Exercises for the lecture on

Introduction into General Relativity

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Sheet 9

Problem 1

Calculate the gravitational-wave luminosity L_{GW} of the Earth-Sun system.

Problem 2

The Crab-Nebula (also known as M1 or NGC 1952) contains a remnant of the Supernova SN 1054 (observed in the year 1054), which is the pulsar PSR B0531+21. Its distance to us is $2000\,\mathrm{pc} \approx 6500\,\mathrm{ly}$. The pulse period is $T = 3.35 \cdot 10^{-2}$ s, which increases with time at a rate of

$$\dot{T} = 4.4 \cdot 10^{-13}\,. \tag{1}$$

Assume the pulsar were a homogeneous ball of radius $10\,\mathrm{Km}$ and a mass 1.5 solar masses. How big would according to (1) be the rate of change of its rotational energy $\dot{E}_{rot}$?

Assume this loss of energy were due to the emission of gravitation waves (which it is not, but let's pretend for the moment). Recall that in Lecture 14 we derived (compare (14.33)))

$$L_{GW} = \frac{32}{5}\frac{G}{c^5}\omega^6(\varepsilon\theta)^2 \tag{2a}$$

for the gravitational-wave luminosity of a rigidly rotating body, where (compare (14.25-26))

$$\theta := I'_1 + I'_2\,, \qquad \varepsilon := \frac{I'_1 - I'_2}{I'_1 + I'_2}\,. \tag{2b}$$

The I'_a are the principal 2nd moments of the mass distribution and the rotation axis is assumed to be the third. How big would you estimate ε to be for $-\dot{E}_{rot} = L_{GW}$ to hold? How big would the maximal gravitational-wave amplitude be on Earth? Tip: Use (14.49).

Refine the above ball-model by assuming the pulsar to be a homogeneous solid ellipsoid with axes a, b, and c:

$$E(a,b,c) := \left\{(x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\right\}\,, \tag{3}$$

where $c = a = 10\,\mathrm{km}$ and $b = a + \delta a$ with $|\delta a| \ll a$. How large would δa have to be in oder for $-\dot{E}_{rot} = L_{GW}$ to hold? Get some information on the maximal height of "mountains" on neutron stars.

Problem 3

As regards the Crab-Nebula, its energy loss is most likely due to magnetic dipole radiation. For that we have $\dot{E}_{\mathrm{rot}} \propto \omega^4$.

Show that, in general, for $\dot{E}_{\mathrm{rot}} \propto \omega^{n+1}$, one has

$$\frac{T\,\ddot{T}}{\dot{T}^2} = 2 - n\,. \tag{4}$$

where the number

$$n := \frac{\omega\,\ddot{\omega}}{\dot{\omega}^2}\,. \tag{5}$$

is the so-called "braking index".

So what observations on the *system* (not its radiation) could - in principle - distinguish between the different channels of energy loss? And in practice?

Problem 4

In Lecture 15 we considered lightlike geodesics in a static stacetime whose spatial metric is conformally flat. We used that the spatial projection of the spacetime geodesic is a geodesic in the optical metric $\hat{g}$ of space, which is also conformally flat: $\hat{g} = n^2\delta$, where δ is the flat (euclidean) metric of space. The discussion of spatial light rays can then be given as if we were is flat euclidean space filled with an optically active medium of diffraction index n. For weak gravitational fields we have $n(\vec{x}) = 1 - 2\phi(\vec{x})/c^2$, where ϕ is the Newtonian gravitational potential. All geometric considerations that follow refer to the flat euclidean metric δ of space.

We consider a situation where the spatial light ray starts and ends in asymptotic regions where ϕ tends to zero, but in the intermediate region encounters regions where $\phi \neq 0$. If $\vec{e}_f$ and $\vec{e}_i$ are, respectively, the final and initial direction of the light ray. Then we derived the following formula for the difference $\vec{\alpha} := \vec{e}_f - \vec{e}_i$:

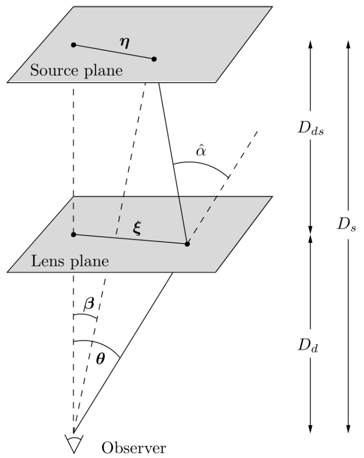
$$\vec{\alpha}(\vec{\xi}) = -\frac{4G}{c^2}\int_{\mathbb{R}^2} d^2\xi'\,\Sigma(\vec{\xi}')\frac{\vec{\xi}-\vec{\xi}'}{\|\vec{\xi}-\vec{\xi}'\|^2}\,, \tag{6a}$$

where $\vec{\xi} = (x, y)$ and

$$\Sigma(\xi) := \int_{-\infty}^{\infty} dz\,\rho(x, y, z)\,. \tag{6b}$$

This formula is valid to leading order in $\|\vec{\alpha}\| \ll 1$ and linear in ϕ/c^2.

The task of this exercise is to derive a "lens map" from equation (6), i.e. a map that relates the image on the Lens plane to the actual location of the objet on the Source plane. See the image below.



The figure on the left displays the geometric relations for gravitational lensing. It shows the Lens plane, parametrised by $\vec{\xi}$, and the Source plane, parametrised by $\vec{\eta}$. θ and β are the angles under which the source is seen, respectively, with and without the lense. D_d and D_s denote, respectively, the distances of the observer to the Lens and Source plane; and $D_{ds} := D_s - D_d$. For small angles α (the figure greately exaggerates the angle) we have $\hat{\alpha} := \|\vec{\alpha}\|$, which is to be considered as function of $\vec{\xi}$.

The lens map assigns to every point on the Lens plane a point of the Source plane. Hence, with the given notation, it gives $\vec{\eta}$ as function of ξ.

Show that

$$\vec{\eta} = \frac{D_s}{D_d}\,\vec{\xi} + D_{ds}\vec{\alpha}(\vec{\xi})\,. \tag{7}$$

Write this in dimensionless form by introducing length parameters ξ_0 in the lens plane and $\eta_0 := (D_s/D_d)\xi_0$ in the source plane. Instead of $(\vec{\xi},\vec{\eta})$ use the variable $\vec{x} := \vec{\xi}/\xi_0$ and $\vec{y} := \vec{\eta}/\eta_0$. Then show, that (7) is equivalent to

$$\vec{y}(\vec{x}) := \vec{\nabla}\varphi(\vec{x})\,, \tag{8a}$$

where

$$\varphi(\vec{x}) := \tfrac{1}{2}\|\vec{x}\|^2 - \psi(\vec{x})\,, \tag{8b}$$

$$\psi(\vec{x}) := \frac{1}{\pi}\int_{\mathbb{R}^2} \ln\big(\|\vec{x}-\vec{x}'\|\big)\,\kappa(\vec{x}')\,d^2x'\,, \tag{8c}$$

$$\kappa(\vec{x}) := \frac{4\pi G}{c^2}\frac{D_d D_{ds}}{D_s}\Sigma(\xi_0\vec{x})\,. \tag{8d}$$

In this equation all vectors refer to $\mathbb{R}^2$

Show that the trace of the Hessian $D^2\psi$ at $\vec{x}$ is given by $2\kappa(\vec{x})$. In particular, it vanishes outside the support of κ. What property of the lens map does that signify?

Problem 5

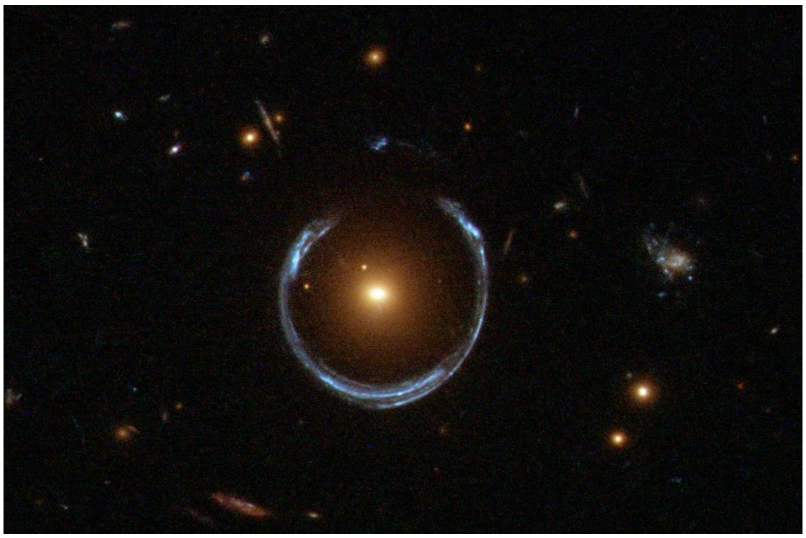
Specialise the lens map of the previous exercise to $\Sigma(\vec{\xi}) = M\,\delta^{(2)}(\vec{\xi})$. Choose the scale ξ_0 to equal the so-called "Einstein radius" R_E,

$$\xi_0 = R_E := \sqrt{2\cdot\frac{2GM}{c^2}\cdot\frac{D_d D_{ds}}{D_s}} \tag{9}$$

and show that then

$$\vec{y}(\vec{x}) = \vec{x}\left(1 - \|\vec{x}\|^{-2}\right)\,. \tag{10}$$

Invert equation (10) for $\|\vec{x}\| \neq 1$ and write down $\vec{x}$ as function of $\vec{y}$. Show that the pre-image of the point $\vec{y} = \vec{0}$ on the Lens plane is a circle of radius R_E; it is called an "Einstein Ring". See the picture below. It was taken by the Hubble-Space-Telescope on December 21, 2011. The lensing mass is called LRG 3-757. Find out more about it!



Picture of LRG 3-757 taken by the Hubble Space Telescope in 2011

Problem 6

Show from Maxwell's equations that light rays are null geodesics along which the polarisation vector is covariantly constant. In addition, derive an equation that describes how the amplitude changes along the light ray.

Proceed as follows: As usual, half of Maxwell's equations, namely $dF = 0$, are solved by $F = dA$; in components:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{11}$$

The other half of Maxwell's equations reads with $J_\nu = 0$ (outside sources):

$$\nabla^\mu F_{\mu\nu} := g^{\mu\lambda}\nabla_\lambda F_{\mu\nu} = 0 \,. \tag{12}$$

We impose the covariant Lorenz gauge

$$\nabla^\mu A_\mu = 0 \,. \tag{13}$$

Equation (12) reads ($\Box := g^{\mu\nu}\nabla_\mu\nabla_\nu$)

$$\Box A_\mu - R_\mu^\nu A_\nu = 0 \,. \tag{14}$$

We note in passing that if $R_\mu^\nu \neq 0$ then the Lorenz gauge does not decouple the equations (unlike the ordinary Lorenz gauge in flat space-time). But this is not important for what follows.

In order to derive the laws of geometric optics in matter-free space-time we consider solutions of (14) obeying (13) for vector potentials of the form

$$A_\mu = (a_\mu + \epsilon b_\mu + O(2)) \exp(i\psi/\epsilon) \,. \tag{15}$$

The amplitude is developed in a power series in ϵ, i.e. a_μ, b_μ are vector fields and $O(2)$ denotes terms of quadratic or higher order in ϵ. ψ ist a real phase-function (the so-called "Eikonal"). We introduce the following quantities

$$k_\mu := \nabla_\mu \psi \qquad \text{wave vector}\,, \tag{16a}$$

$$a := \sqrt{g^{\mu\nu} a_\mu a_\nu} \qquad \text{amplitude}\,, \tag{16b}$$

$$f_\mu := a_\mu / a \qquad \text{polarisation vector}\,. \tag{16c}$$

Insert (15) into (13) and (14) and consider the consequences for $\epsilon \to 0$ (the so-called "geometric-optics-limit"). Order in increasing powers of ϵ, starting with ϵ^{-2}, and set all coefficients individially to zero. Show that for (13) this leads to

$$\epsilon^{-2}: \quad \text{no condition}\,, \tag{17a}$$

$$\epsilon^{-1}: \quad k^\mu a_\mu = 0\,, \tag{17b}$$

and for (14)

$$\epsilon^{-2}: \quad k^\mu k_\mu = 0\,, \tag{18a}$$

$$\epsilon^{-1}: \quad k^\nu \nabla_\nu a_\mu = -\tfrac{1}{2}(\nabla^\nu k_\nu) a_\mu\,, \tag{18b}$$

where (18a) has already been used in (18b).

Take ∇_ν of $k^\mu k_\mu = 0$ and use that fact that k_μ is a gradient field, i.e. (16a), to show

$$k^\nu \nabla_\nu k_\mu = 0\,. \tag{19}$$

Now use (18b) to show

$$k^\nu \nabla_\nu a = -\tfrac{1}{2}\, a\, \nabla^\nu k_\nu\,. \tag{20}$$

This last equation allows you to derive how the amplitude changes along the ray. Consider f_μ from (16c) and conclude, using (18b) and (20), that

$$k^\nu \nabla_\nu f_\mu = 0\,. \tag{21}$$

To sum up, it is now shown that the following results are consequences of Maxwell's equations in the short-wavelength limit, i.e. as $\epsilon \to 0$ and neglecting terms ϵ^n for $n \geq 0$: Light rays are lightlike geodesics (eqns. (18a) and (19)) along which the polarisation vector is perpendicular (eq. (17b)) and parallely transported (eq. (21)). The amplitude changes according to (20) and hence increases/decreases if the divergence of the lightlike vector field k is negative/positive, i.e. if k is focussing/diverging.