Exercises for the lecture on

Introduction into General Relativity

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Sheet 3

Problem 1

The energy-momentum tensor of a *perfect fluid* is given by

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + \Big(-\eta^{\mu\nu} + u^\mu u^\nu/c^2\Big)\ p\,, \tag{1}$$

where ρ denotes the mass-density and p the pressure in the fluid's rest-frame, and u is the fluid's four-velocity.

Show that (1) describes a system which in the reference frame set by u has the following properties:

1) The energy-density is ρc^2;
2) the energy-current density vanishes;
3) the momentum-density vanishes;
4) the momentum-current density is isotropic.

In particular, the material described by (1) does not support shear forces and does not conduct heat. How do you see that?

Problem 2

Show that the condition of vanishing divergence applied to (1) can be written in the following form:

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \dot{u}^\nu\big(\rho + p/c^2\big) + \big(-\eta^{\mu\nu} + u^\mu u^\nu/c^2\big)\nabla_\mu p \\ &\quad + u^\nu\Big[\nabla_\mu\big(u^\mu(\rho + p/c^2)\big) - \dot{p}/c^2\Big]\,. \end{aligned} \tag{2}$$

An overdot always denoted the derivative with respect to proper time of the integral curve of u, i.e., $\dot{u}^\nu := u^\mu\nabla_\mu u^\nu$ and $\dot{p} := u^\mu\nabla_\mu p$.

Show that $\nabla_\mu T^{\mu\nu} = 0$ is equivalent to

$$\dot{u}^\nu\big(\rho + p/c^2\big) + \big(-\eta^{\mu\nu} + u^\mu u^\nu/c^2\big)\nabla_\mu p = 0\,, \tag{3a}$$

$$\nabla_\mu\big(u^\mu\rho\big) + (p/c^2)\,\nabla_\mu u^\mu = 0\,. \tag{3b}$$

Consider the case of vanishing pressure and show that then ρu is a conserved current and that the integral lines of u are geodesics in Minkowski space. (Tip: The integral curves of a vector field are geodesics if and only if the derivative of the vector field with respect to itself vanishes.)

How do you interpret the fact that, according to (3b), the rest-mass current ρu is not preserved if the pressure is non-zero and the fluid is not incompressible (incompressibility here means that $\nabla_\mu u^\mu = 0$)?

(Tip: You may assume that the calculations concerning the divergence operation take place in flat Minkowski space, but they continue to be valid verbatim in GR if you interpret the ∇ as a covariant derivative.)

Problem 3

(Attention: In this exercise we will use distributions. $\delta^{(4)}$ denotes the Dirac-distribution in Minkowski space - with respect to its Lebesgue measure.)

The four-dimensional current-density of a point charge is e which moves along the world-line $z(\tau)$ in Minkowski space (τ is its eigentime) is given by:

$$j^\mu(x) = e \int d\tau \, \delta^{(4)}\big(x - z(\tau)\big)\dot{z}^\mu(\tau)\,. \tag{4}$$

Show that $\nabla_\mu j^\mu = 0$ (in the sense of distributions).

The energy-momentum tensor of a point mass m moving likewise along $z(\tau)$ is:

$$T^{\mu\nu}(x) = m \int d\tau \, \delta^{(4)}\big(x - z(\tau)\big)\, \dot{z}^\mu(\tau)\dot{z}^\nu(\tau)\,. \tag{5}$$

Show that $\partial_\mu T^{\mu\nu} = 0$ (in the sense of distributions) holds, if and only if the wordline satisfies $\ddot{z}^\mu = 0$, i.e. is a geodesic in Minkowski space.

Problem 4

Let (V, η) be a real $n > 2$ dimensional vector space with Lorentz metric; that is, η is a non-degenerate symmetric bilinear form of signature $(1, -1, \cdots, -1)$). Let further $T : V \to V$ be a linear map that is symmetric with respect to η; that is, $\eta(Tv, w) = \eta(v, Tw)$, for all $v, w \in V$. A vector $v \in V - \{0\}$ is called timelike, spacelike, and lightlike if $\eta(v, v)$ is bigger, smaller, and equal to zero, respectively.

Show that if $v \in V$ is an eigenvector for T then its $(n-1)$-dimensional orthogonal complement $\{v\}^\perp := \{w \in V : \eta(w, v) = 0\} \subset V$ is invariant under T (as a set, not pointwise). What does that imply if v is lightlike?

Show further that there exists a η-orthogonal basis of V diagonalising T, if and only if T admits a timelike eigenvector.

Now apply this result to symmetric energy-momentum tensors. How do you interpret the requirement that T (regarded as linear map $V \to V$) possesses a timelike eigenvector? Is this always the case? How would you expect the energy-momentum tensor of a plane electromagnetic wave in vacuum to look like?

Problem 5

Let $T : V \to V$ again be an energy momentum tensor. The following conditions, which are meant to hold for all timelike vectors v, are collectively known as “energy conditions”:

$$\begin{aligned} &\eta(v, Tv) \geq 0 && \text{(weak energy-condition)}, && \text{(6a)} \\ &\eta(v, Tv) - \tfrac{1}{2}\eta(v, v)\mathrm{trace}(T) \geq 0 && \text{(strong energy-condition)}, && \text{(6b)} \\ &\eta(Tv, Tv) \geq 0 \leq \eta(v, Tv) && \text{(energy-dominance condition)}. && \text{(6c)} \end{aligned}$$

Interpret (6a) and (6c) as restrictions on the image of timelike vectors under T. Likewise, interpret (6b) as restriction on the image of the linear map $T' := T - \frac{1}{2}\mathrm{trace}(T)\,\mathrm{id}_V$.

Show that for an ideal fluid (compare (1)) these conditions are equivalent to

- weak energy-condition:

$$\rho \geq 0 \quad \text{and} \quad p \geq -\rho c^2\,, \tag{7a}$$

- strong energy-condition:

$$p \geq \begin{cases} -\rho c^2/3 & \text{if } \rho \geq 0 \\ -\rho c^2 & \text{if } \rho < 0, \end{cases} \tag{7b}$$

- energy-dominance condition:

$$\rho \geq 0 \quad \text{and} \quad -\rho c^2 \leq p \leq \rho c^2\,. \tag{7c}$$

Problem 6

Show that the set $V_+ = \{v \in V : \eta(v, v) > 0\} \subset V$ of timelike vectors has two connected components, $Z = V_+^{\uparrow} \cup V_+^{\downarrow}$. If $n \in V_+^{\uparrow}$ is a chosen reference vector (defining the “future direction”) und $v \in Z$, then $v \in V_+^{\uparrow} \Leftrightarrow \eta(n, v) > 0$.

Show further that the condition (6c) of energy-dominance is equivalent to the condition that $T(V_+^{\uparrow}) \subseteq V_+^{\uparrow}$ and $T(V_+^{\downarrow}) \subseteq V_+^{\downarrow}$ and that this will extend to the corresponding statements for the closures $\bar{V}_+^{\uparrow}$ and $\bar{V}_+^{\downarrow}$.

Let $\{e_0, e_1, \cdots, e_{n-1}\}$ be a η-orthonormal basis of V with timelike e_0. Show that the condition of energy-dominance implies the following inequalities:

$$T_{00} \geq |T_{ab}| \tag{8}$$

for all $a, b \in \{0, 1, \cdots, n-1\}$, where $T_{ab} := \eta(e_a, Te_b)$.
(Tip: Consider expressions of the form $\eta\big(e_0 \pm e_a, T(e_0 \pm e_b)\big)$ and $\eta\big(e_0, T(e_0 \pm e_a)\big)$.)

Can one conclude that the validity of (8) in *some* orthonormal basis implies that T satisfies energy-dominance?