compare with Plancks constant
$\Delta h / h \sim 10^{-8}$ or elementary charge
$\Delta e / e \sim 6 \cdot 10^{-9}$]

Mass density curves space:

$$Kc^2 = \frac{\text{Spatial curvature}}{\text{mass density}}$$

$$= \frac{8\pi G}{c^2}$$

$$= 1{,}87 \cdot 10^{-26} \frac{m^{-2}}{Kg \cdot m^{-3}} \qquad (6.27)$$

$$= 0{,}417 \frac{(AU)^{-2}}{\rho_W} \qquad (6.28)$$

$$= 0{,}75 \frac{(10\,km)^{-2}}{\rho_N} \qquad (6.29)$$

Where

AU = astronomical unit

$\cong 150 \cdot 10^6$ km

ρ_W = density of water = 10^3 Kg $\cdot$ m^{-3}

ρ_N = nuclear density inside core of neutron stars

$= 4 \cdot 10^{17}$ Kg m^{-3}