Exercises for the lecture on

Introduction into General Relativity

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Sheet 4

Problem 1

Consider a space-time metric of the form

$$g = \left(1 + \frac{2\phi(\vec{x})}{c^2}\right) c^2\, dt \otimes dt - d\vec{x} \dot{\otimes} d\vec{x}\,, \tag{1}$$

where $\vec{x} = (x, y, z)$.

Show that the geodesic equations for the spatial coordinates $\vec{x} \circ \gamma = \vec{z}$ are given by

$$\ddot{z}^a(\tau) = -\dot{t}^2(\tau)\, \phi_{,a}(\vec{z}(\tau))\,, \tag{2}$$

where $\phi_{,a} := \partial\phi/\partial x^a$ and a dot denotes the derivative with repect to eigentime (or an affinely related parameter).

Use the geodesic equation for the time component $t(\tau)$ to eliminate $\dot{t}$ as well as to replace the derivatives with respect to τ by derivatives with respect to t (denoted by a dash). Show that this leads to (suppressing the argument t for $\vec{z}$ and $\vec{z}(t)$ for ϕ)

$$\vec{z}'' = -\vec{\nabla}\phi + 2\, \left(1 + \frac{2\phi}{c^2}\right)^{-1} \frac{(\vec{z}' \cdot \vec{\nabla}\phi)\, \vec{z}'}{c^2}\,. \tag{3}$$

Note that this leads to the Newtonian equations of motion in a gravitational potential ϕ if terms v^2/c^2 are neglected. Note also that “space” (i.e. the sections $t = \mathrm{const.}$ in spacetime) is flat, and yet the spatial trajectories are not straight lines. How does that fit with the semi-popular picture of GR as explaining gravity as “curvature”?

Aufgabe 2

We consider the length-functional for timelike curves in the metric (1):

$$\begin{aligned} L(\lambda_1, \lambda_2) &= \int_{\lambda_1}^{\lambda_2} d\lambda\, \sqrt{g_{\alpha\beta}\big(z(\lambda)\big)\dot{z}^\alpha(\lambda)\dot{z}^\beta(\lambda)} \\ &= \int_{\lambda_1}^{\lambda_2} d\lambda\, \sqrt{\left(1 + \frac{2\phi(\vec{z}(t))}{c^2}\right) c^2\dot{t}^2 - \dot{\vec{z}} \cdot \dot{\vec{z}}}\;. \end{aligned} \tag{4}$$

This is invariant under reparametrisations of the curve so that we may just as well use t as parameter along the curve. Then we get

$$\begin{aligned} L(t_1,t_2) &= \int_{t_1}^{t_2} c\,dt \sqrt{\left(1+\frac{2\phi(\vec{z}(t))}{c^2}\right)-\frac{\vec{v}^2}{c^2}} \\ &= c(t_2-t_1)+c\int_{t_2}^{t_2} dt\,\left(\frac{\phi(\vec{z}(t))}{c^2}-\frac{\vec{v}^2}{2c^2}\right)+O(4)\,, \end{aligned} \tag{5}$$

where $\vec{v}$ denotes $d\vec{z}/dt$, i.e. the velocity with respect to Newtonian time. In the second line we expanded the square-root in terms of v/c and ϕ/c and dropped all terms at fourth and higher order.

We now specialise this to the simple potential $\phi(\vec{x}) = gz$ mit $g > 0$, corresponding to a homogeneous Newtonian gravitational field $\vec{g} = -\vec{\nabla}\phi = -g\vec{e}_z$. We consider two identically constructed clocks, U and U', where U stays at rest in the origin of our coordinate system and U' is released at time $t_1 = 0$ with initial velocity v from the origin in a vertical and upward direction (i.e. along the positive z-direction). The spatial trajectory of U' obeys (3), in which you may neglect the term $\propto c^{-2}$ (it will only contribute corrections to order c^{-4} in the length functional, which we consistently neglect).

Assume the two clocks to be synchronised to $\tau = 0$ at the moment $t_1 = 0$ they depart. Calculate the difference in their eigentime at the moment $t_2 > 0$ they meet again at the origin, i.e. after U' has reached its maximal height and fallen back to the spatial point it started from. Use formula (5) and neglect all terms of fourth and higher power in $1/c$. Do you understand - as a matter of principle and before doing any calculation - which of the two clocks should show a larger value of eigentime upon their second encounter, corresponding to that clock having "aged more" than the other one? Try to give an intuitive and a mathematical argument!

Problem 3

Consider a general spacetime metric which for suitable local coordinates $(x^0 = ct, x^1, x^2, x^3)$ takes a form where all coefficients are independent of t:

$$g = g_{00}(\vec{x})\,dx^0\otimes dx^0 + g_{a0}(\vec{x})(dx^0\otimes dx^a + dx^a\otimes dx^0) + g_{ab}(\vec{x})dx^a\otimes dx^b\,. \tag{6}$$

We assume $g_{00} > 0$. Show that an alternative way to write this is

$$g = \phi^2(\vec{x})\,\theta\otimes\theta - h_{ab}(\vec{x})dx^a\otimes dx^b\,, \tag{7a}$$

where

$$\phi = \sqrt{g_{00}}\,,\quad \theta := dx^0 + A_a dx^a \text{ with } A_a := g_{0a}/g_{00}\,,\quad h_{ab} := -g_{ab} + \frac{g_{0a}g_{0b}}{g_{00}}\,. \tag{7b}$$

Show that $K = \partial/\partial x^0$ is a timelike Killing field, i.e. that $L_K g = 0$. (Tip: Apply L_K to the expression on the r.h.s. of (6), using the Leibniz property and that L_K commutes with d.)

Consider the one-form $K^\downarrow := g(K,\cdot)$ and show that $K^\downarrow = \phi^2\theta$ and

$$K^\downarrow \wedge dK^\downarrow = \phi^4\theta \wedge dA\,. \tag{8}$$

Use this to show

$$K^\downarrow \wedge dK^\downarrow = 0 \;\Leftrightarrow\; dA = 0 \;\Leftrightarrow\; \partial_a\left(\frac{g_{0b}}{g_{00}}\right) - \partial_b\left(\frac{g_{0a}}{g_{00}}\right) = 0\,, \tag{9}$$

Use this to prove that if $K^\downarrow \wedge dK^\downarrow = 0$ there exist local coordinates in which all $g_{\mu\nu}$ are independent of time *and* $g_{0a} = 0$. (Tip: Use $dA = 0$ and Poincaré's Lemma to show that there exists a local function f such that $A = df$. Now redefine the time coordinate by $x^0 \mapsto x^0 + f$.)

Problem 4

Let u be a four-velocity field (i.e. a timelike vector field with normalisation $g(u,u,) = c^2$). This may, e.g., be thought of as the four-velocity of a fluid. Its associated acceleration field is $a := \nabla_u u$ (in components: $a^\alpha = u^\beta\nabla_\beta u^\alpha$).

We define the tensor

$$\pi^{\alpha\beta} := g^{\alpha\beta} - \frac{u^\alpha u^\beta}{c^2}\,. \tag{10}$$

Show that $\pi^\alpha_\gamma := \pi^{\alpha\beta}g_{\beta\gamma}$ defines in each tangent space the orthogonal projection into the orthogonal complement of u.

Next we define the following tensors associated to u:

$$\omega_{\alpha\beta} := \tfrac{1}{2}\,\pi^\mu_\alpha\,\pi^\nu_\beta(\nabla_\mu u_\nu - \nabla_\nu u_\mu)\,, \tag{11a}$$

$$\theta_{\alpha\beta} := \tfrac{1}{2}\,\pi^\mu_\alpha\,\pi^\nu_\beta\,(\nabla_\mu u_\nu + \nabla_\nu u_\mu)\,, \tag{11b}$$

$$\theta := \pi^{\alpha\beta}\theta_{\alpha\beta}\,, \tag{11c}$$

$$\sigma_{\alpha\beta} := \theta_{\alpha\beta} - \tfrac{1}{3}\pi_{\alpha\beta}\theta \tag{11d}$$

These have, respectively, the following names: “vorticity”, “shear”, “expansion” and “trace-free shear” of the fluid.

Show that

$$\nabla_\alpha u_\beta = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \tfrac{1}{3}\pi_{\alpha\beta}\,\theta + \frac{u_\alpha a_\beta}{c^2}\,. \tag{12}$$

Show (or argue) that all four terms on the right-hand side are pairwise orthogonal; for example. $\sigma_{\alpha\beta}\omega^{\alpha\beta} = 0$.

Problem 5

We continue the notation from the previous exercise. Show that the rate of change of the expansion θ with respect to proper time along the integral curve of u, i.e. $\dot\theta = \nabla_u\theta$, obeys the following relation, known as *Raychaudhuri equation*:

$$\dot\theta = -\sigma^2 - \tfrac{1}{3}\theta^2 - R_{\alpha\beta}u^\alpha u^\beta + \omega^2 + \nabla_\beta a^\beta \tag{13}$$

where $\omega^2 := \omega_{\alpha\beta}\omega^{\alpha\beta}$ and $\sigma^2 := \sigma_{\alpha\beta}\sigma^{\alpha\beta}$.

(Tip: Recall that generally $u^\beta \nabla_\alpha u_\beta = 0$. Start by showing $\theta = \nabla_\alpha u^\alpha$; then consider $\dot{\theta} = u^\alpha \nabla_\alpha \nabla_\beta u^\beta$ and commute the covariant derivatives $\nabla_\alpha \nabla_\beta$, thereby picking up a curvature (Ricci) term. Then rewrite the term $u^\alpha \nabla_\beta \nabla_\alpha u^\beta$ into $\nabla_\beta(u^\alpha \nabla_\alpha u^\beta) - \nabla_\beta u^\alpha \nabla_\alpha u^\beta$ and use (12) to evaluate the last term.)

Problem 6

Apply the *Raychaudhuri equation* for the following special situation:

1. the vector field is u "geodesic", i.e. satisfies $a^\alpha = 0$;
2. the vector field u is "irrotational", i.e. satisfies $\omega_{\alpha\beta} = 0$;
3. the metric g satisfies Einstein's equations without cosmological constant and with an energy-momentum tensor that satisfies the strong energy-condition.

Show that this implies

$$\dot{\theta} \leq -\tfrac{1}{3}\theta^2 . \tag{14}$$

Now use this inequality to prove the following result: Let $\gamma(\tau)$ be an integral curve of the vector field u; i.e. $\dot{\gamma} = u \circ \gamma$. Suppose there exists a point $\gamma(\tau_*)$ on the integral curve at which θ is properly negative:

$$\theta\big(\gamma(\tau_*)\big) = \theta_* < 0 . \tag{15}$$

Then θ diverges to $(-\infty)$ within a proper time interval of length $3/|\theta_*|$ after τ_*.