Exercises for the lecture on

Introduction into General Relativity

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Sheet 2

Problem 1

Consider a point mass m in the spherically symmetric potential $V(r)$. Assume the mass to have total energy E and angular momentum $\vec{L} = L\vec{e}_z$, where $mr^2\dot{\varphi} = L$. Show that

$$\left(\frac{dr}{d\varphi}\right)^2 \frac{L^2}{r^4} = 2m(E - V(r)) - \frac{L^2}{r^2}\,. \tag{1}$$

For the special potential $V(r) = -\alpha/r$, called "Kepler potential" (because it gives rise to orbits that are conic sections, first proposed to apply to planetary orbits by Johannes Kepler), this can be integrated by elementary means, most easily by using $u = 1/r$ as independent variable. Show that one obtains

$$r(\varphi) = \frac{p}{1 + \epsilon \cos\varphi}\,, \tag{2a}$$

where

$$p := \frac{L^2}{m\alpha} \quad \text{und} \quad \epsilon := \sqrt{1 + \frac{2EL^2}{m\alpha^2}}\,. \tag{2b}$$

This is indeed a conic section. We are interested in $E < 0$, in which case it is an ellipse with "semi-latus rectum" p, semi-major axis $a = p/(1-\epsilon^2) = -\alpha/2E$, and eccentricity ϵ. $\varphi = 0$ corresponds to the point of closest approach, also called *periastron* (and *perihel* if the solar system). The periastron returns periodically in φ periods of 2π; that is, the period in r equals the period in φ and the orbit is spatially closed. This is a special feature (degeneracy) of the Kepler potential.

For general $V(r)$ the orbit will not be closed. Rather, from (1), we can derive the following formula for the excess azimuth of the periastron's recurrence:

$$\begin{aligned} 2\pi + \Delta\varphi &= 2\int_{r_{\min}}^{r_{\max}} \frac{dr\, L/r^2}{\sqrt{2m\big(E - V(r)\big) - L^2/r^2}} \\ &= -2\frac{\partial}{\partial L}\bigg|_E \int_{r_{\min}}^{r_{\max}} dr\,\sqrt{2m\big(E - V(r)\big) - L^2/r^2}\,. \end{aligned} \tag{3}$$

We now consider the potential $V(r) = -\frac{\alpha}{r} + \Delta V(r)$ where ΔV is to be considered as "small perturbation" of the Kepler potential. Derive the following formula for $\Delta\varphi$, valid to linear (leading) order in ΔV:

$$\Delta\varphi = m\frac{\partial}{\partial L}\bigg|_E \left\{\frac{1}{L}\int_0^{2\pi} d\varphi\; r_*^2(\varphi; L, E)\,\Delta V\big(r_*(\varphi; L, E)\big)\right\}\,. \tag{4}$$

Here $r_*(\varphi; L, E)$ denotes the solution (2a) to the unperturbed potential $-\alpha/r$ with valued L and E for angular momentum and energy. Note that in (3) and (4) the expression to the right of the partial derivative is considered as function of L and E, so that the partial derivative with respect to L is to be taken at constant E, as indicated by the symbol $\big|_E$.

Now, calculate $\Delta\varphi$ for perturbations of the form $\Delta V_2(r) = \delta_2/r^2$ and $\Delta V_3(r) = \delta_3/r^3$ and show that, expressed in terms of a and ϵ of the unperturbed ellipse, it assumes the following values, respectively:

$$\Delta_2\varphi \;=\; -2\pi\delta_2 m L^{-2} \quad = \; -2\pi\,\frac{\delta_2/\alpha}{a(1-\epsilon^2)}\,, \tag{5a}$$

$$\Delta_3\varphi \;=\; -6\pi\alpha\delta_3 m^2 L^{-4} \;=\; -6\pi\,\frac{\delta_3/\alpha}{a^2(1-\epsilon^2)^2}\,. \tag{5b}$$

Note: Before taking the partial derivative with respect to L the parameters p and ϵ have to be expressed in terms of L und E by using (2b).

Problem 2

For a time-independent mass distribution $\rho(\vec{x})$ the Newtonian gravitational potential is also time independent and given by

$$\phi(\vec{x}) = -G\int_{\mathbb{R}^3} d^3x'\,\frac{\rho(\vec{x}')}{\|\vec{x}-\vec{x}'\|}\,. \tag{6}$$

(Notation: We distinguish between the gravitational potential ϕ, the physical dimension of which is that of velocity-squared, and potential energy $V = m\phi$ of a mass m in the gravitational potential ϕ, the physical dimension of which is that of energy.)

We assume that the distance to the source is much bigger than the diameter of the source and that we have chosen the origin of our coordinate system somewhere within the source (we will later choose it to be the centre-of-mass). We can then expand the $1/\|\vec{x}-\vec{x}'\|$ in the integrand up to, and including, the second powers in the dimensionless ratios $\vec{x}'/r$, where $r := \|\vec{x}\|$.

Show that this leads to ($\stackrel{2}{=}$ indicates equality up to, and including, the second expansion order)

$$\phi(\vec{x}) \stackrel{2}{=} -G\left\{\frac{M}{r} + \frac{n_a D^a}{r^2} + \frac{1}{2}\frac{(n_a n_b - \frac{1}{3}\delta_{ab})Q^{ab}}{r^3}\right\}, \tag{7a}$$

where $\stackrel{2}{=}$ indicates equality up to, and including, the second order, where $n^a := x^a/r$ and $n_a := \delta_{ab}n^b = n^a$ (here upper-case and lower-case notation is kept for consistency with summation convention) and where

$$M := \int_{\mathbb{R}^3} d^3x\;\rho(\vec{x})\,, \tag{7b}$$

$$D^a := \int_{\mathbb{R}^3} d^3x\;\rho(\vec{x})\;x^a\,, \tag{7c}$$

$$Q^{ab} := \int_{\mathbb{R}^3} d^3x\;\rho(\vec{x})\;\big(3x^a x^b - r^2\delta^{ab}\big)\,, \tag{7d}$$

are the total mass, dipole-, and quandrupole-moments of the mass distribution, respectively.

Show that, as long as $M \neq 0$, it is always possible to choose the coordinate centre in such a way that the dipole moment vanishes.

Problem 3

In this exercise we wish to apply (7) to a mass distribution that is invariant under the rotation about single axis, say the x^3-axis.

Show that in this case the dipole vector $\vec{D}$ always points parallel to the axis of symmetry and that we may always choose the origin of coordinates on that axis such that $\vec{D}$ vanishes. We shall from now on make that choice.

Show further that the quadrupole components are given by the matrix

$$\{Q^{ab}\} = \mathrm{diag}\big(Q, Q, -2Q\big)\,, \tag{8}$$

where $Q := Q^{11} = Q^{22}$. (Tip: That follows from simple symmetry arguments without any calculations.)

Use now (7a) to show that the gravitational potential outside the mass distribution is given in the "quadrupole approximation" (i.e. including mass-multipoles up to, and including, the quadrupole) by

$$\phi(r,\theta) \stackrel{2}{=} -G\,\frac{M}{r}\left[1 + J_2\,\frac{R^2}{2r^2}(1 - 3\cos^2\theta)\right]\,. \tag{9}$$

Here θ is the polar angle and we used polar coordinates for the integration; not that $n^a = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Further, we introduced the dimensionless parameter

$$J_2 := \frac{Q}{MR^2}\,, \tag{10}$$

where R is some characteristic length of the mass distribution which one introduces in order to have dimensionless quantities. Note that (9) does not depend on R, as it cancels out.

Problem 4

We consider a special example of an axisymmetric mass distribution, given by a homogeneously filled interior of the spheroid

$$S(a,b) := \left\{\vec{x} \in \mathbb{R}^3 : \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1\right\}\,. \tag{11}$$

Here we wrote (x, y, z) instead of (x^1, x^2, x^3). Let $B(a, b)$ denote the 3-dimensional interior region bounded by $S(a, b)$. The constant mass density in $B(a, b)$ is denoted by ρ_0.

Show by direct computation that the total mass is

$$M = \int_{B(a,b)} d^3x\, \rho(\vec{x}) = \rho_0 \mathrm{Vol}\big(B(a,b)\big) = \rho_0 \frac{4\pi}{3} a^2 b\,. \tag{12}$$

We know from the previous exercise that the quadrupole tensor of the given mass distribution has the form (8). Show by direct computation that

$$\frac{Q}{M} = \frac{a^2 - b^2}{5}\,. \tag{13}$$

Note that spheroids with $a > b$ are called 'oblate' and those with $a < b$ are called 'prolate'. Hence oblate homogeneously filled spheroids have positive Q (and hence positive J_2) and prolate homogeneously filled spheroids have negative Q (and hence negative J_2).

Let us focus attention to oblate spheroids, i.e. the case $a > b$. The intersection of $S(a,b)$ with any 2-dimensional plane containing the z-axis, e.g., the $y = 0$ plane, is an ellipse with semi-major axis a and semi-minor axis b. Hence its *eccentricity* is $\varepsilon = \sqrt{1-(b/a)^2}$. Hence, taking as characteristic length-scale R in the definition of J_2 the equatorial radius a, we get

$$J_2 = \frac{\varepsilon^2}{5}\,. \tag{14}$$

Note that for ellipses and spheroids instead of the *eccentricity* one often uses the *ellipticity*, that is also called *flattening*, and which is defined by $f := 1 - (b/a)$. Hence we can rewrite J_2 in terms of the "flattening" f as

$$J_2 = \frac{2}{5} f(1 - f/2) \approx \frac{2}{5} f\,, \tag{15}$$

where the last $\approx$ is meant to be the leading-order (here linear) approximation for small f.

Use (15) to estimate J_2 for the Sun using recent values for f that you are asked to find yourself. `https://en.wikipedia.org/wiki/Sun` is an obvious first source, but find other ones and compare.

Problem 5

Consider the motion of a test mass in the potential (9) and show, that if it starts initially in the equatorial plane $\theta = \pi/2$ and tengentially to it, it will always remain within it for all time. You can show this without any calculation by using a "symmetry argument", but beware: the argument has to be mathematically precise!

We restrict attention to orbits in the equatorial plane and use (5) to calculate $(\Delta\varphi)_{\mathrm{Quad}}$ caused by the quadrupole moment. (Why may you use (5) even though it was derived for spherically-symmetric potentials only?) Show that

$$(\Delta\varphi)_{\mathrm{Quad}} = 3\pi J_2 \left[\frac{R}{a(1-\varepsilon^2)}\right]^2\,. \tag{16}$$

Note: Here a and ε are the semi-major axis and the eccentricity of the test-particles's orbit and R is the equatorial radius of the sun.

How big would J_2 have to be in order to explain the residual (i.e. not caused by other planets, like Venus, Earth, and Jupiter) perihelion shift of Mercury, which is approximately 43 arc-seconds per century? Compare this with the estimate for J_2 obtained in exercise 4. Would you think the simple (homogeneous) solar model used here leads to an over- or underestimation of the Sun's quadrupole-moment contribution to Mercury's perihelion shift?

Tip: Note that (16) is the perihelion shift "per revolution", wheres the 43 arc-seconds for Mercury refer to a whole century. For the numerical values you may take

$$R = 695,700\,\mathrm{km}\,, \tag{17a}$$

$$a = 57,909,050\,\mathrm{km}\,, \tag{17b}$$

$$\varepsilon = 0.205630\,. \tag{17c}$$