Exercises for the lecture on

Introduction into General Relativity

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Sheet 5

Problem 1

Let u be a four-velocity field (i.e. a timelike vector field with normalisation $g(u,u,)=c^2$), just like in Problem 4 of Sheet 4. Our notation here will be as there. Let $u^\downarrow := g(u,\cdot) = u_\alpha\, dx^\alpha$ be its corresponding one-form and $\omega := \frac{1}{2}\omega_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ the vorticity two-form, where $\omega_{\alpha\beta}$ is defined like on sheet 4. Show that

$$\omega = \frac{1}{2c^2}\, i_u(u^\downarrow \wedge du^\downarrow) \tag{1}$$

where i_u is the map from k-forms to $k-1$ forms obtained by contracting the first tensor-factor with u.

Problem 2

Consider Minkowski in a global affine chart $(x^0 = ct, x^1 = x, x^2 = y, x^3 = z)$ in which its metric reads

$$g = c^2 dt \otimes dt - \delta_{ab}\, dx^a \otimes dx^b\,. \tag{2}$$

Consider the following vector field:

$$K := \frac{\partial}{\partial t} + \varepsilon_{ab}{}^{c}\Omega^a x^b \frac{\partial}{\partial x^c}\,. \tag{3}$$

Here ε_{abc} equals 1 or -1 depending on whether (abc) is an even or odd permutation of (123) and $\vec{\Omega} = (\Omega^1, \Omega^2, \Omega^3)$ are constant coefficients. Also, spatial indices are lowered and raised with δ_{ab} and its inverse δ^{ab}.

Show that K is a Killing field and that

$$U_K = \{(x^0, \vec{x}) \in M : \|\vec{x}_\perp\| < c/\|\vec{\Omega}\|\} \tag{4}$$

is the open set in Minkowski space where K is timelike. Here $\vec{x}_\perp$ is the component of $\vec{x}$ perpendicular to $\vec{\Omega}$ (in the ordinary $\mathbb{R}^3$-sense).

Let $K^\downarrow := g(K,\cdot)$; show that

$$K^\downarrow \wedge dK^\downarrow = -c\Omega^a \varepsilon_{abc}\, dx^0 \wedge dx^a \wedge dx^b\,. \tag{5}$$

Problem 3

This problem gives a simple and illustrative example for Problem 3 of Sheet 4 and also relates to the previous problem.

We consider Minkowski space in (2+1)-dimensions (we simply suppress one spatial dimension which turns out to be unimportant for what we wish to illustrate). We use planar polar coordinates (r, φ) for the $t = \mathtt{const.}$ sections, so that the metric reads

$$g = c^2 dt \otimes dt - dr \otimes dr - r^2 d\varphi \otimes d\varphi \tag{6}$$

. Now redefine the angular coordinate by

$$\varphi \mapsto \psi := \varphi - \Omega t\,, \tag{7}$$

corresponding to a frame that rigidly rotates with angular velocity Ω against the inertial frame. We restrict attention to the subset $r < c/\Omega$, for otherwise the rotation is impossible.

Rewrite the metric (6) in terms of t, r and ψ and show that it can be put into the form

$$g = \phi^2\, \theta \otimes \theta - h\,, \tag{8a}$$

where

$$\phi = \sqrt{1 - (r\Omega/c)^2}\,, \tag{8b}$$

$$\theta = c dt + A = c dt - \frac{(r\Omega/c)}{1 - (r\Omega/c)^2}\, r d\psi\,, \tag{8c}$$

$$h = dr \otimes dr + \frac{r^2\, d\psi \otimes d\psi}{1 - (r\Omega/c)^2}\,. \tag{8d}$$

Discuss the 2-dimensional Riemannian geometry of h; e.g. the circumference of circles of constant r in comparison to their diameter, and the Riemann curvature tensor (which has only one independent component).

Problem 4

This problem continues the previous one.

Consider the vector field $K = \partial/\partial t$ in (t, r, ψ) coordinates. Show that its orthogonal complement is the kernel of θ. Let γ be a curve in spacetime whose tangent vector is in the kernel of θ. Argue that the points along this curve are obtained by successive Einstein synchronisation, i.e. they are (locally) Einstein simultaneous. Now consider a curve lying entirely on the cylinder $r = R = \mathtt{const.}$ and winding once around it, so as to project to a circle $r = R$ in space . Give an interpretation of the integral of A along that circle. Can you consistently (transitively) Einstein-synchronise clocks that are at rest on a disc that rigidly rotates in Minkowski space? What has transitivity of clock synchronisation to do with whether dA vanishes or not?

One last - unrelated - question: What is the difference between the vector field $\partial/\partial t$ in (t, r, ψ) and in (t, r, ϕ) coordinates?

Aufgabe 5

A static metric can be written in the form

$$g = g_{\alpha\beta}(t, \vec{x})\, dx^\alpha \otimes dx^\beta = \phi^2(\vec{x})\, c^2 dt \otimes dt - \bar{g}_{ab}(\vec{x})\, dx^a \otimes dx^b\,. \tag{9}$$

Show that the Christoffel symbols of (9) are as follows: They vanish if either all or exactly one index is 0, i.e., $\Gamma^0_{00} = \Gamma^0_{ab} = \Gamma^a_{0b} = \Gamma^a_{b0} = 0$, and the other components are

$$\Gamma^a_{00} = \bar{g}^{ab}\phi\phi_{,b}\,, \qquad \Gamma^0_{a0} = \Gamma^0_{0a} = [\ln(\phi)]_{,a}\,, \qquad \Gamma^a_{bc} = \bar{\Gamma}^a_{bc}\,. \tag{10}$$

Here $\bar{\Gamma}^a_{bc}$ are the Christoffel symbols for the metric $\bar{g}$ and $[\cdots]_{,a} = \partial[\cdots]/\partial x^a$.

Now show that the components of the Ricci-tensor for the metric g has the following form:

$$R_{00} = \phi\,\bar{\Delta}\phi\,, \tag{11a}$$
$$R_{0a} = 0\,, \tag{11b}$$
$$R_{ab} = \bar{R}_{ab} - \frac{\bar{\nabla}_a\bar{\nabla}_b\phi}{\phi}\,. \tag{11c}$$

Here $\bar{\nabla}$ is the Levi-Civita covariant derivative for $\bar{g}$ und $\bar{\Delta} := \bar{g}^{ab}\bar{\nabla}_a\bar{\nabla}_b$ is its Laplace-Operator.

Now prove the following theorem: The only static, everywhere regular, and asymptotically Minkowskian (i.e. the metric g tends to the Minkowski metric for $\|\vec{x}\| \to \infty$) solution to Einstein's matter-free field equation without cosmological constant is flat space (Minkowski space). This fact is sometime expresses by saying that Einstein's theory does not admit gravitational-solitons.

Tip: Proceed as follows: From (11a) we have $\Delta\phi = 0$ with $\phi \to 1$ at spatial infinity. Now show that the only solution to this equation that is everywhere regular and approaches the value 1 at infinity is constant everywhere, i.e. $\phi \equiv 1$. Then (11c) implies $\bar{R}_{ab} = 0$. But in 3-dimensions a vanishing Ricci-tensor implies a vanishing Riemann tensor (compare Lecture 8).

Aufgabe 6

Again we consider static metrics (9). An alternative way to write them is as follows:

$$g = \phi^2(\vec{x})\left(c^2 dt \otimes dt - \hat{g}_{ab}(\vec{x})\, dx^a \otimes dx^b\right), \tag{12a}$$

where

$$\bar{g} = \phi^2\hat{g}\,. \tag{12b}$$

We consider geodesics in (12a). Write down the Euler-Lagrange equations of the energy functional for $t(\lambda)$ and $z^a(\lambda)$. From the first you get $\dot{t}\phi^2 = K = \mathrm{const}$. The Euler-Lagrange equation for $z^a(\lambda)$ can be simplified in a twofold way: First by using that $\phi^2\big(c^2\dot{t}^2 - \hat{g}_{ab}\dot{z}^a\dot{z}^b\big) = \kappa = \mathrm{const}.$ (proved in Lecture 5; compare equation

(5.42)). Second, by using t instead of λ as parameter. For that we assume that $\dot{t} \neq 0$, i.e. that the constant K is non zero. Now prove, that the Euler-Lagrange equation for $z^a(t)$ can be cast into the form ($'$ means t-derivative):

$$z''^a + \hat{\Gamma}^a_{bc} \, z'^b z'^c = -\, C \, \hat{g}^{ab} \, (\phi^2)_{,b} \,. \tag{13}$$

here $C := \kappa c^2/2K^2$ and all fields are evaluated at $z(t)$.

This implies the following important theorem: Lightlike geodesics ($\kappa = 0$) in static space-times with metric (12a) are such that their projections into the spatial hypersurfaces $t = const.$ are geodesics with respect to the Riemannian metric $\hat{g}$, where t is an affine parameter. For this reason one often calls $\hat{g}$ the *optical metric* of space. (Note: General spacetimes contain no naturally given spacelike hypersurfaces and hence do not define a natural notion of "spacelike projection". But static spacetimes do have such hypersurfaces: those orthogonal to the Killing vector field defining staticity.)