Lecture 9

The Action Principle in GR.

We wish to understand Einstein's field equation

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} - g_{\alpha\beta}\Lambda = \kappa T_{\alpha\beta} \qquad (9.1)$$

as the result of the requirement that some "action" be stationary

$$\delta \int_\Omega (\mathcal{L}_{grav} + \mathcal{L}_{mat})\, d\mu_g = 0 \qquad (9.2)$$

where $\mathcal{L}_{grav}$ and $\mathcal{L}_{mat}$ are the "Lagrange-densities" for the gravitational field and the matter, respectively. $d\mu_g$ is the volume form on the space-time manifold (M, g), which will depend on g (obviously!).

We first determine $d\mu_g$. Clearly, if $\{e_0, \dots, e_3\}$ is an orthonormal basis-field and $\{\theta^0, \dots, \theta^3\}$ its dual basis-field, then

$$d\mu_g = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \qquad (9.3)$$

because this gives $d\mu_g(e_1, \dots, e_3) = 1$, which is just what we want: An orthonormal parallelpiped spanned by $\{e_0, e_1, e_2, e_3\}$

has unit volume. Let $(\mathcal{U}, \phi)$ be a chart with coordinate functions $X^\alpha : \mathcal{U} \to \mathbb{R}$. Then we can expand each θ^α:

$$\theta^\alpha = \theta^\alpha{}_\beta \, dX^\beta. \qquad (9.4)$$

Hence

$$\begin{aligned} d\mu_g &= \theta^0{}_\alpha \theta^1{}_\beta \theta^2{}_\gamma \theta^3{}_\delta \; dX^\alpha \wedge dX^\beta \wedge dX^\gamma \wedge dX^\delta \\ &= \theta^0{}_{[\alpha} \theta^1{}_\beta \theta^2{}_\gamma \theta^3{}_{\delta]} \; dX^\alpha \wedge dX^\beta \wedge dX^\gamma \wedge dX^\delta \\ &= 4! \, \theta^0{}_{[\alpha} \theta^1{}_\beta \theta^2{}_\gamma \theta^3{}_{\delta]} \; dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \end{aligned} \qquad (9.5)$$

where square brackets denote total anti-symmetrisation:

$$\theta^0{}_{[\alpha} \theta^1{}_\beta \theta^2{}_\gamma \theta^3{}_{\delta]}$$

$$= \frac{1}{4!} \underbrace{\sum_{\sigma \in S_4}}_{\text{Symmetric group of 4 elements}} \operatorname{sign}(\sigma) \, \theta^0{}_{\sigma(0)} \theta^1{}_{\sigma(1)} \theta^2{}_{\sigma(2)} \theta^3{}_{\sigma(3)}$$

$$= \frac{1}{4!} \det\{\theta^\alpha{}_\beta\} \qquad (9.6)$$

Hence

$$\left.\begin{aligned} d\mu_g &= \det\{\theta^\alpha{}_\beta\} \, d^4X \\ \text{where } d^4X &:= dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 \end{aligned}\right\} \qquad (9.7)$$

On the other hand, the metric g has, in terms of $\{\theta^0, \dots, \theta^3\}$ and $\{dx^0, \dots, dx^3\}$, the expansions

$$g = \eta_{\alpha\beta}\, \theta^\alpha \otimes \theta^\beta$$
$$= \eta_{\alpha\beta}\, \theta^\alpha{}_\mu\, \theta^\beta{}_\nu\, dx^\mu \otimes dx^\nu$$
$$= g_{\mu\nu}\, dx^\mu \otimes dx^\nu \tag{9.8}$$

so that

$$g_{\mu\nu} = \eta_{\alpha\beta}\, \theta^\alpha{}_\mu\, \theta^\beta{}_\nu \tag{9.9}$$

Taking the determinant of that, using $\det\{\eta_{\alpha\beta}\} = \det\{\mathrm{diag}(1,-1,-1,-1)\} = -1$

$$\det\{g_{\alpha\beta}\} = -\left[\det\{\theta^\alpha{}_\beta\}\right]^2$$

$$\rightsquigarrow \quad \det\{\theta^\alpha{}_\beta\} = \sqrt{-\det\{g_{\alpha\beta}\}} \tag{9.10}$$

and therefore

$$d\mu_g\big|_u = \sqrt{\det\{-g_{\alpha\beta}\}}\; d^4x \tag{9.11}$$

The gravitational Lagrange-density $\mathcal{L}_g$ must be a scalar function of g alone, have functional derivative linear in the Ricci-tensor. An obvious guess is the Ricci scalar

$$R = g^{\alpha\beta} R_{\alpha\beta} \tag{9.12}$$

$$\begin{aligned} R_{\alpha\beta} &= R^{\lambda}{}_{\alpha\lambda\beta} \\ &= \partial_\lambda \Gamma_\beta{}^\lambda{}_\alpha - \partial_\beta \Gamma_\lambda{}^\lambda{}_\alpha \\ &\quad + \Gamma_\lambda{}^\lambda{}_\sigma \Gamma_\beta{}^\sigma{}_\alpha - \Gamma_\beta{}^\lambda{}_\sigma \Gamma_\lambda{}^\sigma{}_\alpha \end{aligned} \tag{9.13}$$

We now consider a differentiable one-parameter family of Lorentzian metrics

$$\left.\begin{aligned} \gamma : (-\varepsilon, \varepsilon) &\longrightarrow ST^0_2(M) \\ s &\longmapsto \gamma_{\alpha\beta}(s) \end{aligned}\right\} \tag{9.14}$$

such that

$$\gamma_{\alpha\beta}(0) = g_{\alpha\beta} \tag{9.15}$$

$$\left.\frac{d}{ds}\right|_{s=0} \gamma_{\alpha\beta}(s) = \dot{g}_{\alpha\beta} \tag{9.16}$$

Here $g_{\alpha\beta}$ is a Lorentz metric and $\dot{g}_{\alpha\beta}$ some $\in ST^0_2(M)$. We need to calculate

$$\frac{d}{ds}\Big|_{s=0} \int_\Omega R[\gamma(s)]\, d\mu_{\gamma(s)}$$

$$= \frac{d}{ds}\Big|_{s=0} \int_\Omega \gamma^{\alpha\beta}(s)\, R_{\alpha\beta}[\gamma(s)] \left[-\det\{\gamma_{\mu\nu}(s)\}\right]^{1/2} d^4x \tag{9.17}$$

For this we need to know

I. $\frac{d}{ds}\Big|_{s=0} \gamma^{\alpha\beta}(s)$, (9.18)

II. $\frac{d}{ds}\Big|_{s=0} R_{\alpha\beta}[\gamma(s)]$, (9.19)

III. $\frac{d}{ds}\Big|_{s=0} \left[-\det\{\gamma_{\mu\nu}(s)\}\right]^{1/2}$. (9.20)

We first calculate I and III, which are easy, and then II, which is more difficult.

<u>Calculation of I.</u>

$\gamma_{\alpha\beta}(s)$ is a one-parameter family of Lorentzian metrics, hence

$$\gamma^{\alpha\sigma}(s)\, \gamma_{\sigma\beta}(s) = \delta^\alpha{}_\beta \tag{9.21}$$

Taking the $\frac{d}{ds}\Big|_{s=0}$ of that and using the notation (9.15-16), we get

$$\left(\frac{d}{ds}\Big|_{s=0} \gamma^{\alpha\sigma}(s)\right) g_{\sigma\beta} + g^{\alpha\sigma}\, \dot{g}_{\sigma\beta} = 0 \tag{9.22}$$

or

$$\frac{d}{ds}\Big|_{s=0} \gamma^{\alpha\beta}(s) =: \dot{g}^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \dot{g}_{\mu\nu} \qquad (9.23)$$

note the minus sign!

Calculation of III.

Quite generally, if $s \mapsto \gamma_{\alpha\beta}(s)$ is any differentiable curve of $n \times n$ matrices, such that $\gamma_{\alpha\beta}(0) = g_{\alpha\beta}$ is invertible, i.e. $\det\{g_{\alpha\beta}\} \neq 0$, so that $g^{\alpha\beta}$ are the comp. of the inverse matrix to g, then, in matrix notation (supressing indices)

$$\frac{d}{ds}\Big|_{s=0} \det(\gamma(s))$$

$$= \frac{d}{ds}\Big|_{s=0} \det\left(g \cdot g^{-1} \cdot \gamma(s)\right)$$

$$= \det(g) \frac{d}{ds}\Big|_{s=0} \det\left(g^{-1}\gamma(s)\right) \qquad (9.24)$$

but $s \mapsto g^{-1}\gamma(s)$ is a curve in the linear space of endomorphisms through the identity for $s=0$. By the chain rule and the fact that the derivative of the determinant function $\det : \mathrm{End}(V) \to \mathbb{R}$ at $\mathrm{id}_V \in \mathrm{End}(V)$ is the trace function, i.e.

$$D \det\Big|_{id} = \mathrm{trace} \tag{9.25}$$

We get

$$\frac{d}{ds}\Big|_{s=0} \det(\gamma(s))$$

$$= \det(g)\ \mathrm{trace}(g^{-1}\dot{g}) \tag{9.26}$$

Therefore

$$\frac{d}{ds}\Big|_{s=0} \left[-\det\{\gamma_{\alpha\beta}(s)\}\right]^{1/2}$$

$$= \frac{1}{2}\left[-\det\{g_{\alpha\beta}\}\right]^{-1/2} \frac{d}{ds}\Big|_{s=0}(-\det(\gamma(s)))$$

$$= \frac{1}{2}\left[-\det\{g_{\alpha\beta}\}\right]^{1/2}\left(g^{\alpha\beta}\dot{g}_{\beta\alpha}\right) \tag{9.27}$$

or, since in our case g is symmetric

$$= \frac{1}{2}\left[-\det\{g_{\alpha\beta}\}\right]^{1/2} g^{\alpha\beta}\dot{g}_{\alpha\beta}$$

$$= -\frac{1}{2}\left[-\det\{g_{\alpha\beta}\}\right]^{1/2} g_{\alpha\beta}\dot{g}^{\alpha\beta} \tag{9.28}$$

where we used (9.23), which implies

$$g_{\alpha\beta}\dot{g}^{\alpha\beta} = -g_{\alpha\beta}g^{\alpha\mu}g^{\beta\nu}\dot{g}_{\mu\nu}$$

$$= -g^{\mu\nu}\dot{g}_{\mu\nu} = -g^{\alpha\beta}\dot{g}_{\alpha\beta} \tag{9.29}$$

Hence, as the result for III. we get

$$\frac{d}{ds}\Big|_{s=0} d\mu_{\gamma(s)}$$

$$= \frac{1}{2}\left(g^{\alpha\beta}\mathring{g}_{\alpha\beta}\right) d\mu_g$$

$$= -\frac{1}{2}\left(g_{\alpha\beta}\mathring{g}^{\alpha\beta}\right) d\mu_g \qquad (9.30)$$

Calculation of II.

Here, once more, we we employ the trick of using normal coordinates. So let $p \in M$ and $X^\alpha : \mathcal{U} \to \mathbb{R}$ the normal coord. in a n'hood $\mathcal{U}$ of p so that

$$\left.\begin{aligned} g_{\alpha\beta}(p) &= \eta_{\alpha\beta} \\ g_{\alpha\beta,\gamma}(p) &= 0 = \Gamma^{\gamma}_{\alpha\beta}(p) \end{aligned}\right\} \quad (9.31)$$

Then the $\Gamma\Gamma$-terms in (9.13) do not contribute to $d/ds|_{s=0}\, R_{\alpha\beta}[\gamma(s)]$, because they always have an undifferentiated Γ at p. Moreover,

$$\frac{d}{ds}\Big|_{s=0} \Gamma^{\lambda}_{\alpha\beta}[\gamma(s)] =: \dot{\Gamma}^{\lambda}_{\alpha\beta}[g] \qquad (9.32)$$

are tensor fields. This follows explicitly from (7.11) or (7.13) where the inhomogeneous terms vanish when differentiated with respect to s

$$\bar{\Gamma}^{\sigma}_{\mu\nu}(s) = \Gamma^{\lambda}_{\alpha\beta}(s)\,\frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}}\,\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\,\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} + \frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}}\,\frac{\partial^2 x^{\lambda}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}} \qquad \begin{array}{r}(9.33)\\ =(7.11)\end{array}$$

$\frac{d}{ds}\Big|_0$ of that gives

$$\dot{\bar{\Gamma}}^{\sigma}_{\mu\nu} = \dot{\Gamma}^{\lambda}_{\alpha\beta}\,\frac{\partial \bar{x}^{\sigma}}{\partial x^{\lambda}}\,\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}}\,\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \qquad (9.34)$$

The proper differential-geometric statement is: given two connections

$$\nabla, \bar{\nabla} : ST^1_0(M) \times ST^1_0(M) \to ST^1_0(M)$$

then their difference,

$$\Delta := \bar{\nabla} - \nabla \; : ST^1_0(M) \times ST^1_0(M) \to ST^1_0(M)$$

is $C^{\infty}(M,\mathbb{R})$ - linear in both arguments and hence defines an element of $ST^1_2(M)$.

$C^{\infty}(M,\mathbb{R})$ - linearity is easy to see

$$\bar{\nabla}_X (fY) - \nabla_X (fY) = f(\bar{\nabla}_X Y - \nabla_X Y)$$

$$Y(X(f) - X(f)) .$$

See Script on Diff. Geom. for details.

Therefore we get at p in normal coordinates

$$\left[\frac{d}{ds}\Big|_{s=0} R_{\alpha\beta}[\gamma(s)]\right]_p$$

$$= \left[\partial_\lambda \dot{\Gamma}^\lambda_{\beta\alpha} - \partial_\beta \dot{\Gamma}^\lambda_{\lambda\alpha}\right]_p$$

$$= \left[\nabla_\lambda \dot{\Gamma}^\lambda_{\beta\alpha} - \nabla_\beta \dot{\Gamma}^\lambda_{\lambda\alpha}\right]_p$$

$$= \nabla_\lambda \left[\dot{\Gamma}^\lambda_{\beta\alpha} - \delta^\lambda_\beta \dot{\Gamma}^\sigma_{\sigma\alpha}\right]_p \qquad (9.35)$$

where in the second step we were allowed to replace $\partial_\lambda \to \nabla_\lambda$ since the cov der. contains terms proportional to Γ's which vanish at p. But now both sides of this equation are tensorial and hold in normal coordinates; hence they hold generally:

$$\frac{d}{ds}\Big|_{s=0} R_{\alpha\beta}[\gamma(s)]$$

$$= \nabla_\lambda \left[\dot{\Gamma}^\lambda_{\beta\alpha} - \delta^\lambda_\beta \dot{\Gamma}^\sigma_{\sigma\alpha}\right] \qquad (9.36)$$

As an immediate corollary we get

$$g^{\alpha\beta} \left.\frac{d}{ds}\right|_{s=0} R_{\alpha\beta}[\gamma(s)]$$

$$= \nabla_\lambda \dot{V}^\lambda \qquad (9.37)$$

where

$$\dot{V}^\lambda := g^{\alpha\beta} \dot{\Gamma}^\lambda_{\alpha\beta} - g^{\lambda\alpha} \dot{\Gamma}^\sigma_{\sigma\alpha} \qquad (9.38)$$

Taken all this together, i.e. (9.23) for I., (9.30) for III., and (9.36-37) for II. we get

$$\left.\frac{d}{ds}\right|_{s=0} \left(\gamma^{\alpha\beta}(s) R_{\alpha\beta}[\gamma(s)] \, d\mu_{\gamma(s)} \right)$$

$$= \Big\{ \left(R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R \right) \dot{g}^{\alpha\beta}$$

$$+ \nabla_\lambda \dot{V}^\lambda \Big\} \, d\mu_g \qquad (9.39)$$

If we want to include the cosmological constant we add $2\Lambda \, d\mu_g$ to the left-hand side:

$$\frac{d}{ds}\Big|_{s=0}\left[\left(\gamma^{\alpha\beta}(s)R_{\alpha\beta}[\gamma(s)]+2\Lambda\right)d\mu_{\gamma(s)}\right]$$

$$= \left\{\left(G_{\alpha\beta}-\Lambda g_{\alpha\beta}\right)\dot{g}^{\alpha\beta}+\nabla_\lambda \dot{V}^\lambda\right\}d\mu_g \qquad (9.40)$$

The vector field $\dot{V}^\lambda$ contains via the $\dot{\Gamma}$ the $\dot{g}$ and their space-time derivatives. We will calculate the explicit expressions below. But if we assume the $\dot{g}$ and their derivatives to be zero on the boundary $\partial\Omega$ of the integration domain Ω, then, for

$$\left.\begin{aligned} S_{grav}[\Omega, s] &:= \int_\Omega \mathcal{L}_{grav}[\gamma(s)]\, d\mu_{\gamma(s)} \\ \mathcal{L}_{grav} &= \underbrace{A}_{\text{const.}}\,(R+2\Lambda) \end{aligned}\right\} \quad (9.41)$$

we get

$$\left.\begin{aligned} &\frac{d}{ds}\Big|_{s=0} S_{grav}[\Omega, s] \\ &= A\int_\Omega \left(G_{\alpha\beta}-\Lambda g_{\alpha\beta}\right)\dot{g}^{\alpha\beta}\, d\mu_g \end{aligned}\right\} \quad (9.42)$$

Physicists like to write this as

$$\frac{\delta S_{grav}}{\delta g^{\alpha\beta}} = A\left(G_{\alpha\beta}-\Lambda g_{\alpha\beta}\right) \qquad (9.43)$$

The dimensionful constant A is to be determined in such a way so as to give the Integral the right physical dimension (that of an action) and numerical normalisation.

Then, in order to arrive at Einstein's equations (9.1) we set

$$S_{mat}[\Omega, \phi, s] = \int_\Omega \mathcal{L}_{mat}[\phi, \gamma(s)]\, d\mu_{\gamma(s)} \quad (9.44)$$

where ϕ represents collectively all matter variables (fields) and the curve $s \mapsto \gamma(s)$ is inserted at any position where $\mathcal{L}_{mat}$ depends on the spacetime metric g.
Then we must have

$$\frac{d}{ds}\Big|_{s=0} S_{mat}[\Omega, \phi, s]$$

$$= -A\kappa \int_\Omega T_{\alpha\beta}[\phi, g]\, \dot{g}^{\alpha\beta}\, d\mu_g \quad (9.45)$$

for then,

$$0 = \frac{d}{ds}\Big|_{s=0} \left\{ S_{grav}[\Omega, s] + S_{mat}[\Omega, \phi, s] \right\} \quad (9.46)$$

is equivalent to

$$\int_\Omega \left[A(G_{\alpha\beta} - \Lambda g_{\alpha\beta}) - A\kappa T_{\alpha\beta} \right] \dot{g}^{\alpha\beta}\, d\mu_g = 0 \quad (9.47)$$

For this to be true for all symmetric $\dot{g}^{\alpha\beta} \in \mathbf{S}T^2_0(M)$, we must have that $[\cdot\cdot]$ under the integral to vanish (main theorem of variational calculus), and this is equivalent to Einstein's equation.

Note that this entails a definition of energy-momentum tensors for matter which can be applied to actually calculate it for given matter actions (Lagrange-densities), even in the case of special-relativistic theories.

The prescription is this: Take the special-relativistic matter action on Minkowski space (Minkowski-metric η):

$$S_{mat}[\Omega, \phi, \eta] = \int_\Omega \mathcal{L}^{(SRT)}_{mat}[\phi, \eta]\, d\mu_\eta \qquad (9.48)$$

in which all implicit dependences on the Minkowski metric η are made explicit – in particular in the derivative operations: all partial derivatives with respect to affine coordinates in Minkowski space are, in fact, Levi-Civita covariant derivatives with respect to η. Then we replace

$$\eta \longmapsto \gamma(s)\,, \quad \nabla^{(\eta)} \longrightarrow \nabla^{\gamma(s)} \qquad (9.49)$$

($\nabla^{(\eta)}$ = Levi-Civita cov. der. wrt. η
$\nabla^{\gamma(s)}$ = Levi-Civita cov. der. wrt. $\gamma(s)$)

and set

$$\mathcal{L}_{mat}[\phi, \gamma(s)] := \mathcal{L}^{(SRT)}_{mat}[\phi, \eta \mapsto \gamma(s)] \qquad (9.50)$$

$T_{\alpha\beta}[\phi, g]$ is then determined by (9.45). In particular, the special relativistic energy momentum tensor is obtained by

$$T^{(SRT)}_{\alpha\beta}[\phi] := T_{\alpha\beta}[\phi, g=\eta]. \qquad (9.51)$$

Note that because of (9.23), i.e.

$$\dot{g}^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \dot{g}_{\mu\nu}$$

we have

$$\frac{d}{ds}\Big|_{s=0} S_{mat}[\Omega, \phi, s]$$

$$= -A\kappa \int_\Omega T_{\alpha\beta}[\phi, g]\, \dot{g}^{\alpha\beta}\, d\mu_g$$

$$= +A\kappa \int_\Omega T^{\alpha\beta}[\phi, g]\, \dot{g}_{\alpha\beta}\, d\mu_g \qquad (9.52)$$

This is sometimes written as

$$\left.\begin{aligned} \frac{\delta S_{mat}}{\delta g^{\alpha\beta}} &= -A\kappa\, T_{\alpha\beta} \\ \frac{\delta S_{mat}}{\delta g_{\alpha\beta}} &= +A\kappa\, T^{\alpha\beta} \end{aligned}\right\} \qquad (9.53)$$

If we assume the matter Lagrange density $\mathcal{L}_{mat}$, which is a function on M, to depend on $x \in M$ besides the matter fields only on the metric at x and its first derivatives at x, then

$$\frac{d}{ds}\Big|_{s=0} \int_\Omega \mathcal{L}_{mat}[\phi, g^{\alpha\beta} = \gamma^{\alpha\beta}(s)]\, d\mu_{\gamma(s)}$$

$$= \int_\Omega \left\{ \frac{\partial \mathcal{L}_{mat}}{\partial g^{\alpha\beta}} - \partial_\lambda \left(\frac{\partial \mathcal{L}_{mat}}{\partial g^{\alpha\beta}_{\ \ ,\lambda}} \right) - \frac{1}{2} \mathcal{L}_{mat}\, g_{\alpha\beta} \right\} \dot{g}^{\alpha\beta}\, d\mu_g \tag{9.54}$$

hence

$$T_{\alpha\beta} = -A\kappa \left\{ \frac{\partial \mathcal{L}_{mat}}{\partial g^{\alpha\beta}} - \partial_\lambda \left(\frac{\partial \mathcal{L}_{mat}}{\partial g^{\alpha\beta}_{\ \ ,\lambda}} \right) - \frac{1}{2} \mathcal{L}_{mat}\, g_{\alpha\beta} \right\} \tag{9.57}$$

In many cases the dependence of $\mathcal{L}_{mat}$ on g is even ultra-local, i.e. <u>independent</u> of the derivatives. This is e.g. the case in Electromagnetism, where the electromagnetic field is represented by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{9.55}$$

and, in SRT, (recall $dx^0 = c\,dt$)

$$\mathcal{L}_{EM} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2\mu_0} (\vec{E}^2/c^2 - \vec{B}^2) \tag{9.56}$$

Making the η-dependence explicit means to write $\mathcal{L}_{EM}$ in terms of $F_{\mu\nu}$ (all indices down) and η:

$$\mathcal{L}_{EM}[F,\eta] = -\frac{1}{4\mu_0} F_{\alpha\mu} F_{\beta\nu} \eta^{\alpha\beta} \eta^{\mu\nu}. \quad (9.57)$$

Replacing η with g then results in

$$\mathcal{L}_{EM}[F,g] = -\frac{1}{4\mu_0} F_{\alpha\mu} F_{\beta\nu} g^{\alpha\beta} g^{\mu\nu} \quad (9.58)$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}_{EM}}{\partial g^{\alpha\beta}} = -\frac{1}{2\mu_0 c} F_{\alpha\mu} F_\beta{}^\mu$$

$$\left(\frac{\partial \mathcal{L}_{EM}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L}_{EM}\, g_{\alpha\beta} \right)$$

$$= \frac{1}{2\mu_0} \left[-F_{\alpha\mu} F_\beta{}^\mu + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} \right] \quad (9.59)$$

Setting $g = \eta$ again one obtains (or, e.g., the (0 0) component

$$\frac{1}{2\mu_0} \left[-F_{0m} F_0{}^m + \frac{1}{4} \eta_{00} \left(2 F_{0n} F^{0n} + F_{mn} F^{mn} \right) \right]$$

$$= \frac{1}{2\mu_0} \left[-\frac{1}{2} F_{0m} F_0{}^m + \frac{1}{4} F_{ab} F^{ab} \right]$$

$$= \frac{1}{2\mu_0} \left[\frac{1}{2} \vec{E}^2/c^2 + \frac{1}{2} \vec{B}^2 \right]$$

$$\frac{1}{2} \left[\frac{1}{2} \varepsilon_0 \vec{E}^2 + \frac{1}{2\mu_0} \vec{B}^2 \right] = \frac{1}{2} T_{00}^{(EM)}. \quad (9.60)$$

From this we may infer from (9.55) that $-AK/2 = 1$, or

$$A = -\frac{2}{K} = -\frac{c^4}{4\pi G} \qquad (9.61)$$

Hence

$$\mathcal{L}_{grav} = -\frac{2}{K}\,(R + 2\Lambda) \qquad (9.62)$$

Note that this makes sense in terms of physical dimensions

$$[\mathcal{L}_{grav}] = [K^{-1}] \cdot m^{-2} = N \cdot m^{-2} = J \cdot m^{-3} \qquad (9.63)$$

$$= [\text{Energy-density}]$$

if we interpret $d^4x = dt \wedge d^3x$ this gives the dimension of an action. If we interpret $d^4x = dx^0 \wedge d^3x = c\, dt \wedge d^3x$, we will to take A as $-2/cK$ in order to get the correct dimension of an action.

Finally we come back to (9.32) and calculate the derivatives

$$\dot{\Gamma}^{\lambda}_{\alpha\beta} = \frac{d}{ds}\left[\Gamma(\gamma(s))\right]^{\lambda}_{\alpha\beta} \qquad (9.64)$$

and hence the vector field $\dot{V}^{\lambda}$ of (9.38)

Again we employ the normal-coordinate trick, just as before. Then

$$\left[\frac{d}{ds}\Big|_{s=0}\left[\Gamma(\gamma(s))\right]^{\lambda}_{\alpha\beta}\right]_p$$

$$\left[\frac{d}{ds}\Big|_{s=0}\left\{\frac{1}{2}\gamma^{\lambda\sigma}_{(s)}\left[-\gamma_{\alpha\beta,\sigma}(s) + \gamma_{\sigma\alpha,\beta}(s) + \gamma_{\beta\sigma,\alpha}(s)\right]\right\}\right]_p$$

$$= \frac{1}{2} g^{\lambda\sigma}\left(-\dot{g}_{\alpha\beta,\sigma} + \dot{g}_{\sigma\alpha,\beta} + \dot{g}_{\beta\sigma,\alpha}\right)\Big|_p \qquad (9.65)$$

since all other terms vanish due to $g_{\alpha\beta,\gamma}(p) = 0$.

$$= \frac{1}{2} g^{\lambda\sigma}\left(-\nabla_\sigma \dot{g}_{\alpha\beta} + \nabla_\beta \dot{g}_{\sigma\alpha} + \nabla_\alpha \dot{g}_{\beta\sigma}\right)\Big|_p \qquad (9.66)$$

since replacing $\partial_\sigma \to \nabla_\sigma$ just adds terms $\sim \Gamma(p) = 0$. But now both sides are tensorial and hence hold in general:

$$\dot{\Gamma}^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\lambda\sigma}\left(-\nabla_\sigma \dot{g}_{\alpha\beta} + \nabla_\beta \dot{g}_{\sigma\alpha} + \nabla_\alpha \dot{g}_{\beta\sigma}\right) \quad (9.67)$$

Further

$$\dot{V}^{\lambda} = g^{\alpha\beta}\dot{\Gamma}^{\lambda}_{\alpha\beta} - g^{\lambda\alpha}\dot{\Gamma}^{\sigma}_{\sigma\alpha}$$

$$= \frac{1}{2}\Big[g^{\alpha\beta} g^{\lambda\sigma}\left(-\nabla_\sigma \dot{g}_{\alpha\beta} + \nabla_\beta \dot{g}_{\sigma\alpha} + \nabla_\alpha \dot{g}_{\beta\sigma}\right)$$
$$- g^{\lambda\alpha} g^{\sigma\rho}\left(-\nabla_\rho \dot{g}_{\sigma\alpha} + \nabla_\alpha \dot{g}_{\rho\sigma} + \nabla_\sigma \dot{g}_{\alpha\rho}\right)\Big] \quad (9.68)$$

renaming indices in the second group according to $\alpha \to \sigma$, $(\sigma, \rho) \to (\alpha, \beta)$ allows to seperate off the common facter $g^{\alpha\beta} g^{\lambda\sigma}$:

$$\dot{V}^{\lambda} = \frac{1}{2} g^{\alpha\beta} g^{\lambda\sigma}\left(-\nabla_\sigma \dot{g}_{\alpha\beta} + \nabla_\beta \dot{g}_{\sigma\alpha} + \nabla_\alpha \dot{g}_{\beta\sigma}\right.$$
$$\left. + \nabla_\beta \dot{g}_{\alpha\sigma} - \nabla_\sigma \dot{g}_{\beta\alpha} - \nabla_\alpha \dot{g}_{\sigma\beta}\right) \quad (9.69)$$

the 3rd and 6th term in $(\cdot)$ cancel and the others add:

$$\dot{V}^{\lambda} = g^{\alpha\beta} g^{\lambda\sigma}\left(-\nabla_\sigma \dot{g}_{\alpha\beta} + \nabla_\beta \dot{g}_{\alpha\sigma}\right.$$
$$= \left(g^{\lambda\sigma} g^{\alpha\beta} - g^{\lambda\beta} g^{\alpha\sigma}\right) \nabla_\beta \dot{g}_{\alpha\sigma}$$

$$= \left(g^{\lambda\sigma} g^{\alpha\beta} - g^{\lambda\alpha} g^{\beta\sigma}\right) \nabla_\alpha \dot{g}_{\beta\sigma} \quad (9.70)$$

where just exchanged the running labels α and β in the last step.

Another way to write this, using

$$\dot{g}^{\alpha\beta} = - g^{\alpha\mu} g^{\beta\nu} \dot{g}_{\mu\nu}$$

is

$$V^{\lambda} = - \nabla_{\alpha} \dot{g}^{\alpha\lambda} - \nabla^{\lambda} (g^{\alpha\beta} \dot{g}_{\alpha\beta}) \qquad (9.71)$$

We end by stating the minimal coupling scheme, which consists in a prescription of how to couple matter to the gravitational field.

<u>Minimal coupling scheme:</u>

Given a matter-theory in special-relativistic form, i.e. with Poincaré-invariant equations of motion; then, the equations of motion of that very same matter in a gravitational field $g_{\mu\nu}$ is obtained by replacing

$$\left.\begin{aligned} \eta_{\mu\nu} &\longmapsto g_{\mu\nu} \\ \partial_{\mu} &\longmapsto \nabla_{\mu} \end{aligned}\right\} \qquad 9.72$$

Here ∂_{μ} stands for the Levi-Civita connection for the Minkowski metric η and ∇_{μ} for that of g. This scheme is called "minimal" because no curvature-terms are prescribed. We will see that this scheme is not always unique.