Exercises for the lecture on

Introduction into General Relativity

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Sheet 10

Problem 1

Consider a 3-parameter family of timelike geodesics $\gamma(s,\vec{\sigma})$, where s is the proper length along the geodesic labelled by $\vec{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$. This is sometimes called a (3-dimensional) "geodesic congruence". We assume that $\vec{\sigma}$ ranges over some open set $U \subset \mathbb{R}^3$ and that the map

$$\gamma : \mathbb{R} \times U \to M\,, \quad (s,\vec{\sigma}) \mapsto \gamma(s,\vec{\sigma}) \tag{1}$$

is a smooth diffeomorphism of $\mathbb{R} \times U$ onto the image $\mathrm{Im}(\gamma) \subseteq M$ of γ.

For any fixed $\vec{v} \in \mathbb{R}^3$ consider the following vector fields on $\mathrm{Im}(\gamma)$:

$$\dot{\gamma} := \frac{\partial \gamma}{\partial s} = \gamma_*\left(\frac{\partial}{\partial s}\right), \tag{2a}$$

$$\gamma' := v^a \frac{\partial \gamma}{\partial \sigma^a} = \gamma_*\left(v^a \frac{\partial}{\partial \sigma^a}\right) = v^a\,\gamma_*\left(\frac{\partial}{\partial \sigma^a}\right). \tag{2b}$$

That the integral lines of $s \mapsto \gamma(s,\vec{\sigma})$ be geodesics means that

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0\,. \tag{3}$$

Show that

$$L_{\dot{\gamma}}\gamma' = 0\,. \tag{4}$$

Next consider the projection of γ' orthogonal to $\dot{\gamma}$:

$$\gamma'_\perp := \gamma' - g(\gamma',\dot{\gamma})\,\dot{\gamma}\,. \tag{5}$$

Show that

$$L_{\dot{\gamma}}\gamma'_\perp = 0\,. \tag{6}$$

(Hint: (6) follows from (5) once you prove $\dot{\gamma}\big(g(\dot{\gamma},\gamma')\big) = 0$. To show that, use covariant derivatives, the geodesic equation (3), the fact that (5) implies $\nabla_{\dot{\gamma}}\gamma' = \nabla_{\gamma'}\dot{\gamma}$ (for torsion-free connection), and that $\nabla_{\gamma'}\dot{\gamma}$ is perpendicular to $\dot{\gamma}$.)

Finally prove the so-called *Jacobi-Equation of Geodesic Deviation*:

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\gamma'_\perp = R(\dot{\gamma},\gamma'_\perp)\dot{\gamma}\,. \tag{7}$$

(Hint: Recall the definition of curvature: $(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z = R(X,Y)Z$ and use it for $X = Z = \dot{\gamma}$ and $Y = \gamma'_\perp$. Recall also that the connection is torsion-free.)

Problem 2

The purpose of this exercise is to express equation (7) along the integral line of a single geodesic $s \mapsto \gamma(s, \vec{\sigma})$ for some fixed value $\vec{\sigma}$, in terms of components of a conveniently chosen basis. Use an adapted orthonormal $\{e_\alpha : \alpha = 0, 1, 2, 3\}$, i.e. $e_0 = \dot{\gamma}$ and $g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$, which is covariant constant along the selected integral line, i.e $\nabla_{\dot{\gamma}} e_\alpha = 0$ holds along that integral line. Show that (7) is then equivalent to

$$\frac{d^2 n^a}{ds^2} = R^a{}_{00b} n^b \,, \tag{8}$$

where we set $n = n^a e_a := \gamma'_\perp$. How do you interpret this equation?

Problem 3

Write down the Jacobi equation (8) for the metric

$$g = \left(1 + \frac{2\phi(\vec{x})}{c^2}\right) c\,dt \otimes c\,dt - \left(1 - \frac{2\phi(\vec{x})}{c^2}\right) d\vec{x} \dot{\otimes} d\vec{x} \tag{9}$$

in linear order in ϕ/c^2 (Newtonian approximation) and for a geodesic congruence for which at time $t = 0$ all geodesics are running parallel to the t axis (all particles are rest at this moment of time). What is this equation telling you?

Problem 4

Calculate all curvature components to linear order of a linearised, plane gravitational wave (compare Problem 3 of Sheet 7) in $(+)$-mode:

$$\begin{aligned} g = c\,dt \otimes c\,dt &- \big(1 - h_+(z - ct)\big) dx \otimes dx \\ &- \big(1 + h_+(z - ct)\big) dy \otimes dy \\ &- dz \otimes dz \,. \end{aligned} \tag{10}$$

Write down the Jacobi equation (8) for the timelike geodesic congruence of the metric (10) in which the spatial coordinates are all constant along each geodesic. (Hence the parameters $\vec{\sigma}$ may be identified with the coordinates $\vec{x} = (x, y, z)$.) Again: What is this equation telling you?

Problem 5 (for the Bravehearts)

Calculate all Riemann-, Ricci, and Einstein-Tensor components of the most general spherically symmetric metric (R is a constant of dimension "length")

$$\begin{aligned} g &= e^{2a(t,r)} dx^0 \otimes dx^0 \\ &- e^{2b(t,r)} dr \otimes dr \\ &- e^{2c(t,r)} R^2 \big(d\theta \otimes d\theta + \sin^2(\theta) d\varphi \otimes d\varphi\big) \end{aligned} \tag{11}$$

with respect to the orthonormal co-frame $\theta^0 = e^a\, dx^0$, $\theta^1 = e^b\, dr$, $\theta^2 = e^c\, R\, d\theta$, and $\theta^3 = e^c\, R\, \sin(\theta)\, d\varphi$.