Exercises for the lecture on

Introduction into General Relativity

by DOMENICO GIULINI

Sheet 8

Problem 1

Compute the mean energy-current density of a monochromatic plane gravitational wave in the $(+)$ – mode; that is $h_+(z-ct) = A\ \cos\big[\omega(z-ct)\big]$, with $\omega = 2\pi\nu$, where ν is the frequency. What are the typical values obtained for amplitude $A = 10^{-21}$ and frequency $100\,\mathrm{Hz}$?

Tip: Use expression (13.41) of Lecture 13 for the mean energy-momentum tensor of a plane wave in TT-gauge.

Problem 2

Like in Lecture 14 we consider a homogeneous rod (or slab) of total mass M, length L and of quadratic cross section $q = a^2$. It rotates with constant angular velocity ω about an axis through its midpoint which is perpendicular to the length-direction of the rod. In Lecture 14 we derived the following formula for the total GW-luminosity (assuming $a \ll L$)

$$L_{GW}^{(\mathrm{rod})} = \frac{2}{45} \cdot \frac{G}{c^5} \cdot \omega^6 \cdot M^2 \cdot L^4 \,. \tag{1}$$

Show that the centrifugal stress (force per area) acting along the cross section through the midpoint is given by

$$\sigma = \tfrac{1}{2}\rho v^2 \,, \tag{2}$$

where $\rho = M/(qL)$ is the mass-density inside the rod and $v = \omega L/2$ is the velocity of the rod's ends.

Show that if the rod is made from a material the breaking-stress of which is σ_{max}, then $L_{GW}^{(\mathrm{rod})}$ is bounded from above by

$$L_{GW}^{(\mathrm{max})} = \frac{1024}{45} \cdot \frac{G}{c^5} \cdot \frac{q^2 \cdot \sigma_{max}^3}{\rho} \,. \tag{3}$$

Typical values are

$\star$	ρ	σ_{max}
steel	$7,85\ \mathrm{g/cm^3}$	$2100\ \mathrm{N/mm^2}$
fiberglass	$2,5\ \mathrm{g/cm^3}$	$4800\ \mathrm{N/mm^2}$

Calculate L_{GW}^{max} for these cases as well as the corresponding values v_{max} of the velocities of the ends. Note that both quantities are independent of the length L (as long as our assumption $a \ll L$ holds).

Problem 3

We again consider the rotating rod of the previous problem, but now we are interested in the amplitude of the emitted gravitational wave. In Lecture 14 we showed that the amplitude of a circular polarised wave that is emitted parallel to the axis of rotation ist given by

$$A(r) = \frac{4G\omega^2}{c^4} \cdot \frac{1}{r} \cdot \theta' . \tag{4}$$

Here θ' is the moment of inertia of the rod with respect to the chosen axis, which is $\theta'_3 = \frac{1}{12} M L^2$. Show that it is bounded above by

$$A_{max}(r) = \frac{L}{r} \cdot \underbrace{\frac{8}{3} \frac{G q \sigma_{max}}{c^4}}_{A_*} . \tag{5}$$

Note that contrary to (3) this does not depend on ρ!

Calculate A_* for steel and fiberglass and discuss the possibility to produce detectable gravitational waves in the laboratory.

Problem 4

Once more we again consider the rotating rod of the previous two problems. This time we are interested in the polarisations. In Lecture 14 we saw that the rotating rod emits circular polarised waves parallel to the axis of rotation and linear polarised waves perpendicular to it. Now we are interested in the polarisations if the line of sight contains an angle α with the plane of rotation (i.e. an angle $(\pi - \alpha)$ with the rotation axis).

Proceed as follows: In Lecture 14 we obtained the following expression for the amplitude:

$$h^{TT}(t, \vec{x}) = \left(\frac{2}{3}\right) \left(\frac{2GM}{c^2 r}\right) \left(\frac{L\omega}{2c}\right)^2 \Big[\cos(2\omega t)\,\Psi_+ + \sin(2\omega t)\,\Psi_\times\Big]^{TT} \tag{6a}$$

where in terms of the basis $\{\theta^a : a = 1, 2, 3\}$ dual to the orthonormal basis $\{e_a : a = 1, 2, 3\}$ of euclidean space with e_3 pointing in the direction of the rotation axis, we have

$$\begin{aligned} \Psi_+ &:= \theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2 , \\ \Psi_\times &:= \theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1 . \end{aligned} \tag{6b}$$

Now apply the TT projection for a line of sight parallel to

$$n = \sin(\alpha) e_1 + \cos(\alpha) e_3 \tag{7}$$

Note that this is a line of sight with inclination angle α against the axis of rotation or, said differently, inclination angle $(\pi - \alpha)$ against the plane of rotation of the rod.

Using this n you must project each tensor-factor in Ψ_+ and $\Psi_\times$ into the plane perpendicular to n, which we call $\{n\}^\perp$. The projection operator is $P_n^\perp = \mathrm{id} - n \otimes n^\downarrow$. On the dual vectors θ^a the projection acts like $\theta^a \mapsto \theta^a \circ P_n^\perp$. On the plane $\{n\}^\perp$ you can use the orthonormal basis $\bar{e}_1 := \cos(\alpha) e_1 - \sin(\alpha) e_3$ and $\bar{e}_2 := e_2$, and their corresponding dual basis $\bar{\theta}^1 := \cos(\alpha)\theta^1 - \sin(\alpha)\theta^3$ and $\bar{\theta}^2 := \theta^2$. Use these to form $\bar{\Psi}_+ := (\bar{\theta}^1 \otimes \bar{\theta}^1 - \bar{\theta}^2 \otimes \bar{\theta}^2)$ and $\bar{\Psi}_\times := (\bar{\theta}^1 \otimes \bar{\theta}^2 + \bar{\theta}^2 \otimes \bar{\theta}^1)$. Show that $\Psi_+ \circ P_n^\perp \otimes P_n^\perp = \cos^2(\alpha) \bar{\theta}^1 \otimes \bar{\theta}^1 - \bar{\theta}^2 \otimes \bar{\theta}^2$, which upon suptraction of its trace part results in $\Psi_+^{TT} = \frac{1}{2}(1 + \cos^2(\alpha))\bar{\Psi}_+$. Similarly (easier, in fact) you get $\Psi_+^{TT} = \cos(\alpha)\bar{\Psi}_\times$. As a result you get from (6a)

$$h^{TT}(t, r) = \bar{h}_+(t, r)\bar{\Psi}_+ + \bar{h}_\times(t, r)\,\bar{\Psi}_\times \,. \tag{8a}$$

where

$$\bar{h}_+(t, r) = \left(\frac{2}{3}\right)\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \cdot \frac{1 + \cos^2(\alpha)}{2} \cdot \cos(2\omega t)\,, \tag{8b}$$

$$\bar{h}_\times(t, r) = \left(\frac{2}{3}\right)\left(\frac{2GM}{c^2 r}\right)\left(\frac{L\omega}{2c}\right)^2 \cdot \cos(\alpha) \cdot \sin(2\omega t)\,. \tag{8c}$$

This can be said to be an "elliptically polarised" wave. Explain why! Calculate the eccentricity of the "ellipse" as a function of α.