Exercises for the lecture on

Introduction into General Relativity

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Sheet 11

Problem 1

Recall the exterior Schwarzschild metric in an orthonormal frame

$$g = \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \,, \tag{1}$$

where ($r_S = 2m = 2GM/c^2$):

$$\begin{aligned} \theta^0 &= \left(1 - \frac{r_S}{r}\right)^{1/2} c\,dt \,, & \theta^1 &= \left(1 - \frac{r_S}{r}\right)^{-1/2} dr \,, \\ \theta^2 &= r\, d\theta \,, & \theta^3 &= r\, \sin(\theta)\, d\varphi \,. \end{aligned} \tag{2}$$

In Lecture 18 we have calculated its curvature components in the orthonormal frame (2). The result is (compare equations (18.29)):

$$\frac{r_S}{r^3} = R_{0101} = -2R_{0202} = -2R_{0303} = 2R_{1212} = 2R_{1313} = -R_{2323} \tag{3}$$

Consider a radial Lorentz transformation with rapidity $\alpha = \tanh^{-1}(v/c)$, under which the basis $\theta^0, \theta^1, \theta^2, \theta^3$ is mapped to another orthonormal basis $\hat\theta^0, \hat\theta^1, \hat\theta^2, \hat\theta^3$, according to

$$\hat\theta^0 = \cosh(\alpha)\theta^0 + \sinh(\alpha)\theta^1 \,, \hat\theta^1 = \cosh(\alpha)\theta^1 + \sinh(\alpha)\theta^0 \,, \hat\theta^2 = \theta^2 \,, \hat\theta^3 = \theta^3 \,. \tag{4}$$

Show that the components $\hat R_{\alpha\beta\mu\nu}$ of the curvature tensor with respect this new basis equal (3).

Problem 2

Consider the Jacobi equation of geodesic deviation which we discussed on Sheet 10. In components with respect to a parallel-transported basis it may be written in the form

$$\frac{d^2 n^a}{d\tau^2} = c^2 R^a{}_{00b} n^b \,. \tag{5}$$

Here we used proper time τ rather than proper length as parameter, which accounts for the factor c^2 on the right-hand side.

We apply this equation to a family of geodesics, each of which is radially infalling in the exterior Schwarzschild geometry. Hence the curvature components are given by (3), even in the rest frame of the radially infalling observer, as was just shown.

Show that

$$\frac{d^2n^1}{d\tau^2} = \frac{c^2 r_S}{r^3} n^1 , \tag{6a}$$
$$\frac{d^2n^2}{d\tau^2} = -\frac{c^2 r_S}{2r^3} n^2 , \tag{6b}$$

and identically for n^3 instead of n^2.

Apply this equation to the free fall of an elastic body. Neighbouring mass-elements would drift apart radially according to (6a) and towards each other in the transversal directions according to (6b), if the elastic forces of the body would not keep them in place. We like to calculate these forces.

Consider a solid rod of length ℓ, constant rest-mass density ρ, and square-shaped cross section of area $q = a^2$, where $a \ll \ell$ is the side of the square. The rod is freely falling with its length oriented radially.

Show that the longitudinal tension (force per unit area) that acts on the cross section at the rod's midpoint is, according to (6a), given by

$$T_{\|} = \frac{\rho c^2}{8} \cdot \left(\frac{\ell}{r_S}\right)^2 \cdot \left(\frac{r_S}{r}\right)^3 . \tag{7}$$

Hint: The calculation of the total tension at the midpoint is somewhat similar to the calculation of the tension in the rotating rod that we discussed in Problem 2 of Sheet 8.

Problem 3

In this extercise we shall apply equation (7).

Imagine we drop the rod of the precious exercise of length $\ell = 25\,\mathrm{m}$ made of steel (i.e. like a railway line) with density $7.85\mathrm{g/cm}^3$ and breaking tension $\sigma_{max} = 2100\mathrm{N/mm}^2$ onto a neutron star of mass 1.5 solar masses and radius 10 km. We assume the geometry outside the neutron star to be given by the Schwarzschild metric, which is a good approximation if the neutron star has negligible spin.

Derive a lower limit for the number of pieces the railway line will be torn into before hitting the star's surface.

Next imagine a person (yourself?) jumping into a black hole - feets first. This means We assume the exterior Schwarzschild metric to be valid down to, and below, the Schwarzschild radius $r = r_S$. We set $\rho = 1 \cdot \mathrm{g} \cdot \mathrm{cm}^{-3}$ (density of water approximating that of a human body), $\ell = 180\,\mathrm{cm}$ (length of a human body), and transversal dimension $b = 33\,\mathrm{cm}$ (average width of a human body). Show that the longitudinal tension (7) can be written as

$$T_{\|}/E = Z \left(\frac{M_\odot}{M} \cdot 10^4\right)^2 . \tag{8}$$

Here $M_\odot = 2 \cdot 10^{33}\,\mathrm{g}$ is the mass of the Sun and $E = 10^5\,\mathrm{g}/(\mathrm{s}^2\,\mathrm{cm})$ a unit of tension (the "pain limit") corresponding to the weight of a mass of 100 kg in the gravitational field of the earth equally distributed over a cross section of area $b^2 = 10^3\,\mathrm{cm}^2$. This E is here thought of as the human “pain limit” for such stresses. Calculate the numerical factor Z and estimate, the lower mass-limit, above which the the jump through the horizon $r = r_S$ of black hole is “bearable”. What would you feel if you jumped through the horizon of the black hole at the centre of our Galaxy?

In the following problem we shall calculate an upper bound for your remaining lifetime after having passed the horizon.

Problem 4

We consider the metric

$$\begin{aligned} g = \left(1 - \frac{r_S}{r}\right) c\,dt \otimes c\,dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr \otimes dr \\ -r^2\bigl(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi\bigr)\,, \end{aligned} \tag{9}$$

for $r < r_S$. Prove that any timelike curve (not just geodesics!) having entered the region $r < r_S$ from the outside will hit $r = 0$ within an eigentime less than

$$\tau_{\mathrm{max}} = \frac{\pi}{2} \cdot \frac{r_S}{c}\,. \tag{10}$$

What effect would it have if you fired the boosters of your spaceship in the (hopeless) attempt to escape your imminent “spaghettification” by the black hole?

Estimate your maximal remaining lifetime after passing the horizon of the black hole in the centre of our Galaxy, which has 4.6 million solar masses?

Hint: In order to derive (10) start from

$$\left(1 - \frac{r_S}{r}\right) c^2\dot{t}^2 - \left(1 - \frac{r_S}{r}\right)^{-1} \dot{r}^2 - r^2\bigl(\dot{\theta}^2 + \sin^2(\theta)\,\dot{\varphi}^2\bigr) = 1 \tag{11}$$

which is valid for any timelike curve $x^\alpha(s) = \bigl(t(s), r(s), \theta(s), \varphi(s)\bigr)$ parametrised in terms of proper length s. First show: If $r(s_0) < r_S$ and $\dot{r}(s_0) < 0$ then $\dot{r}(s) < 0$ for all $s > s_0$. Show then that $(-\dot{r}) > c\sqrt{(r_S/r) - 1}$ and hence (10) by integration.

Problem 5

Rewrite the Schwarzschild solution in terms of coordinates (u, v, θ, φ), where (θ, φ) are as before and (u, v) are related to (t, r) via

$$u(ct, r) := ct - r_*(r)\,, \tag{12a}$$
$$v(ct, r) := ct + r_*(r)\,, \tag{12b}$$

where

$$r_*(r) := r + r_S \cdot \ln\left(\left|\frac{r}{r_S} - 1\right|\right)\,. \tag{12c}$$

These are called the “Eddington-Finkelstein-Coordinates”. Note that the above transformation formulae make sense for $r > r_S$ as well as for $r < r_S$.

Show that for $r > r_S$ they have the following interpretation: The curve $u = k = \mathrm{const.}$ is a radially outgoing light ray intersecting the $t = 0$ hypersurface at $r = r_*^{-1}(-k)$. Likewise, the curve $v = k = \mathrm{const.}$ is a radially ingoing light ray intersecting the $t = 0$ hypersurface at at $r = r_*^{-1}(k)$.

Can you give an interpretation in the case $r < r_S$?

Draw the curves $u = 0$ and $v = 0$ in the (ct, r)-half-plane $r > 0$.

Instead of $(ct, r, \theta, \varphi) \mapsto (u, v, \theta, \varphi)$ consider the following two transformations where only ct is replaced by u or v, respectively, and r is retained:

$$(ct, r, \theta, \varphi) \mapsto (u, r, \theta, \varphi) \quad \text{“outgoing Eddington-Finkestein Coordinates”} \tag{13a}$$

$$(ct, r, \theta, \varphi) \mapsto (v, r, \theta, \varphi) \quad \text{“ingoing Eddington-Finkestein Coordinates”.} \tag{13b}$$

Write the metric in these two systems of coordinated and show that for all $r > 0$ the metric coefficients are regular and satisfy $\det\{g_{\alpha\beta}\} \neq 0$ (except for the trivial zeros at the poles $\theta = 0, \pi$).

Draw a picture with the light-cones in the (u, r) and (v, r) coordinates each. (Hint: Determine first which linear combinations of $\partial/\partial u$ and $\partial/\partial r$ in the first, and which linear combinations of $\partial/\partial v$ and $\partial/\partial r$ in the second case are lightlike.)

Problem 6

Again we consider the exterior Schwarzschild solution for $r > r_S$ in new coordinates. This time we replace t by a new time coordinate $T(t, r)$ according to

$$cT(t, r) := ct + f(r)\,. \tag{14}$$

The function f shall be determined by the requirement that the hypersurfaces $T = \mathrm{konst}$ be flat (i.e. their induced 3-dimensional Riemannian metric is flat).

Show that this is fulfilled if f obeys the differential equation

$$f'(r) = \frac{\sqrt{r/r_S}}{(r/r_S) - 1}\,, \tag{15}$$

which for $r > r_S$ is solved by

$$f(r) = r_S \cdot \left\{ 2\sqrt{r/r_S} + \ln\left(\frac{\sqrt{r/r_S} - 1}{\sqrt{r/r_S} + 1} \right) \right\}. \tag{16}$$

Show that the exterior Schwarzschild metric then assumes the form

$$\begin{aligned} g = &\big(1 - r_S/r\big)\, c dT \otimes c dT \\ &- \sqrt{r_S/r}\,\big(c dT \otimes dr + dr \otimes c dT\big) \\ &- dr \otimes dr - r^2 (d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi)\,, \end{aligned} \tag{17}$$

which is regular at $r = r_S$ (why?).

Draw the the curve $T = 0$ in the (ct, r)− half plane $r > r_S$.

The coordinates (T, r, θ, φ) are called "Gullstrand-Painlevé-coordinates" and T the "Gullstrand-Painlevé-Time".

Consider radially infalling geodesics obeying $\dot{r} \to 0$ for $r \to \infty$. Show that their four-velocities are

$$u = \left(1 - \frac{r_S}{r}\right)^{-1} \frac{\partial}{\partial t} - c\sqrt{\frac{r_S}{r}} \frac{\partial}{\partial r}, \tag{18a}$$

which in (T, r)-coordinates take the form

$$u = \frac{\partial}{\partial T} - c\sqrt{\frac{r_S}{r}} \frac{\partial}{\partial r}. \tag{18b}$$

Consider the corresponding one-form field $u^{\downarrow\flat} := g(u, \cdot)$ and show that it takes the following form in Schwarzschild coordinates

$$u^\flat = c^2\, dt + c \cdot \frac{\sqrt{r_S/r}}{(r_S/r) - 1}\, dr, \tag{19a}$$

and hence

$$u^\flat = c^2\, dT. \tag{19b}$$

use (19b) and (18b) to justify the following two statements: 1) The geodesic vector field u is everywhere perpendicular to the hypersurfaces of constant T: 2) The eigentime τ along an integral curve of u between its intersection points with the hypersurfaces $T = T_1$ and $T = T_2 > T_1$ is just $\tau = (T_2 - T_1)$.

Problem 7

We consider 2-dimensional Minkowski space in standard coordinates (ct, x). In the region $x > |ct|$ we intriduce new "hyperbolic polar coordinates" (λ, ξ), where $\xi > 0$, through

$$\begin{aligned} ct(\lambda, \xi) &= \xi \cdot \sinh(\lambda), \\ x(\lambda, \xi) &= \xi \cdot \cosh(\lambda). \end{aligned} \tag{20}$$

Show that in terms of them the Minkowski metric reads

$$\eta := c dt \otimes c dt - dx \otimes dx = \xi^2\, d\lambda \otimes d\lambda - d\xi \otimes d\xi. \tag{21}$$

Show that for fixed ξ the curve $\lambda \mapsto \big(ct(\lambda, \xi), x(\xi, \lambda)\big)$ is an integral curve of the Killing field $K = x\partial_{ct} + ct\partial_x$ that generates boost in x-direction, with initial condition $ct(\lambda = 0, \xi) = 0$. The parameter λ is just the rapidity of the generated boost.

Show that the parameter λ (also called "Killing time") is related to the eigentime τ along that curve by

$$\tau = \frac{\xi}{c} \cdot \lambda, \tag{22}$$

if we choose $\tau = 0$ for $\lambda = 0$.

Show further that the modulus of the acceleration is given by

$$a = \frac{c^2}{\xi}\,. \tag{23}$$

Consider every point in the intervall $[\xi_1, \xi_2]$ on the x-axis ($t = 0$) being transported by the same parameter value λ along the flow of K. This map is an isometry, i.e. preserves dinstances. The acceleration $a = c^2/\xi$ as well as the eigentime $\tau = \xi/c$ depends on the point ξ, but the product $a \cdot \tau = c\lambda$ is independent of it. What does that tell you as regards the question of how to accelerate an extended material body in, say, x-direction without deforming it? What would happen to the body if all of its points were moved with the same acceleration?

Show that for fixed λ the curves $\xi \mapsto \big(ct(\lambda, \xi)\,,\, x(\xi, \lambda)\big)$, $\xi \geq 0$, are half-lines intersecting the integral lines of K perpendicularly. (Draw the lines of constant λ and ξ.) According to the way one speaks in Special Relativity, the lines of constant λ consist each of simultaneous events for the observer moving along K. On the other hand, all these lines intersect at the origin, so that, following this way of speaking, the origin is "simultaneous" to all other events in the region $x > |ct|$. Is that something to worry about?

One could rewrite (20) by replacing λ according to (22) by τ and parametrise the region $x > |ct|$ by (τ, ξ) instead of (λ, ξ). How would the curves of constant eigentime τ look like? Why do they look so different to the curves of constant λ?

Finally, consider the timelike wordline of an "observer" at constant ξ. Characterise the coordinates (ct, x) of those points that can causally influence the observer (at any point on his/her wordline) and also the set of points that the observer can causally influence. The intersection of the complements of these two sets is a set called the "causal complement" of the observer. How do you interpret the fact that observers exist whose causal complement is non-empty? What about "realistic" observers (with finite fuel reservoirs) whose acceleration phase lasts only a finite time, i.e. whose wordlines are straight outside a bounded interval in proper time?

Can you imagine what all this has to do with the Schwarzschild geometry?