Sheet 1 : Solutions

Exercise 1.

In a gravitational field

$$\vec{g}(t,\vec{X}) = -g\,\vec{e}_z$$

where $g = \text{const.} > 0$

a mass obeys Newton's equation relative to an inertial system and timescale:

$$m_i \ddot{\vec{X}} = -m_g \; g\,\vec{e}_z + \vec{F} \qquad (1.1.1)$$

We have

m_i = inertial mass
m_g = (passive) gravitational mass
$\vec{F}$ = other forces

Suppose the mass is forced to move with vertical acceleration a downward

$$\ddot{\vec{X}} = -a\,\vec{e}_z \qquad (1.1.2)$$

Then, according to (S1.1), the force that must be acting on the mass is

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$$\vec{F} = (g\, m_g - a\, m_i)\, \vec{e}_z \qquad (1.1.3)$$

This is the force with which the floor of the decending elevator pushes the feet of the person standing on the bathroom scale. It equals the weight shown on the scale. It vanishes for

$$a = g \;\; m_g / m_i$$

and would even reverse sign, i.e., the person would be pulled down rather than pushed up, if $a > g\, m_g/m_i$.

This we apply to the two bodies B_1 and B_2, with inertial and gravitational masses $(m_i)_{1,2}$ and $(m_g)_{1,2}$. In the given gravitational field their vertical free-fall ($\vec{F} = 0$) would be given by

$$\ddot{\vec{X}}_{1,2} = - \left(\frac{m_g}{m_i}\right)_{1,2} g\, \vec{e}_z \qquad (1.1.4)$$

Thus is their accelerations are

$$a_{1,2} = g \left(\frac{m_g}{m_i}\right)_{1,2} \qquad (1.1.5)$$

Suppose $a_2 > a_1$, i.e.

$$\frac{(mg)_2}{(mi)_2} > \frac{(mg)_1}{(mi)_1} \qquad (1.1.6)$$

and suppose B_1 where forced by the action of B_2 via the rubber band to move with acceleration $a = a_2$. Then according to (1.1.3), this force must be

$$\vec{F}_{21} = \text{force acting on } B_2 \text{ by } B_1$$

$$\overset{(3)}{=} (g\,(mg)_1 - a_2\,(mi)_1)\,\vec{e}_z$$

$$\overset{(5)}{=} \left[g\,(mg)_1 - g\,\frac{(mg)_2}{(mi)_2}\,(mi)_1 \right] \vec{e}_z$$

$$= g(mg)_1 \left[1 - \frac{(mg/mi)_2}{(mg/mi)_1} \right] \vec{e}_z \qquad (1.1.7)$$

By Newtons third law, the force $\vec{F}_{12}$ by which B_1 acts on B_2 is the negative of that:

$$\vec{F}_{12} = g(mg)_1 \left[\frac{(mg/mi)_2}{(mg/mi)_1} - 1 \right] \vec{e}_z \qquad (1.1.8)$$

Since we assumed $a_2 > a_1$, i.e. equation (6), the term in [..] is positive, meaning that B_1 pulls B_2 upwards or B_2 pulls B_1 downwards.

Hence we see: To assume that there forces between the parts of a freely-falling compound-body to vanish, whatever the parts are, is equivalent to assume that

$$\frac{m_g}{m_i} = \text{universal constant}$$

So, from a Newtonian perspective at least, Galilei's argument is

Exercise 2

$$(1.2.1) \quad (m_i)_1 \ddot{\vec{X}}_1 = (m_g^{(p)})_1 (m_g^{(a)})_2 \, G \frac{\vec{X}_2 - \vec{X}_1}{\|\vec{X}_2 - \vec{X}_1\|^3}$$

$$(1.2.2) \quad (m_i)_2 \ddot{\vec{X}}_2 = (m_g^{(p)})_2 (m_g^{(a)})_1 \, G \frac{\vec{X}_1 - \vec{X}_2}{\|\vec{X}_1 - \vec{X}_2\|}$$

Momentum Conservation

$$\dot{\vec{P}} = (m_i)_1 \ddot{\vec{X}}_1 + (m_i)_2 \ddot{\vec{X}}$$

$$= \left[(m_g^{(p)})_1 (m_g^{(a)})_2 - (m_g^{(p)})_2 (m_g^{(a)})_1 \right] G \frac{\vec{X}_2 - \vec{X}_1}{\|\vec{X}_2 - \vec{X}_1\|}$$

$$\overset{!}{=} \vec{0}$$

$$\Longleftrightarrow \left(\frac{m_g^{(p)}}{m_g^{(a)}} \right)_1 = \left(\frac{m_g^{(p)}}{m_g^{(a)}} \right)_2 \qquad (1.2.3)$$

for all pairs of masses.

Energy conservation

Multiply (1.2.1) with $\dot{\vec{X}}_1$ and (1.2.2) with $\dot{\vec{X}}_2$ and add both equations so obtained:

$$\frac{d}{dt}\left[\frac{1}{2}(m_i)_1 \dot{\vec{X}}_1^2 + \frac{1}{2}(m_i)_2 \dot{\vec{X}}_2^2\right]$$

$$= -(m_g^{(p)})_1 (m_g^{(a)})_2 \, G \frac{(\vec{X}_1 - \vec{X}_2)\cdot \dot{\vec{X}}_1}{\|\vec{X}_1 - \vec{X}_2\|^3}$$

$$+ (m_g^{(p)})_2 (m_g^{(a)})_1 \, G \frac{(\vec{X}_1 - \vec{X}_2)\, \dot{\vec{X}}_2}{\|\vec{X}_1 - \vec{X}_2\|^3}$$

$$\overset{(11)}{=} (m_g^{(p)})_1 (m_g^{(a)})_2 \, G \, \frac{d}{dt}\left(\|\vec{X}_1 - \vec{X}_2\|^{-1}\right) \qquad (1.2.9)$$

Hence $\frac{d}{dt}(E_{kin} + E_{pot}) = 0$

where $E_{kin} = \frac{1}{2}\left[(m_i)_1 \dot{\vec{X}}_1^2 + (m_i)_2 \dot{\vec{X}}_2^2\right]$

$$E_{pot} = -G \frac{(m_g^{(p)})_1 (m_g^{(a)})_2}{\|\vec{X}_1 - \vec{X}_2\|}$$

$$= -G \frac{(m_g^{(p)})_2 (m_g^{(a)})_1}{\|\vec{X}_1 - \vec{X}_2\|} \qquad (1.2.10)$$

Let us from now on assume that

$$m_i = m_g^{(a)} = m_g^{(p)} = m \qquad (1.2.11)$$

for both masses. Then momentum and energy conditions are valid, independently of whether the masses are positive or negative. Let us assume

$$m_1 = -m_2 = m > 0 \tag{1.2.12}$$

Then the equations of motion become

$$\left.\begin{aligned} \ddot{\vec{X}}_1 &= -G\, m_2 \frac{\vec{X}_1 - \vec{X}_2}{\|\vec{X}_1 - \vec{X}_2\|^3} \\ &= G\, m_1 \frac{\vec{X}_1 - \vec{X}_2}{\|\vec{X}_1 - \vec{X}_2\|^3} \\ &= \ddot{\vec{X}}_2 \end{aligned}\right\} \tag{1.2.13}$$

Hence $\frac{d^2}{dt^2}(\vec{X}_1 - \vec{X}_2) = 0$ or

$$\vec{X}_2(t) - \vec{X}_1(t) = \vec{d} + \vec{e}\, t \tag{1.2.14}$$

where $\vec{d}$ and $\vec{e}$ are t-independent.

1. Case $\vec{e} = 0 \Rightarrow \vec{X}_2 - \vec{X}_1 = \vec{d}$ (1.2.15)

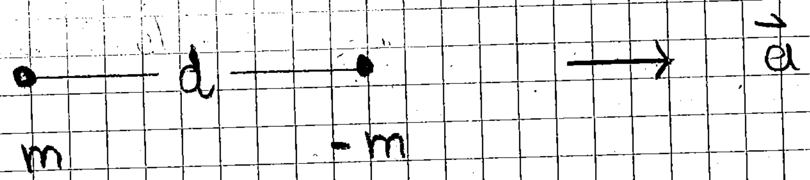
Have
$$\left.\begin{aligned} \ddot{\vec{X}}_1(t) &= G\, m_1 \frac{\vec{d}}{d^3}, \quad d = \|\vec{d}\| \\ &=: \vec{a} \end{aligned}\right\} \text{(1.2.1}$$

Then

$$\vec{X}_1(t) = \frac{1}{2}\vec{a}t^2 + \vec{V}t + \vec{c}$$

$$\vec{X}_2(t) = \frac{1}{2}\vec{a}t^2 + \vec{V}t + \vec{c} + \vec{d} \qquad (1.2.17)$$

Hence both particles undergo constant accelerations $\vec{a}$ while keeping constant distance $\vec{d}$. $\vec{a}$, $\vec{V}$ and $\vec{c}$ are integration constants, where $\vec{a} \sim \vec{d}$.



Note that we still have momentum and energy conserved.

Case 2. Let now $\vec{d} = \vec{0}$, then

$$\ddot{\vec{X}}_1(t) = -G m_1 \frac{\vec{e}}{\|\vec{e}\|^3} t^{-2}$$

$$=: \vec{a}\, t^{-2} \qquad (1.2.18)$$

$$\rightsquigarrow \dot{\vec{X}}_1(t) = \left(1 - \frac{1}{t}\right)\vec{a} + \vec{V}_1 \qquad (1.2.19)$$

with initial condition $\dot{\vec{X}}(t=1) = \vec{V}$.

$$\curvearrowright \quad \vec{X}_1(t) = [(t-1) - \ln(t)]\,\vec{a}$$
$$+ (t-1)\vec{V}_1 + \vec{C}_1 \qquad (1.2.20)$$

With initial condition

$$\dot{\vec{X}}_1(t=1) = \vec{V}_1\,, \quad \vec{X}_1(t=1) = \vec{C}_1$$

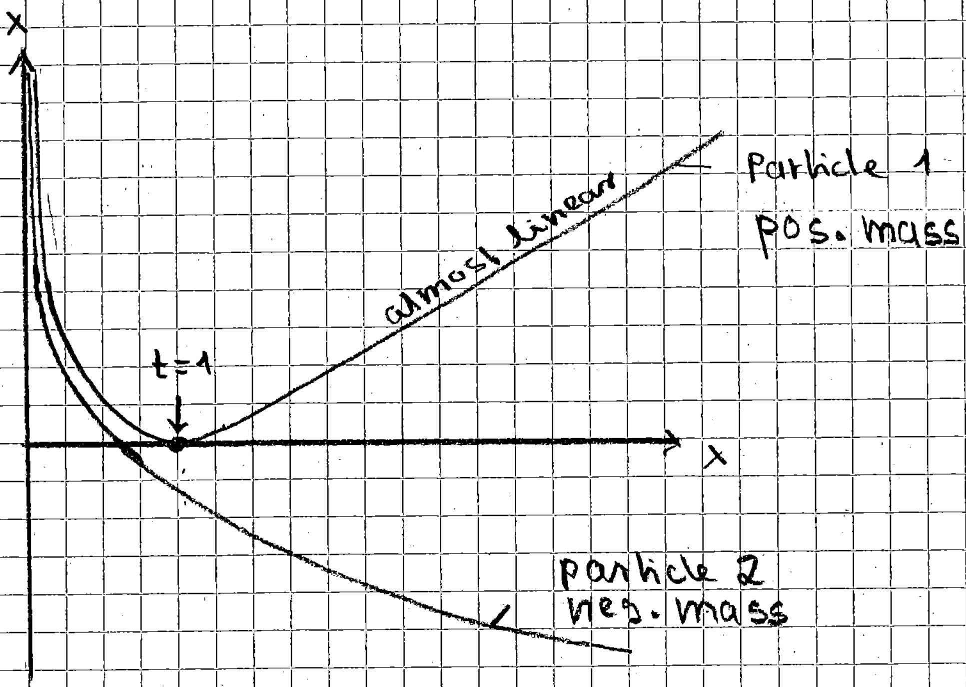
We choose $\vec{V}_1 = 0 = \vec{C}_1$, so that at $t=1$ particle 1 is at rest at origin. We also choose $\vec{e} = -\vec{e}_x$ (particle moves on x-axis) and $Gm = 1$ (choice of mass scale). Then our solution becomes

$$\left.\begin{aligned} X_1(t) &= (t-1) - \ln(t) \geq 0 \quad \forall t \in (0,\infty] \\ X_2(t) &= X_1(t) - t = -(1+\ln(t)) \end{aligned}\right\} (1.2.21)$$

where we just wrote down the x-coordinates for both particles, since both move on x-axis. Their velocities and accelerations are

$$\left.\begin{aligned} \dot{X}_1(t) &= \left(1 - \frac{1}{t}\right), \quad \dot{X}_2 = -\frac{1}{t} \\ \ddot{X}_1(t) &= \ddot{X}_2(t) = \frac{1}{t^2} > 0 \end{aligned}\right\} (1.2.22)$$

This motion is even more weird than the "run-away" solution of case 1.



Note that the case of two masses of opposite sign in Newtonian gravity is not analogous to that of two charges of like charges repelling each other via Coulomb interaction, since in the gravity case the inertial masses, too, have opposite sign.

Exercise 3

$$\Delta \phi = 4\pi G \varrho$$

$$\left.\begin{aligned} \int_{\mathbb{R}^3} \varrho \, d^3x &= \frac{1}{4\pi G} \int_{\mathbb{R}^3} \Delta\phi \, d^3x \\ &= \frac{1}{4\pi G} \lim_{r\to\infty} \left\{ \int_{S^2(r)} (\vec{\nabla}\phi)\cdot\vec{n} \, d\Omega \right\} \end{aligned}\right\} \quad (1.3.1)$$

Gauß' theorem

2-sphere of radius r

Normal to $S^2(r)$

Force density $\vec{f}$ is

$$\vec{f} = -\varrho \vec{\nabla}\phi = -\frac{1}{4\pi G} \Delta\phi \, \vec{\nabla}\phi \quad (1.3.2)$$

Define the tensor

$$t_{ab} := \frac{1}{4\pi G}\left(\nabla_a\phi \nabla_b\phi - \frac{1}{2}\delta_{ab}\nabla^c\phi\nabla_c\phi\right) \quad (1.3.3)$$

then

$$\nabla^a t_{ab} = \frac{1}{4\pi G}\Big(\Delta\phi\nabla_b\phi + \nabla_a\phi\nabla^a\nabla_b\phi - \frac{1}{2}\nabla_b\nabla^c\phi\nabla_c\phi - \frac{1}{2}\nabla^c\phi\nabla_b\nabla_c\phi\Big) \quad (1.3.4)$$

Since $\nabla_a\nabla_b\phi = \nabla_b\nabla_a\phi$ (we assume ϕ to be C^2, i.e. twice continuously differentiable), the 2^{nd} term cancels the sum of the 3^{rd} and 4^{th} term. Hence, also using $\Delta\phi = 4\pi G \varrho$, we get

$$\nabla^a t_{ab} = \varrho \nabla_b \phi = -f_b . \qquad (1.3.5)$$

Let $\vec{n}(\vec{x})$ be a vector field that satisfies

$$\nabla_a n_b = \nabla_b n_a \qquad (1.3.6)$$

Examples are the constant field, where in cartesian coordinates $n^a = n_a = \text{const}$, or the rotation field

$$\vec{n}(\vec{x}) = \vec{w} \times \vec{x} \qquad (1.3.7)$$

where $\vec{w}$ is constant. In the latter case $n_a(\vec{x}) = \varepsilon_{abc} w^b x^c$, and

$$\nabla_a n_b = \nabla_a \varepsilon_{bcd} w^c x^d$$

$$= \varepsilon_{bca} w^c \qquad (1.3.8)$$

which is antisymmetric in (a,b), so (1.3.6) is fullfilled.

Given a $\vec{n}(\vec{x})$ satisfying (1.3.6), we form

$$\int_{\mathbb{R}^3} f_b n^b \, d^3x = -\int_{\mathbb{R}^3} (\nabla^a t_{ab}) n^b \, d^3x \qquad (1.3.9)$$

$$= -\int_{\mathbb{R}^3} \nabla^a (t_{ab} n^b)\, d^3x$$

$$+ \int_{\mathbb{R}^3} t_{ab} (\nabla^a n^b)\, d^3x \qquad (1.3.10)$$

Since t_{ab} is symmetric, i.e. $t_{ab} = t_{ba} = \frac{1}{2}(t_{ab} + t_{ba}) := t_{(ab)}$, we have

$$t_{ab} \nabla^a n^b = t_{ab} \tfrac{1}{2}(\nabla^a n^b + \nabla^b n^a)$$
$$= 0 \quad \text{by } (1.3.6).$$

Hence

$$\int f_b n^b \, d^3x = -\int_{\mathbb{R}^3} \nabla^a (t_{ab} n^b)\, d^3x$$

$$= -\lim_{r\to\infty} \left\{ \int_{S^2(r)} \nu^a t_{ab} n^b \, d\Omega \right\} \qquad (1.3.11)$$

where now the normal to $S^2(r)$ is called ν^a.

Now, this limit of surface integrals vanishes since ϕ satisfies $\Delta\phi = 4\pi G \rho$ with ρ of compact support; hence

$$\phi(\vec{x}\to\infty) \sim \frac{1}{r}, \quad \vec{\nabla}\phi(\vec{x}\to\infty) \sim \frac{1}{r^2} \qquad (1.3.12)$$

and

$$t_{ab}(x \to \infty) \sim |\nabla \phi|^2 \sim \frac{1}{r^4} \qquad (1.3.13)$$

so even if n grows for large r like $\sim r$, like (1.3.7), the integrand of the surface integral (1.3.11) falls off like $1/r^3$ whereas the surface volume element grows like r^2. Hence, for $r \to \infty$, the limit is zero.

$$\int_{\mathbb{R}^3} f_b n^b d^3x = 0 \qquad (1.3.14)$$

For $\vec{n} = \vec{e}_x, \vec{e}_y, \vec{e}_z$ this shows that the total force in x, y and z-direction vanishes. For $\vec{n} = \vec{\omega} \times \vec{X}$ we have $\vec{n} \cdot \vec{f} = \vec{\omega} \cdot (\vec{X} \times \vec{f})$ and hence, since $\vec{\omega}$ is constant

$$0 = \int \vec{f} \cdot \vec{n} \, d^3x = \vec{\omega} \cdot \int (\vec{X} \times \vec{f}) \, d^3x \qquad (1.3.15)$$

and hence the total torque in $\vec{\omega}$-direction vanishes. Since $\vec{\omega}$ was arbitrary, the torque-vector vanishes.

Exercise 4

This is a conceptually very interesting exercise, since it shows how to certain aspects of GR through very simple and plausible modifications of Newtonian gravity. If you want to know more about it, see

https://arxiv.org/pdf/1306.5966.pdf

The idea is simple: Modify

$$\Delta \phi = 4\pi G \rho \qquad (1.4.1)$$

by replacing ρ with the sum of mass-density for matter and the (energy density of grav. field)/c^2. That is: we also want to include gravity as its own source, thereby rendering the field equations non-linear. The replacement is

$$\rho \mapsto \rho + \left(- \frac{1}{8\pi G c^2} \| \vec{\nabla} \phi \|^2 \right) \qquad (1.4.2)$$

Hence we replace (1.4.1) with

$$\Delta \phi = 4\pi G \left[\rho - \frac{1}{8\pi G c^2} \| \vec{\nabla} \phi \|^2 \right] \qquad (1.4.3)$$

A very useful observation is that this equation can be rewritten in a simpler form by using ψ instead of ϕ as dependent variable, where

$$\left.\begin{aligned} \psi &:= \exp(\phi/2c^2) \\ \rightsquigarrow \quad \vec{\nabla}\psi &= \frac{1}{2c^2}(\vec{\nabla}\phi)\,\psi \\ \rightsquigarrow \quad \Delta\psi &= \frac{1}{2c^2}(\Delta\phi)\,\psi + \left(\frac{1}{2c^2}\right)^2 \|\vec{\nabla}\phi\|^2\,\psi \end{aligned}\right\} \text{(1.4.E)}$$

Multiplying (1.4.3) with $\psi/2c^2$ and writing the $(\vec{\nabla}\phi)^2$-term to the left side shows that it is equivalent to

$$\underbrace{\frac{\psi}{2c^2}\left[\Delta\phi + \frac{1}{2c^2}\|\vec{\nabla}\phi\|^2\right]}_{\Delta\psi} = \frac{2\pi G}{c^2}\,\psi\,\rho$$

Hence (1.4.3) is equivalent to the <u>linear</u> equation

$$\Delta\psi = \frac{2\pi G}{c^2}\,\psi\,\rho \qquad (1.4.5)$$

Note that it is still ϕ, not ψ, that is the gravitational potential, i.e. we have $\ddot{\vec{x}} = -\nabla\phi$ (<u>not</u> $\ddot{\vec{x}} = -\nabla\psi$).

So the fact that φ obeys a linear equation – mathematically useful as it is – does not change the fact that the physical quantity obeys a non linear equation.

In order to find the general, spherically symmetric solution for mass density

$$\rho(\vec{x}) = \begin{cases} \sigma = \text{const.} & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \quad (1.4.6)$$

we use the linear equation for $\varphi = \varphi(r)$ (no dependence on θ, ϕ). For the Laplacian in polar coord., the radial part is

$$\Delta \varphi(r) = \varphi''(r) + \frac{2}{r}\varphi' = \frac{1}{r}(r\varphi)'' \quad (1.4.7)$$

where $' = \frac{d}{dr}$

Hence, in the spherically symmetric case, (1.4.5) reads

$$\left.\begin{aligned} (r\varphi)'' &= w^2 (r\varphi) \\ \text{where } w^2 &:= \begin{cases} 2\pi G\sigma/c^2 & \text{for } r \leq R \\ 0 & \text{for } r > R \end{cases} \end{aligned}\right\} \quad (1.4.8)$$

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The boundary conditions are such that $\phi(r \to \infty) \to 0$, which for ψ means $\psi(r \to \infty) \to 1$. Note that ϕ and ψ have different physical dimension: ϕ that of $(\text{velocity})^2$, whereas ψ is dimensionless.

Hence, for $r > R$ we have

$$(r\psi)'' = 0 \iff r\psi = a + br$$

$$\Rightarrow \psi(r) = 1 + \frac{a}{r} \qquad (1.4.9)$$

where we have chosen $b = 1$ so that $\psi(r \to \infty) \to 1$

For $r \leq R$ have

$$(r\psi)'' = \omega^2 (r\psi)$$

$$\Rightarrow r\psi = A\cosh(\omega r) + B\sinh(\omega r)$$

$$\Rightarrow \psi(r) = \frac{A\cosh(\omega r) + B\sinh(\omega r)}{r}$$

Finiteness at $r = 0$ (no singularity inside the star) implies $A = 0$

$$\Rightarrow \psi(r) = B \cdot \frac{\sinh(\omega r)}{r} \qquad (1.4.10)$$

then $\psi(0) = \omega B < \infty$.

Continuity at $r = R$ implies that (1.4.9) and (1.4.10) must match if evaluated at $r = R$:

$$1 + \frac{a}{R} = B \, \frac{\sinh(\omega R)}{R}$$

$$\Rightarrow \quad a = -R + B \sinh(\omega R) \qquad (1.4.11)$$

Moreover, we require φ' to match at $r = R$:

$$-\frac{a}{R^2} = -\frac{B}{R^2} \sinh(\omega R) + \frac{B\omega}{R} \cosh(\omega R) \qquad (1.4.12)$$

We have to solve (11) and (12) for a and B. Inserting the expression (11) for a in (12) gives an equation for B only:

$$\frac{1}{R} = \frac{B\omega}{R} \cosh(\omega R)$$

$$\rightsquigarrow \quad B = [\omega \cosh(\omega R)]^{-1} \qquad (1.4.13)$$

Inserted back into (11) then gives a:

$$a = -R + \omega^{-1} \tanh(\omega R) \qquad (1.4.14)$$

For $r > R$ have:

$$\psi(r) = 1 + \frac{a}{r} = 1 - \frac{R}{r} + \frac{\tanh(\omega R)}{\omega r}$$

$$=: 1 - \frac{R_g}{2r} \qquad (1.4.15)$$

where, for later convenience, we defined the constant

$$R_g := 2R\left(1 - \frac{\tanh(\omega R)}{\omega R}\right) \qquad (1.4.16)$$

For $r \leq R$ have:

$$\psi(r) = \frac{\sinh(\omega r)}{\omega r \cosh(\omega R)} \qquad (1.4.17)$$

The active gravitational mass M is given by $\frac{1}{4\pi G}$ times the flux of $\nabla \phi$ to infinity. From the middle equation (1.4.4) we see that for $r \to \infty$ $\vec{\nabla}\phi = 2c^2 \vec{\nabla}\psi$ + terms falling-off faster than $1/r^2$. Hence

$$M = \lim_{r \to \infty} \left\{ \frac{1}{4\pi G} \int (\vec{\nabla} \phi) \cdot \vec{n} \, d\Omega \right\}$$

$$= \lim_{r \to \infty} \left\{ \frac{c^2}{2\pi G} \int (\vec{\nabla} \psi) \cdot \vec{n} \, d\Omega \right\}$$

$$\overset{15}{=} \frac{c^2}{2\pi G} \frac{R g}{2} 4\pi$$

$$= \frac{c^2}{G} R g$$

$$\overset{16}{=} \frac{2 c^2 R}{G} \left[1 - \frac{\tanh(\omega R)}{\omega R} \right] \tag{1.4.18}$$

Recall the meaning of ω:

$$\left.\begin{aligned} \omega &= \frac{\sqrt{2\pi G \sigma}}{c} \\ \sigma &= \text{mass density of star} \end{aligned}\right\} \tag{1.4.19}$$

So for given R = radius of star, its active gravitational mass $M(R, \sigma)$ strictly monotonically increasing but boundet function of σ with upper bound

$$M(R, \sigma \to \infty) = M_{max}(R) := \frac{2 c^2 R}{G} \tag{1.4.20}$$

How can that be? Why is it impossible to increase the gravity of a star by putting more matter into its volume? Well, because the negative gravitational binding, that we added to the source ρ, also gravitates. From a certain density on (for given R), the addition of any further particle of mass m to the star does not add anything, because the binding energy subtracts mc^2.

But the inequality

$$M < \frac{2c^2 R}{G}$$

can also be read as

$$R > \frac{GM}{2c^2}$$

saying that any mass distribution (here taken to be constant inside the star) whose active gravitational mass is M cannot have a radius smaller than $GM/2c^2$. In particular, there cannot be solutions to (1.4.3) with $M > 0$ and δ-like ρ.