Exercises for the lecture on

Introduction into General Relativity

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Sheet 1

Problem 1

In his last book, the so-called 'Discorsi' from 1638, Galileo Galilei outlined a hought-experiment that apparently allows him to conclude rigorously and without ever performing any real experiments the validity of UFF (Universality of Free Fall). This means that in a static and homogeneous gravitational field bodies suffer the same acceleration, independent of their mass and composition (chemical, physical, or otherwise).

In modernised terminology, the argument runs as follows: Let there be two bodies, B_1 and B_2 of masses m_1 and $m_2 > m_1$, respectively. Let the gravitational field be $\vec{g} = -g\vec{e}_z$, where $g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$ (actually, the numerical value does not matter here). Galilei aims to produce a contradiction to a claim he attributes to Aristotle, roughly saying that the free-fall acceleration is a monotonously increasing function of the mass. Now, suppose this is indeed the case. Then B_1 and B_2 will fall in the negative z-direction with accelerations a_1 and a_2, respectively, where $a_2 > a_1$. Now suppose we glue B_1 on top of B_1, so as to produce an new compound body, B_3, of mass $m_3 = m_1 + m_2 > m_2 > m_1$. Then, according to our hypothesis, the compound body (i.e. its centre of mass) will fall with still greater acceleration than B_2:

$$a_1 < a_2 < a_3 \,. \tag{1}$$

On the other hand, let us think for a moment of the glue between the bodies as a short elastic link, like a rubber band connecting B_1 to B_2. Then, since $a_2 > a_1$, B_2 will overtake B_1 and the rubber-band connection will be set under tension, trying to accelerate B_1 above a_1 and to decelerate B_2 below a_2. By making the rubber band tighter and shorter we can approximate the glued situation to any degree of accuracy without loosing any of the principal arguments. In this way we are led to another conclusion, namely that the combined systems (i.e. its center of mass) will fall with an acceleration in between a_1 and a_2:

$$a_1 < a_3 < a_2 \tag{2}$$

Now, having "derived" both equations (1) and (2), we arrive at the desired contradiction. It implies that at least one of our initial hypotheses must be false. However, since we apparently only used the hypothesis that the free-fall acceleration is a monotonously increasing function of mass, and since the argument would just be the same if it were assumed to be a monotonously decreasing function, and since the argument would still apply to masses within any mass-intervall in which the acceleration is locally monotonic, we seem to to be able to conclude rigorously that the acceleration cannot in fact depend on the mass at all: It must be the same for *all* masses, depending only on the gravitational field itself. This, finally, is how Galilei claims to be able to defeat the idea of Aristotle by pure reason, i.e. without any experiment.

Ingenious! Don't you think so? (Recall that this argument predates the formulation of Newtonian mechanics.)

Analyse the argument in the context of Newtonian mechanics, keeping inertial- and (passive) gravitational mass well distinguished. Lookout for hidden assumptions Galilei's argument might contain.

The following exercise might help you on the right track. Imagine a person of inertial mass m_i and (passive) gravitational mass m_g standing on an ordinary bathroom scale. Both are placed in an elevator that momentarily descends in the direction of the gravitational field with acceleration $a < g$ (for $a \geq g$ the person would take-off the scale). Calculate the weight (i.e. the force) shown by the scale at this moment of time. What lesson can you draw from this calculation concerning Galilei's thought experiment?

Problem 2

Two mass points move under the influence of their mutual gravitational force according to Newton's equations of motion and law of gravity. We carefully distinguish between active and passive gravitational mass.

Show that the total momentum is preserved if and only if the ratio between active and passive gravitational mass is the same for both. Show that in this case there is also energy conservation.

Now assume equality between inertial and gravitational masses. Show that in this case everything said so far is still valid the mass of one of the bodies is negative and positive for the other one. Give explicit solutions to the equations of motion for this system of two bodies in the case that their masses are equal in modulus and opposite in sign.

Problem 3

In this problem inertial = passive-gravitational = active-gravitational mass. We let ρ denote the mass density and ϕ the gravitational potential. Newton's field equation is

$$\Delta\phi = 4\pi\, G\, \rho\,. \tag{3}$$

We assume ρ to be of compact support.

Show that the integral of ρ over all of space equals the active gravitational M, which equals the (appropriately normalised) flux of the gravitational field 'at infinity':

$$M := \lim_{r\to\infty} \left\{ \frac{1}{4\pi G} \int_{S^2(r)} \vec{\nabla}\phi \cdot \vec{n}\; do \right\}\,. \tag{4}$$

Here $S^2(r)$ is a two sphere with radius r centred at the origin.

The gravitational force-density by which the mass-distribution acts onto itself is

$$\vec{f} = -\rho\, \vec{\nabla}\phi\,. \tag{5}$$

Show that it can be written as the divergence of a symmetric tensor of rank 2:

$$f_a = -\nabla^b t_{ab}\,, \tag{6}$$

where

$$t_{ab} = \frac{1}{4\pi G}\big(\nabla_a\phi\nabla_b\phi - \tfrac{1}{2}\delta_{ab}\nabla_c\phi\nabla^c\phi\big)\,. \tag{7}$$

Use this to show that the total force and total torque by which the distribution acts on itself vanish.

Problem 4

Just like in electrostatics, we can assign an energy density ϵ to the Newtonian gravitational field. It is given by

$$\epsilon(\vec{x}) = \frac{-1}{8\pi G}\|\vec{\nabla}\phi(\vec{x})\|^2\,. \tag{8}$$

The opposite sign results from the attractivity of forces between like masses. Can you derive this expression?

If we assume on top of Newton's equations that any energy E corresponds to an inertial mass m according to $E = mc^2$ and that inertial and gravitational (active and passive) mass equals inertial mass, we can attempt to "improve" Newton's equation (3) by adding to the source ρ on the right-hand side the corresponding term ε/c^2 from the field itself:

$$\Delta\phi = 4\pi G\left(\rho - \frac{1}{8\pi G c^2}\|\vec{\nabla}\phi\|^2\right)\,. \tag{9}$$

Determine all spherically symmetric solutions of this equation with asymptotic behaviour $\phi(r \to \infty) \to 0$ and mass distribution:

$$\rho(\vec{x}) = \begin{cases} \sigma = \text{const.} & \text{for } r \leq R\,,\\ 0 & \text{for } r > R\,. \end{cases} \tag{10}$$

(Tip: The field-redefinition $\psi := \exp(\phi/2c^2)$ linearises (9) which is then easily solved for $r > R$ and $r < R$. Require finiteness of solution at $r = 0$ und C^1 at $r = R$.)

The active gravitational mass M is still defined by the (appropriately normalised) Flux of the gravitational field, i.e. by (4). Now this is *not* equal to the space integral of ρ. Show for the solution just obtained that M, considered as function of the star's constant density σ and its radius, is given by (we set $\omega := \sqrt{2\pi G\sigma/c^2}$):

$$M(\sigma, R) = \frac{2c^2R}{G}\left(1 - \frac{\tanh(\omega R)}{\omega R}\right)\,. \tag{11}$$

Use that to show the following inequality, which does not depend on σ:

$$M < \frac{2c^2 R}{G}\,. \tag{12}$$

[One may show, that this inequality remains valid even if the star's density is not constant - though still spherically symmetric - i.e. depends on r.]

How do you interpret this result? Why is it impossible to increase M arbitrarily by putting more matter in the volume inside a sphere of fixed radius R - quite in contrast to the Newtonian case? Can you give a solution to (9) where $\rho(\vec{x}) = \delta^{(3)}(\vec{x})$, in analogy to the "fundamental solution" $\propto 1/r$ of (3) ?