Exercises for the lecture on

Introduction into General Relativity

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Sheet 7

Problem 1

Since Problem 6 of Sheet 5 has not yet been done, it will be repeated here as Problem 1.

Problem 2

In Lecture 12 we showed that outside matter the linearised gravitational field $h_{\alpha\beta} = g_{\alpha\beta}\eta_{\alpha\beta}$ can be made to satisfy the following complete set of gauge-conditions (i.e. there exist no residual gauge transformations),

$$k^\alpha \tilde{h}_{\alpha\beta}(k) = 0\,, \tag{1a}$$

$$v^\alpha \tilde{h}_{\alpha\beta}(k) = 0\,, \tag{1b}$$

$$\eta^{\alpha\beta} \tilde{h}_{\alpha\beta}(k) = 0\,, \tag{1c}$$

where v is any fixed timelike vector and where $\tilde{h}_{\alpha\beta}$ is the Fourier transform of $h_{\alpha\beta}$. This is called the *transverse-tracless gauge*.

Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of Minkowski space, with dual basis $\{\theta^0, \theta^1, \theta^2, \theta^3\}$. We choose $v = e_0$ and consider the amplitude $\tilde{h}(k)$ for $k = k_* := \kappa(\omega/c)(e_0 + e_3)$, where κ is some constant equal to $k^0 = k^3$. This amplitude corresponds to a plane wave propagating at the velocity of light in e_3-direction.

Show that (1) imply for the tensor $\tilde{h} = \tilde{h}_{\alpha\beta}\theta^\alpha \otimes \theta^\beta \in V^* \otimes V^*$ that

$$\tilde{h}(k_*) = h_+\big(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2\big) + h_\times\big(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1\big)\,, \tag{2}$$

where h_+ and $h_\times$ are independent components. Characterise the 1-dimensional subspaces in $V^* \otimes V^*$ to which the amplitudes h_+ and $h_\times$ correspond and show that they are orthogonal.

Now consider Lorentz transformations that fix e_0 and e_3, i.e. spatial rotations in the plane $\mathrm{span}\{e_1, e_2\}$, which we think of as being oriented in the (12) sense. Show that they are given by

$$R(\varphi) = \cos(\varphi)\big(e_1 \otimes \theta^1 + e_2 \otimes \theta^2\big) + \sin(\varphi)\big(e_2 \otimes \theta^1 - e_1 \otimes \theta^2\big) \tag{3}$$

corresponding to a positive rotation by angle φ.

Show that the action T_φ of $R(\varphi)$ on $\tilde{h}(k_*)$ is given by

$$T_\varphi\big(\tilde{h}(k_*)\big) = h'_+\big(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2\big) + h'_\times\big(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1\big)\,, \tag{4}$$

where

$$\begin{pmatrix} h'_+ \\ h'_\times \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix} . \tag{5}$$

This means that an orthogonal rotation in V in the 2-plane perpendicular to the direction of propagation corresponds to an orthogonal transformation in the 2-dimensional subspace of $V^* \otimes V^*$ spanned by the directions of the amplitudes h_+ and $h_\times$ by *twice* the angle (both in positive directions, if the orientations are chosen as indicated: (12) in the first and $(+\times)$ in the second case). Can you explain the two meanings of the word "orthogonal" in the previous sentence?

Problem 3

Consider a linearised metric $g = \eta + h$ of a plane-gravitational wave in the transverse-traceless gauge. As before we take $k \propto (e_0 + e_3)$, i.e. the spatial direction of propagation is parallel to the third axis and oriented in the positive direction.

Show that the metric reads

$$\begin{aligned} g = c\,dt \otimes c\,dt &- \big(1 - h_+(z-ct)\big) dx \otimes dx \\ &- \big(1 + h_+(z-ct)\big) dy \otimes dy \\ &- dz \otimes dz \\ &+ h_\times(z-ct)\big(dx \otimes dy + dy \otimes dx\big) \end{aligned} \tag{6}$$

where the argument $(z-ct)$ is meant to indicate that the functions h_+ and $h_\times$ depends on (t, x, y, z) only through the combination $z - ct$.

Write down all components of the geodesic equation and show that they are solved by all spatial coordinates x, y, z being constant. Consider amplitude-functions whose support is contained on the negative real axis. Consider a large set of test particles distributed more or less uniformly on the circle $\{x^2 + y^2 = R^2\}$ in the plane $z = 0$. The particles are at fixed spatial coordinates for $t < 0$. What happens to them for $t > 0$, after being hit by the gravitational wave? Are they starting to "move"? If so, how much and in what directions? Discuss the h_+ and $h_\times$ amplitudes separately. (Tip: Deduce anything you say as much as you can from the equations; avoid folklore!)

Problem 4 (for DiffGeom lovers)

This problem is closely related to previous problems, like Problem 4 of Sheet 6, which it generalises and specialises at the same time: It generalises from Minkowski to arbitrary stationary curved spacetimes, but is specialises to stationary observers. It also relates closely to Problem 4 of Sheet 4 and Problem 1 of Sheet 5.

Consider a stationary spacetime (M, g, K), where M is a smooth manifold M, g a Lorentzian Metrik, and K a timelike Killing field K: $L_K g = 0$. In an open neighbourhood of M we choose a field of "adapted stationary orthonormal frames". Here, as usual, orthogonality means that

$$g(e_\alpha, e_\beta) = \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)\,, \tag{7a}$$

"adapted" means that

$$e_0 = K/\sqrt{g(K,K)}\,, \tag{7b}$$

and "stationary" means that

$$L_K e_\alpha = 0\,. \tag{7c}$$

Prove that this is always possible; i.e., if $p \mapsto \{e_\alpha(p) : \alpha = 0,1,2,3\} \subset T_p(M)$ is a smooth field of orthonormal bases for all $p \in \Sigma$, where $\Sigma \subset M$ is a spacelike hypersurface, and if γ is an integral curve of K that intersects Σ at p, and if we propagate the $e_\alpha(p)$ along γ by requiring (7c), then (7a) and (7b) will continue to hold along γ.

Now consider a "stationary observer" moving along a worldline γ that is an integral curve of K, but parametrised by proper length s; i.e. $\dot\gamma = K/\sqrt{g(K,K)} = e_0$. The observer carries along γ a "gyroscope" that is characterised by a "spin" vector-field $S \in ST_\gamma(M)$ *over* γ, obeying

$$g(\dot\gamma, S) = 0 \tag{8a}$$

and

$$F_\gamma S := \nabla_{\dot\gamma} S + g(\ddot\gamma, S)\,\dot\gamma - g(\dot\gamma, S)\,\ddot\gamma = 0 \tag{8b}$$

Show (or argue) that (8b) indeed preserves (8a) and that S has constant length $\|S\| = \sqrt{-g(S,S)}$ along γ. Hence we may write

$$S = S^a e_a\,, \tag{9}$$

where the e_a are *any* three orthonormal vectors perpendicular to $\dot\gamma = e_0$ (at this point the e_a it need not be the stationary basis introduced above). We write $\vec{S} := (S^1, S^2, S^3)$.

Show that (8b) is equivalent to

$$\dot{\vec{S}} = \vec\omega_T \times \vec{S}\,, \tag{10a}$$

where

$$\vec\omega_T := \left(\omega_T^1\,,\, \omega_T^2\,,\, \omega_T^3\right) := \left(\omega_{03}^2\,,\, \omega_{01}^3\,,\, \omega_{02}^1\right) \tag{10b}$$

are the connection coefficient from $\nabla_{e_\alpha} e_\beta = \omega^\gamma_{\alpha\beta} e_\gamma$.

Show further that if we now specialise the spatial basis vectors e_a to be stationary, i.e. obey (7c), then, defining as usual $K^\downarrow := g(K,\cdot)$, we have ($\varepsilon^{123} = 1$ etc.)

$$\omega_T^a = \tfrac{1}{4}\varepsilon^{abc}\, dK^\downarrow(e_b, e_c)/\|K\|\,. \tag{11}$$

Finally deduce from this the following geometric statement that is independent of any choice of bases: The spin vector $S \in ST_\gamma(M)$ precesses against the frame of stationarity (defined by a timelike Killing field) by the angular velocity $\omega_T \in ST_\gamma(M)$

$$\omega_T = -\tfrac{1}{2}\Big[\star\big(u^\downarrow \wedge du^\downarrow\big)\Big]^\uparrow \tag{12}$$

where $\star$ is the Hodge duality-map and $u = K/\sqrt{g(K,K)}$ is the normalised Killing field. Alternatively, introducing the angular-velocity 2-form $\omega := i_u \star \omega_T^\downarrow$, we have

$$\omega := i_u \star \omega_T^\downarrow = -\tfrac{1}{2} i_u \big(u^\downarrow \wedge du^\downarrow\big) = -\tfrac{1}{2} \pi \otimes \pi\big(du^\downarrow\big) \tag{13}$$

where π is the projection perpendicular to u (as in Problem 4 of Sheet 4). Note that on p-forms in n dimensions and metric with n_- negative directions (signature (n_+, n_-) with $n = n_+ + n_-$), we have $\star \circ \star = (-1)^{p(n-p)+n_-}$, which applied to our case $p = 3$, $n = 4$, and $n_= 3$ gives $\star \circ \star = id$. See DiffGeom-Notes (7.45,7.51).

Compare this with Problem 1 on Sheet 5 (the difference in the factor c^2 results from the fact that there u was normalised to c rather than 1.) This shows that for stationary observers the vorticity 2-form introduced earlier ist equivalent to $-\omega_T$. This sign-difference makes sense since “vorticity” is the angular velocity of the fluid against the inertial frame defined by torque-free suspended gyroscopes, whereas according to our definition here the Thomas precession ω_T represents the angular velocity of torque-free suspended gyroscopes against the stationary frame (realised by the observer “flowing” along the integral curves of the timelike Killing field. Sometimes ω_T is defined oppositely, i.e. as rotation of the stationary observer against the system defined by gyroscopes (e.g. in Straumann), in which case ω_T has the opposite sign to ours. We have chosen our convention because our ω_T corresponds to what is directly observed: The motion of torque-free suspended gyroscopes against the stationary frame that is (approximately) realised by the best-matched average rotational rest-frame of the background of “fixed stars”.