Lecture 8

More on the curvature tensor

$$\left.\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} \\ &\quad + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\nu\beta} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\beta} \end{aligned}\right\} \quad (8.1)$$

Symmetries: Are best discussed in normal coordinates where $\Gamma = 0$, $\partial\Gamma \neq 0$.
The rationale behind this is as follows: Any tensor relation that holds in a particular coordinate system also holds in any other coordinate system and is therefore generally valid.

Let $p \in M$ and x^α a normal coord. system so that

$$g_{\alpha\beta}(p) = \eta_{\alpha\beta} \qquad (8.2)$$

$$\partial_\gamma g_{\alpha\beta}(p) = 0 \qquad (8.3)$$

Then, at point p, have

$$R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2}\eta^{\alpha\sigma}\left(-g_{\nu\beta,\sigma\mu} + g_{\sigma\nu,\beta\mu} + \underline{g_{\beta\sigma,\nu\mu}} + g_{\mu\beta,\sigma\nu} - g_{\sigma\mu,\beta\nu} - \underline{g_{\beta\sigma,\mu\nu}}\right)$$

$$\left.\begin{aligned} R_{\alpha\beta\mu\nu} = -\frac{1}{2}\big(& g_{\alpha\mu,\beta\nu} + g_{\beta\nu,\alpha\mu} \\ & - g_{\alpha\nu,\beta\mu} - g_{\beta\mu,\alpha\nu}\big) \end{aligned}\right\} \quad (8.4)$$

From this we immediately infer

$$R_{\alpha\beta\mu\nu} = - R_{\alpha\beta\nu\mu} \tag{8.5}$$

$$R_{\alpha\beta\mu\nu} = - R_{\beta\alpha\mu\nu} \tag{8.6}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \tag{8.7}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \tag{8.8}$$

These are tensor-relations and hence hold generally.

<u>Counting of components</u>

(8.5-7) say that $R_{\alpha\beta\mu\nu}$ are the components of a symmetric bilinear form on the space $\Lambda^2 V$

$$\Lambda^2 V \times \Lambda^2 V \longrightarrow \mathbb{R}$$

$$(\Theta, \Lambda) \longmapsto R_{\alpha\beta\mu\nu}\, \Theta^{\alpha\beta}\, \Lambda^{\mu\nu} \tag{8.9}$$

If $\dim(V) = n$ then

$$\dim(\Lambda^2 V) = N := \tfrac{1}{2} n(n-1) \tag{8.10}$$

$$\dim\big(\underbrace{(\Lambda^2 V)^* \vee (\Lambda^2 V)^*}_{\substack{\text{symm. bilinear} \\ \text{forms on } \Lambda^2 V}}\big) = \tfrac{1}{2} N(N+1) \tag{8.11}$$

The combination on the left-hand side of (8.8) is totally antisymmetric in all indices $\alpha\beta\mu\nu$. For example:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta}$$

Exchanging α and β gives

$$R_{\beta\alpha\mu\nu} + R_{\beta\nu\alpha\mu} + R_{\beta\mu\nu\alpha}$$

$$= \underset{\substack{\uparrow \\ (8.6)}}{-} R_{\alpha\beta\mu\nu} \underset{\substack{\uparrow \\ (8.7)(8.5)}}{-} R_{\alpha\mu\nu\beta} \underset{\substack{\uparrow \\ (8.7)(8.6)}}{-} R_{\alpha\nu\beta\mu}$$

Hence (8.8) gives $\binom{n}{4}$ independent conditions (as many as there are possibilities to pick 4 distinguishable elements out of a set of n).

Hence, in n-dimensions, have

$$\# \text{Riem} = \frac{1}{2} N(N+1) - \binom{n}{4} \qquad (8.12)$$

$$= \frac{1}{4} n(n-1)\left[\frac{1}{2} n(n-1) + 1\right] - \binom{n}{4}$$

$$= \frac{1}{4} n(n-1)\left[\frac{1}{2} n(n-1) + 1 - \frac{(n-2)(n-3)}{6}\right]$$

$$= \frac{1}{4} n(n-1)\left[\frac{1}{3} n^2 + \frac{1}{3} n\right]$$

$$= \underline{\underline{\frac{1}{12} n^2 (n^2-1)}} \qquad (8.13)$$

$$\# \text{Riem} = \begin{cases} 1 & \text{for } n=2 \\ 6 & \text{for } n=3 \\ 20 & \text{for } n=4 \\ 50 & \text{for } n=5 \end{cases} \qquad (8.14)$$

Note in $n=3$ dim. Riemann and Ricci tensor have same number of components. In fact, in $n=3$ dim. Riemann tensor can be expressed through Ricci-tensor. This corresponds to the fact that for $n=3$ Einstein equations determine all of curvature by $T^{\alpha\beta}$ $\Rightarrow$ g is flat outside support of T $\Rightarrow$ no gravitational degrees of freedom and no grav. waves.

In $n=4$ dim have $\#\text{Ric} = 10$ and $\#\text{Riem} = 20$. So Einstein equations determine only <u>half</u> of the curvature components.

The cyclic symmetry

$$R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\nu\beta\mu} + R^{\alpha}{}_{\mu\nu\beta} = 0 \qquad (8.15)$$

is also called the <u>1st. Bianchi Identity</u>

There is also a 2nd Bianchi Identity
It involves covariant derivatives of
the Riemann - tensor

$$\left.\begin{aligned}\nabla_\lambda R^\alpha{}_{\beta\mu\nu} &= \partial_\lambda \partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \partial_\lambda \partial_\nu \Gamma_\mu{}^\alpha{}_\beta \\ &\quad + \text{Terms with at least one} \\ &\qquad \text{factor} \sim \Gamma\end{aligned}\right\} \quad (8.16)$$

Hence at p in normal coordinates

$$\nabla_\lambda R^\alpha{}_{\beta\mu\nu} + \nabla_\nu R^\alpha{}_{\beta\lambda\mu} + \nabla_\mu R^\alpha{}_{\beta\nu\lambda}$$

$$\begin{aligned} = \; & \underbrace{\partial^2_{\lambda\mu} \Gamma_\nu{}^\alpha{}_\beta}_{1} - \underbrace{\partial^2_{\lambda\nu} \Gamma_\mu{}^\alpha{}_\beta}_{2} \\ + \; & \underbrace{\partial^2_{\nu\lambda} \Gamma_\mu{}^\alpha{}_\beta}_{2} - \underbrace{\partial^2_{\nu\mu} \Gamma_\lambda{}^\alpha{}_\beta}_{3} \\ + \; & \underbrace{\partial^2_{\mu\nu} \Gamma_\lambda{}^\alpha{}_\beta}_{3} - \underbrace{\partial^2_{\mu\lambda} \Gamma_\nu{}^\alpha{}_\beta}_{1} \qquad = 0 \end{aligned}$$

Since $\partial^2_{\mu\nu} = \partial^2 / \partial x^\mu \partial x^\nu$ is symmetric.

This shows

$$\boxed{\nabla_\lambda R^\alpha{}_{\beta\mu\nu} + \nabla_\nu R^\alpha{}_{\beta\lambda\mu} + \nabla_\mu R^\alpha{}_{\beta\nu\lambda} = 0}$$

2nd Bianchi - Id. (8.17)

If we contract the 2nd Bianchi Identity on α and μ we get

$$\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} + \nabla_\alpha R^\alpha{}_{\beta\nu\lambda} = 0$$

i.e.

$$\nabla_\alpha R^\alpha{}_{\beta\nu\lambda} = \nabla_\nu R_{\lambda\beta} - \nabla_\lambda R_{\nu\beta} \qquad (8.18)$$

One more contraction on β and λ (i.e. multiplication with $g^{\beta\lambda}$ using $\nabla_\alpha g^{\beta\lambda} = 0$)

$$\nabla_\alpha R^\alpha{}_\nu = \nabla_\nu R - \nabla_\lambda R^\lambda{}_\nu$$

$$\Leftrightarrow \quad \nabla_\alpha \left(R^\alpha{}_\nu - \tfrac{1}{2}\delta^\alpha{}_\nu R\right) = 0 \qquad (8.19)$$

$$\left.\begin{aligned} &\Leftrightarrow \quad \nabla_\alpha G^{\alpha\beta} = 0 \\ &\text{with} \quad G^{\alpha\beta} = R^{\alpha\beta} - \tfrac{1}{2} g^{\alpha\beta} R \end{aligned}\right\} \quad (8.20)$$

That the Einstein-Tensor is covariant divergence-free is a consequence of the 2nd Bianchi Identity (twice contracted).

Now we consider the commutator of two covariant derivatives

$$[\nabla_\mu, \nabla_\nu]\, T^{\alpha_1 \dots \alpha_l}{}_{\beta_1 \dots \beta_m} \qquad (8.21)$$

in normal coordinates at a point. Let first T be a vector field X

$$[\nabla_\mu, \nabla_\nu] X^\alpha = \nabla_\mu (\partial_\nu X^\alpha + \Gamma_\nu{}^\alpha{}_\beta X^\beta) - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \partial_\nu X^\alpha + \partial_\mu \Gamma_\nu{}^\alpha{}_\beta X^\beta + \text{Terms} \sim \Gamma$$

$$- (\mu \leftrightarrow \nu)$$

$$= (\partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \partial_\nu \Gamma_\mu{}^\alpha{}_\beta) X^\beta$$

$$= R^\alpha{}_{\beta\mu\nu} X^\beta$$

Since this is a tensorial equation it holds in all coord. systems

$$[\nabla_\mu, \nabla_\nu] X^\alpha = R^\alpha{}_{\beta\mu\nu} X^\beta \qquad (8.22)$$

Next we take a covector

$$[\nabla_\mu, \nabla_\nu] \Theta_\beta = \nabla_\mu (\partial_\nu \Theta_\beta - \Gamma_\nu{}^\alpha{}_\beta \Theta_\alpha) - (\mu \leftrightarrow \nu)$$

$$= \partial_\mu \partial_\nu \Theta_\beta - \partial_\mu \Gamma_\nu{}^\alpha{}_\beta \Theta_\alpha + \text{Terms} \sim \Gamma$$

$$- (\mu \leftrightarrow \nu)$$

$$= -(\partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \partial_\nu \Gamma_\mu{}^\alpha{}_\beta)\,\Theta_\alpha$$

$$= -R^\alpha{}_{\beta\mu\nu}\,\Theta_\alpha$$

Again: Since this is a tensorial equation it holds in all coord. systems

$$[\nabla_\mu, \nabla_\nu]\,\Theta_\beta = -R^\alpha{}_{\beta\mu\nu}\,\Theta_\alpha \qquad (8.23)$$

In the general case we get more terms of the same sort, one for each upper and one for each lower index. The result is

$$\left.\begin{aligned} &[\nabla_\mu, \nabla_\nu]\, T^{\alpha_1 \dots \alpha_l}{}_{\beta_1 \dots \beta_m} \\ &= \sum_{i=1}^{l} R^{\alpha_i}{}_{\lambda\mu\nu}\, T^{\alpha_1 \dots \alpha_{i-1} \lambda \alpha_{i+1} \dots \alpha_l}{}_{\beta_1 \dots \beta_m} \\ &- \sum_{i=1}^{m} R^{\lambda}{}_{\beta_i \mu\nu}\, T^{\alpha_1 \dots \alpha_l}{}_{\beta_1 \dots \beta_{i-1} \lambda \beta_{i+1} \dots \beta_m} \end{aligned}\right\} \qquad (8.24)$$

Application and Example

Suppose we have a vector field X on our Lorentz manifold (M, g) such that

$$L_X g = 0 \tag{8.25}$$

Such an X is called a <u>Killing (vector) field</u>.

In passing we note that Killing fields form a sub-Lie-algebra of the Lie-algebra of all vector fields. This is an immediate consequence of the general identity

$$[L_X, L_Y] = L_{[X,Y]} \tag{8.26}$$

that we proved in the lecture on Diff-Geom. Indeed, if X, Y are Killing, i.e. $L_X g = L_Y g = 0$ implies $[L_X, L_Y] g = 0$ and (8.26) then $L_{[X,Y]} g = 0$. So if X and Y are Killing their commutator is again Killing. What we are going to show next will imply that the Lie-algebra of Killing vector fields on a n-dimensional manifold is at most $\frac{1}{2} n(n+1)$-dimensional.

Now, we have seen in Example 4) on page L7.14 that

$$(L_X g)_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha \qquad (8.27)$$

That X be Killing hence means that $\nabla_\alpha X_\beta$ is antisymmetric

$$\nabla_\alpha X_\beta = -\nabla_\beta X_\alpha \qquad (8.28)$$

On the other hand, we have just derived that

$$\nabla_\mu \nabla_\nu X_\beta = \nabla_\nu \nabla_\mu X_\beta - R^\alpha{}_{\beta\mu\nu} X_\alpha \qquad (8.29)$$

We consider the second covariant derivative of X which we modify by first aplying (8.28) and then (8.29), and that three times running

$$\nabla_\mu \nabla_\nu X_\beta$$

$$= -\nabla_\mu \nabla_\beta X_\nu$$

$$= -\nabla_\beta \nabla_\mu X_\nu + R^\alpha{}_{\nu\mu\beta} X_\alpha \qquad (8.30)$$

this we do two more times ...

$$= \nabla_\beta \nabla_\nu X_\mu + R^\alpha{}_{\nu\mu\beta} X_\alpha$$

$$= \nabla_\nu \nabla_\beta X_\mu + (R^\alpha{}_{\nu\mu\beta} - R^\alpha{}_{\mu\beta\nu}) X_\alpha$$

(.. once more ..)

$$= -\nabla_\nu \nabla_\mu X_\beta + (\cdots)^\alpha X_\alpha$$

$$= -\nabla_\mu \nabla_\nu X_\beta + (R^\alpha{}_{\nu\mu\beta} - R^\alpha{}_{\mu\beta\nu} + R^\alpha{}_{\beta\nu\mu}) X_\alpha \qquad (8.31)$$

Hence

$$2 \nabla_\mu \nabla_\nu X_\beta = (R^\alpha{}_{\nu\mu\beta} + R^\alpha{}_{\beta\nu\mu} - R^\alpha{}_{\mu\beta\nu}) X_\alpha \qquad (8.32)$$

By the 1st. Bianchi Identity (8.8) we have

$$R^\alpha{}_{\nu\mu\beta} + R^\alpha{}_{\beta\nu\mu} = - R^\alpha{}_{\mu\beta\nu}$$

which allows to write the term in brackets on the right-hand side of (8.32) as $-2 R^\alpha{}_{\mu\beta\nu} = 2 R^\alpha{}_{\mu\nu\beta}$. Hence

$$\boxed{\nabla_\mu \nabla_\nu X_\beta = R^\alpha{}_{\mu\nu\beta} X_\alpha} \qquad (8.33)$$

This equation is satisfied by any Killing field X.

Note that the Killing equation (8.28) merely states that the symmetric part of the first derivatives are determined by the fields:

$$\partial_\alpha X_\beta + \partial_\beta X_\alpha = 2\Gamma^\gamma_{\alpha\beta} X_\gamma$$

(8.33) implies that the fields determine the second and, in fact, all higher derivatives. Hence the only freedom in specifying X is its values $X^\alpha(p)$ (n values) and the antisymmetric part of its derivative:
$\nabla_\alpha X_\beta - \nabla_\beta X_\alpha = \partial_\alpha X_\beta - \partial_\beta X_\alpha = (dX)_{\alpha\beta}$.
Together this makes $n + \frac{1}{2} n(n-1) =$

$$\# \text{ Killing} = \frac{1}{2} n(n+1) \qquad (8.34)$$

freely specifiable initial conditions. Since all equations are linear, there is an at most $\frac{1}{2} n(n+1)$ - dimensional linear space of solutions.

Finally we turn to the decomposition of the Riemann tensor. We wish to write $R^{\alpha}{}_{\beta\mu\nu}$ in a form that displays that part of it that is determined by the Ricci-tensor. The remainder is then that part which is <u>not</u> determined by Einsteins equations (locally). This remainder is called the <u>Weyl-tensor</u>. Since the Ricci tensor is the trace of the Riemann tensor

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta} \qquad (8.35)$$

and the Ricci scalar is the trace of that,

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^{\lambda\sigma}{}_{\lambda\sigma} \qquad (8.36)$$

the Weyl-tensor will be the trace-free part of the Riemann tensor. The trace part of Riem will be a tensor of the same rank and the same symmetries built from g and Ric and linear in the latter

Now, given two symmetric tensors of rank 2, h and k, a tensor of rank 4 with the symmetries of Riem can be formed using the following

Kulkarni - Nomizu Product $\owedge$

$$(h \owedge k)_{\alpha\beta\mu\nu} =$$

$$h_{\alpha\mu} k_{\beta\nu} + h_{\beta\nu} k_{\alpha\mu}$$
$$- h_{\alpha\nu} k_{\beta\mu} - h_{\beta\mu} k_{\alpha\nu} \tag{8.37}$$

One easily shows that

$$(h \owedge k)_{\alpha\beta\mu\nu} = -(h \owedge k)_{\alpha\beta\nu\mu}$$
$$= -(h \owedge k)_{\beta\alpha\mu\nu}$$
$$= (h \owedge k)_{\mu\nu\alpha\beta} \tag{8.38}$$

and

$$(h \wedge k)_{\alpha\beta\mu\nu} + (h \owedge k)_{\alpha\nu\beta\mu} + (h \owedge k)_{\alpha\mu\nu\beta} = 0 \tag{8.39}$$

From Ric and R we can form the combinations

$$g \owedge R \quad \text{and} \quad R \cdot g \owedge g$$

Hence we set for $a, b \in \mathbb{R}$

$$\text{Riem} = \text{Weyl} + a\, g \owedge \text{Ric} + b\, R\, g \owedge g \tag{8.40}$$

and determine a and b by $\text{trace}(\text{Weyl}) = 0$.

Note that

$$(g \owedge g)_{\alpha\beta\mu\nu} = 2\,(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \quad (8.41)$$

hence we have in components

$$R_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu}$$
$$+ a\left(g_{\alpha\mu} R_{\beta\nu} + g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\nu} R_{\beta\mu} - g_{\beta\mu} R_{\alpha\nu}\right)$$
$$+ 2b R\,(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) \qquad (8.42)$$

Taking the trace on (α, μ), leaving the dimension $= n$ open, we get from $W^{\lambda}{}_{\alpha\lambda\beta} = 0$:

$$R_{\beta\nu} = a\left((n-2) R_{\beta\nu} + g_{\beta\nu} R\right)$$
$$+ 2b R\,(n-1)\, g_{\beta\nu} \qquad (8.43)$$

Which is an identity in Ric and R only if

$$a\,(n-2) = 1 \quad \text{and} \quad a + 2b\,(n-1) = 0$$

$$\Rightarrow \quad \left.\begin{aligned} a &= \frac{1}{n-2} \\ b &= -\frac{1}{2(n-1)(n-2)} \end{aligned}\right\} (8.44)$$

Therefore

$$\text{Weyl} = \text{Riem} - \frac{g \owedge \text{Ric}}{n-2} + R\,\frac{g \owedge g}{2(n-1)(n-2)} \tag{8.45}$$

or in components

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu}$$
$$- \frac{1}{(n-2)}\Big\{ g_{\alpha\mu} R_{\beta\nu} + g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\nu} R_{\beta\mu} - g_{\beta\mu} R_{\alpha\nu} \Big\}$$
$$+ \frac{R}{(n-1)(n-2)} \Big\{ g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \Big\} \tag{8.46}$$

Another way to write this is to use the trace-free part of the Ricci tensor on the right hand side, i.e

$$\widetilde{\text{Ric}} := \text{Ric} - \frac{1}{n}\, g\, R \tag{8.47}$$

Then

$$\text{Weyl} = \text{Riem} - \frac{g \owedge \widetilde{\text{Ric}}}{n-2} - R\,\frac{g \owedge g}{2n(n-1)} . \tag{8.48}$$

The Weyl-Tensor has all the symmetries (8.5-8) of the Riemann tensor and in addition satisfies

$$W^{\lambda}{}_{\alpha\lambda\beta} = 0 \qquad (8.49)$$

which, because of $W^{\lambda}{}_{\alpha\lambda\beta} = W^{\lambda}{}_{\beta\lambda\alpha}$ are $\frac{1}{2}n(n+1)$ independent conditions. Hence it has

$$\# \text{Weyl} = \frac{1}{12} n^2(n^2-1) - \frac{1}{2}n(n+1) \qquad (8.50)$$

independent components

$$\# \text{Weyl} = \begin{cases} 0 & n=3 \\ 10 & n=4 \\ 35 & n=5 \end{cases} \qquad (8.51)$$

The Weyl tensor vanishes identically in $n=3$ dimensions so that

$$\text{Riem} = \frac{1}{2}\, g \otimes \text{Ric} - \frac{1}{4} R\, g \otimes R \qquad (8.52)$$
$$(\text{for } n=3)$$

A Riemannian or Semi-Riemannian manifold whose Weyl and traceless Ricci tensor both vanish, i.e.

$$\text{Weyl} = \tilde{\text{Ric}} = 0, \qquad (8.53)$$

is said to be of <u>constant curvature</u>.

Constant curvature means that the Riemann tensor is <u>pure trace</u>
From (8.48) we infer (Weyl = $\tilde{R}ic = 0$) :

$$\mathrm{Riem} = R \, \frac{g \owedge g}{2n(n-1)} \tag{8.54}$$

or in components

$$R_{\alpha\beta\mu\nu} = \frac{R}{n(n-1)} \left(g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \right) \tag{8.55}$$

The Ricci-tensor is then

$$R_{\alpha\beta} = \frac{1}{n} R \, g_{\alpha\beta} \tag{8.56}$$

and the Einstein tensor

$$G_{\alpha\beta} = \frac{1}{n} R g_{\alpha\beta} - \frac{1}{2} R \, g_{ab}$$

$$= \frac{2-n}{2n} R \, g_{\alpha\beta} \tag{8.57}$$

From the twice contracted 2. Bianchi-Identity we obtained (8.20),

$$\nabla_\alpha G^{\alpha\beta} = 0 \, ,$$

which for (8.57) implies ($\nabla g = 0$)

$$R = \mathrm{const.} \tag{8.58}$$

One more remark on the Weyl tensor:
It has the remarkable and useful property to be <u>conformally</u> <u>invariant</u>. By this the following is meant:

Let g and g' be two conformally related C^2-metrics on the manifold M, that is, there is a function $\phi \in C^2(M, \mathbb{R})$ such that

$$g' = \exp(\phi)\, g \qquad (8.59)$$

then

$$\mathrm{Weyl}[g'] = \mathrm{Weyl}[g] \qquad (8.60)$$

i.e.

$$W^{\alpha}{}_{\beta\mu\nu}[g'] = W^{\alpha}{}_{\beta\mu\nu}[g] \qquad (8.61)$$

Note that if we define $W_{\alpha\beta\mu\nu}[g'] := g'_{\alpha\lambda} W^{\lambda}{}_{\beta\mu\nu}[g']$ and $W_{\alpha\beta\mu\nu}[g] = g_{\alpha\lambda} W^{\lambda}{}_{\beta\mu\nu}[g]$, then this implies

$$W_{\alpha\beta\mu\nu}[g'] = \exp(\phi)\, W_{\alpha\beta\mu\nu}[g] \qquad (8.62)$$

Because of (8.60) the Weyl tensor is also called "conformal curvature" and usually denoted by the letter $\mathbf{C}$, i.e. $C^{\alpha}{}_{\beta\mu\nu}$ instead of $W^{\alpha}{}_{\beta\mu\nu}$.