Exercises for the lecture on

Introduction into General Relativity

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Sheet 6

Problem 1

Consider the linearised Einstein equations in De Donder gauge for the static and spherically-symmetric mass-distribution

$$\rho(r) = \begin{cases} \rho_0 & \text{for } R_1 < r < R_2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $\rho_0 > 0$ is constant.

Show that for $r < R_1$ the coefficients $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ differ from $\eta_{\alpha\beta}$ only by constant rescalings of the time and the space coordinates and that spacetime is flat in that region. Are the metrics $g_{\alpha\beta}$ and $\eta_{\alpha\beta}$ gauge equivalent in that region?

Problem 2

In Lecture 11 we derived for the "mixed components" $(h_{01}, h_{02}, h_{03}) =: \vec{h}$ of the linearised metric $g_{\alpha\beta} - \eta_{\alpha\beta} =: h_{\alpha\beta}$ the following expression (compare lecture-notes, formula (11.46)):

$$\vec{h}(\vec{x}) = \frac{4G\rho_0}{c^3}\, \vec{\Omega} \times \int d^3x' \frac{\vec{x}'}{\|\vec{x} - \vec{x}'\|}\,. \tag{2}$$

This is valid for for a spherically symmetric and locally constant mass-distribution ρ_0 that rigidly rotates with angular velocity $\vec{\Omega}$. The spatial integral is to be extended over the region with non-zero mass distribution. We also derived the formula (compare lecture-notes, formula (11.56)):

$$\int d^3x' \frac{\vec{x}'}{\|\vec{x} - \vec{x}'\|} = \int dr' \int_{S^2(r')} d^2\Omega' \frac{\vec{x'}}{\|\vec{x} - \vec{x}'\|} = \vec{x}\, \frac{4\pi}{3} \int dr' \begin{cases} r'^4/r^3 & \text{for } r > r' \\ r' & \text{for } r' > r \end{cases} \tag{3}$$

where $S^2(r') := \{\vec{x}' \in \mathbb{R}^3 : \|\vec{x}'\| = r'\}$.

In Lecture 11 we applied this to the region outside a spherical star, i.e. we considered the case $r > r'$. In this exercise we want to consider the opposite case, where $r' > r$. This case is realised if the mass is concentrated in a spherical shell like in (1), but now rotating with constant angular velocity $\vec{\Omega}$. The coefficients $\vec{h}$ are to be computed inside the inner shell, i.e. for $r < R_1$. (Tip: Just use the stated formulae; you do not need to know anything else from Lecture 11.)

Calculate the gravitomagnetic field $\vec{B} = -c\vec{\nabla} \times \vec{h}$ inside the shell and show that the geodesic equation for a test particle acquires a term that just looks like the Coriolis force with a prefactor depending on the mass distribution. Assume $\rho_0 = 10^{-29} \cdot \mathrm{g} \cdot \mathrm{cm}^{-3}$ (average mass-density in our universe). How thick would the mass shell have to be, assuming $R_1 = 0$, so that this prefactor becomes equal to unity?

Problem 3

In Lecture 11 we showed that outside a stationary, spherically symmetric and homogeneous mass distribution that rigidly rotates with constant angular velocity Ω, a torque-free supported top will precess with angular velocity $\vec{\omega}_{TL} = -\vec{B}/2$ relative to the stationary frame, where $\vec{B} = -c\vec{\nabla} \times \vec{h}$ is the gravitomagnetic field (compare lecture-notes, formula (11.74)).

Use formula (11.76) from the lecture-notes to calculate $\vec{\omega}_{TL}$ as a function of latitude on the surface of the Earth, assuming its mass $m_E = 6 \cdot 10^{24}$ kg to be homogeneously distributed over a ball of radius 6.4×10^3 km.

Recall that changes $\Delta\Omega$ in angular velocity of frames can be detected by means of the Sagnac-Effect as optical phase shifts $\Delta\varphi$. The relation is (compare solutions-notes for problem 4 on sheet 5, formula 5.4.11):

$$\Delta\varphi = \frac{8\pi A}{\lambda c} \Delta\Omega \tag{4}$$

The *Federal Agency of Cartography and Geodesy* runs a Geodetic Observatory at Wettzell in south-east Germany, which includes a $4\,\mathrm{m} \times 4\,\mathrm{m}$ square-shaped HeNe-ring-laser operating at a wavelength of 632.8 nm. With that they can measure variations in the length of a day down to 0.1 milliseconds.

Would that laser - in principle - be able to detect the Earth's gravitomagnetic field? Or how large would the side-length of such a square-ring-laser have to be in oder to achieve that in the perfectly ideal case, neglecting all possible kinds of perturbations and noise?

Problem 4 (for DiffGeom lovers)

Consider a timelike wordline $s \mapsto z(s)$ in Minkowski space, where we choose proper length s as parameter; hence $\eta(\dot{z}, \dot{z}) = 1$. We think of this worldline as that of a pointlike particle with intrinsic angular momentum, called "spin". The spin vector S defines a vector field over the map z (compare chapter 5.2 of DiffGeom lecture-notes) whose values are orthogonal to $\dot{z}$, i.e. $\eta(\dot{z}, S) = 0$. This means that $S(s)$ is contained in the orthogonal complement of $\dot{z}(s) \in T_{z(s)}(M)$ which is a spacelike hyperplane that is often referred to as the "instantaneous rest space" of the particle.

The law according to which the spin vector S is transported along the wordline is given by the requirement that its Fermi derivative along z is zero (compare chapter 5.8 DiffGeom lecture-notes, in particular eqn. (5.77)):

$$F_z S = 0 \Leftrightarrow \Big(\nabla_{\dot{z}} + \big(\dot{z} \otimes \ddot{z}^{\downarrow} - \ddot{z} \otimes \dot{z}^{\downarrow}\big)\Big) S = 0\,. \tag{5}$$

Show that this is equivalent to

$$\dot{S} = -\dot{z}\,\eta(\ddot{z}, S)\,. \tag{6}$$

In order to evaluate this equation, we express the four-vectors $\dot{z}$ and S in terms of components. To that end we introduce a fixed affine frame in Minkowski space, which we call the "laboratory frame", and write

$$\dot{z} = \gamma(1, \vec{\beta}) \tag{7}$$

$$S = \left(\gamma\vec{\beta}\cdot\vec{S}\,,\ \vec{S} + \frac{\gamma^2}{\gamma+1}(\vec{\beta}\cdot\vec{S})\vec{\beta}\right). \tag{8}$$

Here $\beta = \vec{v}/c$ is the velocity of the particle (in units of c) with respect to the laboratory frame and $\gamma = 1/\sqrt{1-\beta^2}$. In contrast, as is obvious from (7), $\vec{S}$ are not the spatial components of S with respect to the laboratory frame, but rather with respect to that instantaneous rest frame of the particle that is obtained from the laboratory frame by a pure boost with boost-parameter $\vec{\beta}$. Prove this!

Now, insert (7) and (8) into (6) and show that this 4-vector-equation is equivalent to

$$\dot{\vec{S}} = \vec{\omega}_T \times \vec{S}\,, \tag{9a}$$

where

$$\vec{\omega}_T := \frac{\gamma^2}{\gamma+1}\,\dot{\vec{\beta}} \times \vec{\beta} \tag{9b}$$

is the so-called "Thomas-Frequency" (More precisely, it becomes the Thomas frequency if we redefine the overdot to be the derivative with respect to proper time τ rather than proper length $s = c\tau$, in which case (9a) still holds without any additional factors of c if the same redefinition is made in (9b).)

(Tip: First evaluate the time component of (6) (using (7) and (8)) to express $(\dot{\vec{S}}\cdot\vec{\beta})$ in terms of expressions containing $\vec{S}$ but not $\dot{\vec{S}}$. Then turn to the three space components of (6) and solve for $\dot{\vec{S}}$ by eliminating all terms containing $(\dot{\vec{S}}\cdot\vec{\beta})$ by the previously derived expression. Stay confident and keep calm.)

Recall the interpretation of the components $\vec{S}$ and use it to give an interpretation of (9a) in terms of geometrically defined structures. What is the spin vector rotating against?

Problem 5 (for *true* DiffGeom lovers)

Recall the definition of the Fermi covariant derivative in terms of projection operators (DiffGeom lecture-notes, eq. (5.71)). Recall also the definition of the Levi-Civita covariant derivative for embedded surfaces in euclidean space (DiffGeom lecture-notes, chapter 3.5). Convince yourself that the latter actually works in higher dimensions and other signatures, as long as the hypersurface is non-lightlike (i.e. its normal is nowhere lightlike). In particular, it works for the hyperboloidal hypersurfce of normalised timelike vectors in the 4-dimensional vector space with Minkowskian inner product. Now show that (6) is equivalent to parallel transport of S along the hodograph $s \mapsto \dot{z}(s)$

(the curve in the space of velocities) with respect to the Levi-Civita connection on the unit spacelike hyperboliod. Give an interpretation of the Thomas precession as Riemannian holonomy on the negatively curved 3-dimensional Riemannian manifold of four-velocities.