Lecture 7

Einstein's field equation relate tensors ($R_{\alpha\beta}$, $T_{\alpha\beta}$ etc.) pointwise on a 4-dim. manifold M with Lorentz metric g; the equations are "local".

We recall the transformation property of tensors: Let $T \in ST^{l}_{m}(M)$, i.e.

$$T = T^{\alpha_1 \dots \alpha_l}{}_{\beta_1 \dots \beta_m} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\alpha_l}} \otimes dx^{\beta_1} \otimes \dots \otimes dx^{\beta_m} \tag{7.1}$$

Then, if $\bar{x}^{\alpha} = \bar{x}^{\alpha}(x)$ and $x^{\alpha} = x^{\alpha}(\bar{x})$ are the chart transition maps

$$\left.\begin{aligned} \frac{\partial}{\partial x^{\alpha}} &= \frac{\partial \bar{x}^{\gamma}}{\partial x^{\alpha}} \frac{\partial}{\partial \bar{x}^{\gamma}} \\ dx^{\beta} &= \frac{\partial x^{\beta}}{\partial \bar{x}^{\delta}} d\bar{x}^{\delta} \end{aligned}\right\} \tag{7.2}$$

so that

$$T = \bar{T}^{\gamma_1 \dots \gamma_l}{}_{\delta_1 \dots \delta_m} \frac{\partial}{\partial \bar{x}^{\gamma_1}} \otimes \dots \otimes \frac{\partial}{\partial \bar{x}^{\gamma_l}} \otimes d\bar{x}^{\delta_1} \otimes \dots \otimes d\bar{x}^{\delta_m} \tag{7.3}$$

where

$$\left.\begin{aligned} \bar{T}^{\gamma_1 \dots \gamma_l}{}_{\delta_1 \dots \delta_m}(\bar{x}) = {} & T^{\alpha_1 \dots \alpha_l}{}_{\beta_1 \dots \beta_m}(x) \frac{\partial \bar{x}^{\gamma_1}}{\partial x^{\alpha_1}} \dots \frac{\partial \bar{x}^{\gamma_l}}{\partial x^{\alpha_l}} \\ & \times \frac{\partial x^{\beta_1}}{\partial \bar{x}^{\delta_1}} \dots \frac{\partial x^{\beta_m}}{\partial \bar{x}^{\delta_m}} \end{aligned}\right\} \tag{7.4}$$

Many of the geometric quantities we consider involve the Christoffel-symbols

$$\Gamma^{\lambda}_{\alpha\beta} := \frac{1}{2} g^{\lambda\sigma} \left(-g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} \right) \quad (7.5)$$

(as usual we write $g_{\alpha\beta,\sigma} = \partial_\sigma g_{\alpha\beta} = \partial g_{\alpha\beta} / \partial x^\sigma$).
which we encountered in the geodesic eqn.

$$\ddot{z}^{\lambda} + \Gamma^{\lambda}_{\alpha\beta} \dot{z}^{\alpha} \dot{z}^{\beta} = 0 . \quad (7.6)$$

The Christoffel symbols represent a geometric object (a connection) which is not a tensor on M. To see how the coefficients $\Gamma^{\lambda}_{\alpha\beta}$ behave under changes of charts we rewrite (7.6) in terms of coordinates $\bar{x}$:

$$\left.\begin{aligned} \bar{z}^{\alpha} &= \bar{x}^{\gamma} \circ \gamma = (\bar{\phi} \circ \phi^{-1})^{\gamma} (z) \\ z^{\lambda} &= x^{\lambda} \circ \gamma = (\phi \circ \bar{\phi}^{-1})^{\lambda} (\bar{z}) \end{aligned}\right\} (7.7)$$

Then

$$\left.\begin{aligned} \dot{z}^{\lambda} &= \frac{\partial x^{\lambda}}{\partial \bar{x}^{\sigma}} \dot{\bar{z}}^{\sigma} \\ \ddot{z}^{\lambda} &= \frac{\partial x^{\lambda}}{\partial \bar{x}^{\sigma}} \ddot{\bar{z}}^{\sigma} + \frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} \dot{\bar{z}}^{\mu} \dot{\bar{z}}^{\nu} \end{aligned}\right\} (7.8)$$

and (7.6) turns into

$$\frac{\partial x^{\lambda}}{\partial \bar{x}^{\sigma}} \ddot{\bar{z}}^{\sigma} + \left(\Gamma^{\lambda}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} + \frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} \right) \dot{\bar{z}}^{\mu} \dot{\bar{z}}^{\nu} = 0$$

(7.9)

or equivalently

$$\ddot{\bar{z}}^{\sigma} + \left(\frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\alpha\beta} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} + \frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} \right) \dot{\bar{z}}^{\mu} \dot{\bar{z}}^{\nu} = 0 \qquad (7.10)$$

But this must equal the geodesic equation in $\bar{x}$ coordinates

$$\ddot{\bar{z}}^{\sigma} + \bar{\Gamma}^{\sigma}_{\mu\nu} \dot{\bar{z}}^{\mu} \dot{\bar{z}}^{\nu} = 0$$

hence

$$\bar{\Gamma}^{\sigma}_{\mu\nu} = \underbrace{\Gamma^{\lambda}_{\alpha\beta} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}}}_{\text{homogeneous part (like a tensor)}} + \underbrace{\frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}}}_{\text{inhomogeneous part}} \qquad (7.11)$$

Note: $$\frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \frac{\partial}{\partial \bar{x}^{\mu}} \frac{\partial x^{\lambda}}{\partial \bar{x}^{\nu}}$$

$$= - \frac{\partial^2 \bar{x}^{\sigma}}{\partial x^{\lambda} \partial x^{s}} \frac{\partial x^{s}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\lambda}}{\partial \bar{x}^{\nu}}$$

$$= - \frac{\partial^2 \bar{x}^{\sigma}}{\partial x^{\alpha} \partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \qquad (7.12)$$

This allows to write (7.11) in a more convenient form

$$\bar{\Gamma}^{\sigma}_{\mu\nu} = \frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \left(\Gamma^{\lambda}_{\alpha\beta} - \frac{\partial x^{\lambda}}{\partial \bar{x}^{\varrho}} \frac{\partial^2 \bar{x}^{\varrho}}{\partial x^{\alpha} \partial x^{\beta}} \right) \qquad (7.13)$$

This differs from (7.11) just in the way we write the inhomogeneous part.

Note: In differential geometry a geometric object is characterised by a definite transformation property of the coefficients which represent it. A tensor has a linear homogeneous transformation law, but that is not a neccessary condition for the qualification as a geometric object. The Christoffel symbols, too, represent a geometric object – the Levi-Civita connection.

Let $p \in M$ be a fixed point and x^{α} the coordinates in a n'hood $\mathcal{U}$ of p. Consider the following coordinate transformation in $\mathcal{U}$

$$\bar{x}^{\lambda}(q) := x^{\lambda}(q) - x^{\lambda}(p) + \frac{1}{2} \Gamma^{\lambda}_{\alpha\beta}(p) \left(x^{\alpha}(q) - x^{\alpha}(p) \right) \left(x^{\beta}(q) - x^{\beta}(p) \right) \qquad (7.14)$$

Then

$$\left. \frac{\partial \bar{x}^{\lambda}}{\partial x^{\varrho}} \right|_p = \delta^{\lambda}_{\varrho}, \quad \left. \frac{\partial^2 \bar{x}^{\varrho}}{\partial x^{\alpha} \partial x^{\beta}} \right| = \Gamma^{\varrho}_{\alpha\beta}(p) \qquad (7.15)$$

Equation (7.13) then immediately gives

$$\bar{\Gamma}^{\sigma}_{\mu\nu}(p) = 0 \qquad (7.16)$$

In addition, we may always achieve

$$\left.\begin{aligned} g_{\alpha\beta}(p) &= g\left(\frac{\partial}{\partial \bar{x}^{\alpha}}, \frac{\partial}{\partial \bar{x}^{\beta}}\right)\Big|_p \\ &= \eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1). \end{aligned}\right\} \quad (7.17)$$

[This can be achieved by a linear transformation $\bar{x}^{\alpha} \mapsto \bar{\bar{x}}^{\alpha} = \Lambda^{\alpha}{}_{\beta}\bar{x}^{\beta}$ under which the Γ's transform linear-homogeneously and therefore stay zero once they are zero.]

<u>Proposition and Definition</u>. For any point $p \in M$ there exists a chart $(\mathcal{U}, \phi)$ with $p \in \mathcal{U}$, $\phi : \mathcal{U} \to \mathbb{R}^n$, $\phi = x^{\alpha} e_{\alpha}$, $x^{\alpha} : \mathcal{U} \to \mathbb{R}$ coord. functions, with

$$\left.\begin{aligned} x^{\alpha}(p) = 0, \quad g_{\alpha\beta}(p) &= \eta_{\alpha\beta} \\ g_{\alpha\beta,\gamma}(p) &= 0 \end{aligned}\right\} \quad (7.18)$$

Such a chart is called a Riemannian Normal coordinate system at p.

Note that

$$g_{\alpha\beta,\gamma}(p) = 0 \iff \Gamma^{\lambda}_{\alpha\beta}(p) = 0$$

In fact, the first derivatives of g and Γ are just related by linear invertible maps:

$$\Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma} \left(-g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} \right) \qquad (7.19)$$

Since $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$ there are

$$\frac{1}{2} n(n+1) \cdot n$$

$$= 40 \quad \text{for } n = 4$$

independent components in n dimensions, just like $g_{\alpha\beta,\gamma} = g_{\beta\alpha,\gamma}$.

From (7.19)

$$g_{\lambda\sigma} \Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2} \left(-g_{\alpha\beta,\sigma} + g_{\sigma\alpha,\beta} + g_{\beta\sigma,\alpha} \right)$$

$$+ \quad g_{\lambda\alpha} \Gamma^{\lambda}_{\sigma\beta} = \frac{1}{2} \left(-g_{\sigma\beta,\alpha} + g_{\alpha\sigma,\beta} + g_{\beta\alpha,\sigma} \right)$$

$$\Rightarrow \quad g_{\alpha\sigma,\beta} = g_{\lambda\alpha} \Gamma^{\lambda}_{\sigma\beta} + g_{\lambda\sigma} \Gamma^{\lambda}_{\alpha\beta} \qquad (7.20)$$

The Γ-symbols are just the coefficients of a covariant derivative

$$\nabla_{\frac{\partial}{\partial x^\alpha}}\left(\frac{\partial}{\partial x^\beta}\right) = \Gamma^{\lambda}_{\alpha\beta}\,\frac{\partial}{\partial x^\lambda} \qquad (7.21)$$

$$\nabla_{\frac{\partial}{\partial x^\alpha}}\left(dx^\beta\right) = -\,\Gamma_{\alpha}{}^{\beta}{}_{\lambda}\, dx^\lambda \qquad (7.22)$$

Here ∇ is a map

$$\nabla : ST^{l}_{m}(M) \to ST^{l}_{m+1}(M)$$

or

$$\nabla : ST^{1}_{0}(M) \times ST^{l}_{m}(M) \to ST^{l}_{m}(M)$$

$$(X, T) \mapsto \nabla_X T$$

satisfying ($\forall f \in C^\infty(M)$, $X \in ST^1_0(M)$, $T \in ST^l_m(M)$):

$$\left.\begin{aligned}
&\nabla_X f = X(f) \\
&T \mapsto \nabla_X T \quad \text{is } \mathbb{R}\text{-linear} \\
&X \mapsto \nabla_X T \quad \text{is } C^\infty(M)\text{-linear} \\
&\nabla_X (T \otimes T') = \nabla_X T \otimes T' + T \otimes \nabla_X T' \\
&C \circ \nabla_X = \nabla_X \circ C \quad (C = \text{contraction}).
\end{aligned}\right\} \quad (7.23)$$

The Levi-Civita connection is characterized by

1.) vanishing Torsion $T \in \mathcal{T}^1_2(M)$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] = 0 \qquad (7.24)$$

2.) metricity

$$\nabla_X g = 0 \qquad (7.25)$$

Then ∇ is uniquely determined

$$\begin{aligned}
+ \quad \nabla_X (g(Y,Z)) &= \underbrace{g(\nabla_X Y, Z)}_{1} + \underbrace{g(Y, \nabla_X Z)}_{2} \\
+ \quad \nabla_Y (g(Z,X)) &= \underbrace{g(\nabla_Y Z, X)}_{3} + \underbrace{g(Z, \nabla_Y X)}_{1} \\
- \quad \nabla_Z (g(X,Y)) &= \underbrace{g(\nabla_Z X, Y)}_{2} + \underbrace{g(X, \nabla_Z Y)}_{3}
\end{aligned} \qquad (7.26)$$

$$X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))$$

$$= g(\nabla_X Y, Z) + g(\nabla_Y X, Z)$$

$$g(\nabla_X Z - \nabla_Z X, Y) + g(X, \nabla_Y Z - \nabla_Z Y)$$

$$= 2\, g(\nabla_X Y, Z)$$

$$+ g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X)$$

$$(7.27)$$

where we used $T = 0$.

Hence

$$g(\nabla_X Y, Z)$$

$$= \frac{1}{2}\Big\{\big(-Z(g(X,Y)) + X(g(Y,Z)) + Y(g(Z,X))\big)$$

$$+ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\Big\} \qquad (7.28)$$

If $\{e_\alpha : \alpha = 0,1,2,3\}$ are local basis fields, so that

$$[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta}\, e_\gamma \qquad (7.29)$$

and

$$\nabla_{e_\alpha} e_\beta = \omega^\gamma_{\alpha\beta}\, e_\gamma \qquad (7.30)$$

$$g(e_\alpha, e_\beta) = g_{\alpha\beta} \qquad (7.31)$$

Then, for $X = e_\alpha$, $Y = e_\beta$, $Z = e_\gamma$, (7.28) becomes

$$\omega^\lambda_{\alpha\beta}\, g_{\lambda\gamma}$$

$$= \frac{1}{2}\big(-e_\gamma(g_{\alpha\beta}) + e_\alpha(g_{\beta\gamma}) + e_\beta(g_{\gamma\alpha})\big)$$

$$+ C^\lambda_{\alpha\beta}\, g_{\lambda\gamma} - C^\lambda_{\beta\gamma}\, g_{\lambda\alpha} + C^\lambda_{\gamma\alpha}\, g_{\lambda\beta}$$

Multiplication with $g^{\sigma\gamma}$ gives

$$W^{\sigma}_{\alpha\beta} = \frac{1}{2} g^{\sigma\gamma} \left(-e_\gamma(g_{\alpha\beta}) + e_\alpha(g_{\beta\gamma}) + e_\beta(g_{\gamma\alpha}) \right)$$

$$+ \frac{1}{2} \left(C^{\sigma}_{\alpha\beta} - g^{\sigma\gamma} C^{\lambda}_{\beta\gamma} g_{\lambda\alpha} + g^{\sigma\gamma} C^{\lambda}_{\gamma\alpha} g_{\lambda\beta} \right) \qquad (7.32)$$

The right-hand side is completely determined once g and the basis is specified. Hence the coefficients $W^{\sigma}_{\alpha\beta}$ are uniquely determined and hence ∇ is.

In a coordinate basis $e_\alpha = \frac{\partial}{\partial x^\alpha}$ we have $C^{\gamma}_{\alpha\beta} = 0$ and

$$W^{\sigma}_{\alpha\beta} = \Gamma^{\sigma}_{\alpha\beta} \qquad (7.33)$$

$\Rightarrow$ The Christoffel symbols are the coefficients of the unique cov. derivative with $T = 0$, $\nabla g = 0$ with respect to a chart basis field $\partial/\partial x^\alpha$.

Examples: $X = X^\alpha \frac{\partial}{\partial x^\alpha}$, $\nabla_X T = ?$

a) $T = Y = Y^\beta \frac{\partial}{\partial x^\beta}$ (Vector field)

$$\Rightarrow \nabla_X Y = \nabla_{X^\alpha \frac{\partial}{\partial x^\alpha}} \left(Y^\beta \frac{\partial}{\partial x^\beta} \right) = X^\alpha \frac{\partial Y^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$$

$$+ X^\alpha Y^\beta \Gamma^{\gamma}_{\alpha\beta} \frac{\partial}{\partial x^\gamma} = X^\alpha \left(Y^\gamma{}_{,\alpha} + \Gamma^{\gamma}_{\alpha\beta} Y^\beta \right) \frac{\partial}{\partial x^\gamma}$$

We write

$$\nabla_{\frac{\partial}{\partial x^\alpha}}(y) =: (\nabla_\alpha y^\gamma)\frac{\partial}{\partial x^\gamma} \qquad (7.34)$$

so that

$$\nabla_X y = X^\alpha (\nabla_\alpha y^\gamma)\frac{\partial}{\partial x^\gamma} \qquad (7.35)$$

then

$$\nabla_\alpha y^\gamma = y^\gamma{}_{,\alpha} + \Gamma_\alpha{}^\gamma{}_\beta\, y^\beta \qquad (7.36)$$

b) $T = \alpha = \alpha_\beta\, dX^\beta$ (co-vector field)

$$\nabla_X \alpha = \nabla_{X^\alpha \frac{\partial}{\partial x^\alpha}}(\alpha_\beta\, dX^\beta) = X^\alpha \frac{\partial \alpha_\beta}{\partial X^\alpha}\, dX^\beta$$

$$- X^\alpha \alpha_\beta\, \Gamma_\alpha{}^\beta{}_\gamma\, dX^\gamma$$

$$= X^\alpha (\alpha_{\beta,\alpha} - \alpha_\gamma \Gamma^\gamma{}_{\alpha\beta})\, dX^\beta$$

Again we write

$$\nabla_{\frac{\partial}{\partial X^\alpha}}(\alpha) =: (\nabla_\alpha \alpha_\gamma)\, dX^\gamma \qquad (7.37)$$

then $\nabla_X \alpha = X^\alpha (\nabla_\alpha \alpha_\gamma)\, dX^\gamma$ (7.38)

with

$$\nabla_\alpha \alpha_\beta = \alpha_{\beta,\alpha} - \Gamma^\gamma{}_{\alpha\beta}\, \alpha_\gamma \qquad (7.39)$$

c) Let now generally $T \in ST^{l}_{m}(M)$

$$T = T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_m} \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes dx^{\beta_m} \qquad (7.40)$$

Then

$$\nabla_X T = X^{\gamma} \left(\nabla_{\gamma} T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_m} \right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \dots \otimes dx^{\beta_m} \qquad (7.41)$$

with

$$\nabla_{\gamma} T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_m} = \frac{\partial}{\partial x^{\gamma}} T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_m}$$
$$+ \sum_{i=1}^{l} \Gamma^{\alpha_i}_{\gamma \lambda} T^{\alpha_1 \dots \alpha_{i-1} \lambda \alpha_{i+1} \dots \alpha_l}_{\beta_1 \dots \beta_m}$$
$$- \sum_{i=1}^{m} \Gamma^{\lambda}_{\gamma \beta_i} T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_{i-1} \lambda \beta_{i+1} \dots \beta_m} \qquad (7.42)$$

The coefficients $\nabla_{\gamma} T^{\alpha_1 \dots \alpha_l}_{\beta_1 \dots \beta_m}$ are that of $\nabla T \in ST^{l}_{m+1}(M)$, i.e. a tensor.

The formulae for the coordinate expressions for $\nabla_X T$ should be compared with those for the Lie-derivative $L_X T$

Note the similarities in the systematics and differences in signs!

$$
\left.
\begin{aligned}
(\nabla_X T)^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m}
&= X^\gamma \partial_\gamma T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&+ \sum_{i=1}^{l} X^\gamma \Gamma^{\alpha_i}{}_{\gamma\lambda} T^{\alpha_1 \ldots \alpha_{i-1} \lambda \alpha_{i+1} \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&- \sum_{i=1}^{m} X^\gamma \Gamma^{\lambda}{}_{\gamma \beta_i} T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_{i-1} \lambda \beta_{i+1} \ldots \beta_m}
\end{aligned}
\right\} \quad (7.43)
$$

$$
\left.
\begin{aligned}
(L_X T)^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m}
&= X^\gamma \partial_\gamma T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&- \sum_{i=1}^{l} X^{\alpha_i}{}_{,\lambda} T^{\alpha_1 \ldots \alpha_{i-1} \lambda \alpha_{i+1} \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&+ \sum_{i=1}^{m} X^{\lambda}{}_{,\beta_i} T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_{i-1} \lambda \beta_{i+1} \ldots \beta_m}
\end{aligned}
\right\} \quad (7.44)
$$

$$
\left.
\begin{aligned}
&= X^\gamma \nabla_\gamma T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&- \sum_{i=1}^{l} (\nabla_\lambda X^{\alpha_i}) T^{\alpha_1 \ldots \alpha_{i-1} \lambda \alpha_{i+1} \ldots \alpha_l}{}_{\beta_1 \ldots \beta_m} \\
&+ \sum_{i=1}^{m} (\nabla_{\beta_i} X^\lambda) T^{\alpha_1 \ldots \alpha_l}{}_{\beta_1 \ldots \beta_{i-1} \lambda \beta_{i+1} \ldots \beta_m}
\end{aligned}
\right\} \quad (7.45)
$$

Examples

1.) $$\nabla_\gamma g_{\alpha\beta} = g_{\alpha\beta,\gamma} - g_{\lambda\beta}\Gamma^\lambda_{\gamma\alpha} - g_{\alpha\lambda}\Gamma^\lambda_{\gamma\beta}$$
$$= 0 \qquad (7.46)$$

2.) $$\nabla_\gamma g^{\alpha\beta} = g^{\alpha\beta}{}_{,\gamma} + \Gamma_\gamma{}^\alpha{}_\lambda g^{\lambda\beta} + \Gamma_\gamma{}^\beta{}_\lambda g^{\alpha\lambda}$$
$$= 0 \qquad (7.47)$$

3.) $$\nabla_\gamma \delta^\alpha{}_\beta = \delta^\alpha{}_{\beta,\gamma} + \Gamma_\gamma{}^\alpha{}_\lambda \delta^\lambda{}_\beta - \Gamma^\lambda_{\gamma\beta}\delta^\alpha_\lambda$$
$$= 0 \qquad (7.48)$$

4.) $$\begin{aligned}(L_X g)_{\alpha\beta} &= X^\gamma g_{\alpha\beta,\gamma} - X^\lambda{}_{,\alpha} g_{\lambda\beta} - X^\lambda{}_{,\beta} g_{\alpha\lambda} \\ &= X^\gamma \nabla_\gamma g_{\alpha\beta} + \nabla_\alpha X^\lambda g_{\lambda\beta} + \nabla_\beta X^\lambda g_{\alpha\lambda} \\ &= 0 + \nabla_\alpha(X^\lambda g_{\lambda\beta}) + \nabla_\beta(X^\lambda g_{\alpha\lambda}) \\ &= \nabla_\alpha X_\beta + \nabla_\beta X_\alpha \\ &= 2\nabla_{(\alpha} X_{\beta)} \qquad (7.49)\end{aligned}$$

5.) Let T be a symmetric, divergence free tensor: $T^{\alpha\beta} = T^{\beta\alpha}$, $\nabla_\alpha T^{\alpha\beta} = 0$, then

$$\begin{aligned}\nabla_\alpha(T^{\alpha\beta}X_\beta) &= T^{\alpha\beta}\nabla_\alpha X_\beta \\ &= T^{\alpha\beta}\nabla_{(\alpha}X_{\beta)} \\ &= (1/2)\, T^{\alpha\beta}(L_X g)_{\alpha\beta} \qquad (7.50)\end{aligned}$$

Hence: if X satisfies $L_X g = 0 \Rightarrow T^{\alpha\beta}X_\beta$ is divergence-free.