Sheet 10: Solutions

$$\left.\begin{aligned} \gamma : \mathbb{R} \times \mathcal{U} &\to M \quad , \quad \mathcal{U} \subseteq \mathbb{R}^3 \\ (s, \vec{\sigma}) &\longmapsto \gamma(s, \vec{\sigma}) \end{aligned}\right\} \qquad (10.1.1)$$

$$\dot{\gamma} = \frac{\partial \gamma}{\partial s} = \gamma_* \frac{\partial}{\partial s} \qquad (10.1.2)$$

$$\begin{aligned} \gamma' = v^a \frac{\partial \gamma}{\partial \sigma^a} &= \gamma_* \left(v^a \frac{\partial}{\partial \sigma^a} \right) \\ &= v^a \, \gamma_* \frac{\partial}{\partial \sigma^a} \end{aligned} \qquad (10.1.3)$$

$$\begin{aligned} [\dot{\gamma}, \gamma'] &= v^a \left[\gamma_* \frac{\partial}{\partial s}, \gamma_* \frac{\partial}{\partial \sigma^a} \right] \\ &= v^a \, \gamma_* \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial \sigma^a} \right] \\ &= 0 \quad \text{since } \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial \sigma^a} \right] = 0 \end{aligned} \qquad (10.1.4)$$

Compare equation (4.51) of my Diff Geom. notes: $F_* [X, Y] = [F_* X, F_* Y]$.

Since $[X, Y] = L_X Y = - L_Y X$, this shows

$$L_{\dot{\gamma}} \gamma' = 0 \qquad (10.1.5)$$

For each $\vec{\sigma} \in \mathcal{U} \subseteq \mathbb{R}^3$, $s \mapsto \gamma(s, \vec{\sigma})$ is a geodesic. Hence $\dot{\gamma}$ is a geodesic vector field:

$$\nabla_{\dot\gamma}\dot\gamma = 0 \qquad (10.1.6)$$

The parameter s is chosen to be the arc length:

$$g(\dot\gamma, \dot\gamma) = 1 \qquad (10.1.7)$$

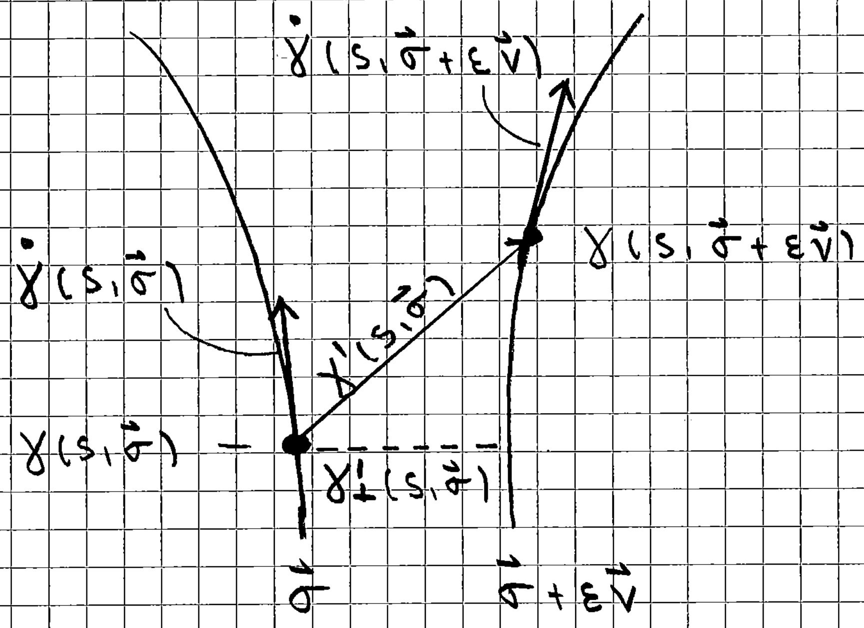
The orthogonal projection of γ' perpendicular to $\dot\gamma$ is

$$\gamma'_\perp := \gamma' - g(\gamma', \dot\gamma)\,\dot\gamma \qquad (10.1.8)$$

Note that

$$\gamma'(s, \vec\sigma) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \gamma(s, \vec\sigma + \varepsilon\vec V) \qquad (10.1.9)$$

which means that $\gamma'(s, \vec\sigma)$ is the vector pointing from $\gamma(s, \vec\sigma)$ to $\gamma(s, \vec\sigma + \varepsilon\vec V)$ for small ε, to first order in ε. It should be understood as the vector connecting "infinitesimally close" neighbouring geodesics in the direction defined by $\vec V$.
For anything that follows only the direction defined by $\vec V$, not the modulus, matters.
This is expressed by the linearity of the final equation in γ'. $\gamma'_\perp$ is the projection of γ' in the "rest frame" of $\dot\gamma$.



(10.1.10)

The vector $\gamma'_\perp(s,\vec\sigma)$ is the simultaneous distance vector of the geodesic labelled by $\vec\sigma$ and that labelled by $\vec\sigma+\varepsilon\vec v$ to first order in ε. Its first and second derivative along $s\mapsto\gamma(s,\vec\sigma)$ gives the relative velocity and acceleration (with respect to proper length s along γ), respectively, with which the "infinitesimally neighbouring" geodesic $s\mapsto\gamma(s,\vec\sigma+\varepsilon\vec v)$ recedes from the geodesic $s\mapsto\gamma(s,\vec\sigma)$.

Now, as γ', $\gamma'_\perp$ too is Lie dragged by $\dot\gamma$:

$$\begin{aligned}
L_{\dot\gamma}\gamma'_\perp &= L_{\dot\gamma}\left(\gamma'-g(\gamma',\dot\gamma)\dot\gamma\right)\\
&= -L_{\dot\gamma}\left(g(\gamma',\dot\gamma)\right)\dot\gamma\\
&= -\nabla_{\dot\gamma}\left(g(\gamma',\dot\gamma)\right)\dot\gamma\\
&= -g(\nabla_{\dot\gamma}\gamma',\dot\gamma)\,\dot\gamma \qquad (10.1.11)
\end{aligned}$$

Zero torsion implies

$$\nabla_{\dot\gamma}\gamma' - \nabla_{\gamma'}\dot\gamma - [\dot\gamma, \gamma'] = 0$$

Using $[\dot\gamma, \gamma'] = 0$ (10.1.4) get

$$\nabla_{\dot\gamma}\gamma' = \nabla_{\gamma'}\dot\gamma \qquad (10.1.12)$$

$$\Rightarrow \quad L_{\dot\gamma}\gamma'_\perp = -g(\nabla_{\gamma'_\perp}\dot\gamma, \dot\gamma)\dot\gamma$$

$$= -\frac{1}{2}\gamma'_\perp[\underbrace{g(\dot\gamma, \dot\gamma)}_{\equiv 1}]\dot\gamma$$

$$= 0 \qquad (10.1.13)$$

By the definition of the curvature tensor, we have

$$\nabla_{\dot\gamma}\nabla_{\gamma'_\perp}\dot\gamma - \nabla_{\gamma'_\perp}\nabla_{\dot\gamma}\dot\gamma - \nabla_{[\dot\gamma, \gamma'_\perp]}\dot\gamma = R(\dot\gamma, \gamma'_\perp)\dot\gamma$$

But $\nabla_{\dot\gamma}\dot\gamma = 0$ (10.1.6), $[\dot\gamma, \gamma'_\perp] = 0$ (10.1.13) and $\nabla_{\gamma'_\perp}\dot\gamma = \nabla_{\dot\gamma}\gamma'_\perp$ ($T = 0$ and (10.1.13)). Hence

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\gamma'_\perp = R(\dot\gamma, \gamma'_\perp)\dot\gamma \qquad (10.1.14)$$

Problem 2

For basis $\{e_\alpha : \alpha = 0,1,2,3\}$ with

$$g(e_\alpha, e_\beta) = \eta_{\alpha\beta} := \mathrm{diag}(1,-1,-1,-1)$$

and $e_0 = \dot{\gamma}$, with

$$\nabla_{\dot{\gamma}}\, e_\alpha = 0 \qquad (10.2.1)$$

(parallely propagated, adapted, ortho-normal basis), we have for

$$\gamma'_\perp = n^\alpha e_\alpha = n^a e_a \qquad (10.2.2)$$

(no term $\sim e_0$ since $\gamma'_\perp \perp e_0 = \dot{\gamma}$)

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}}\, n^a e_a = e_a \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}}\, n^a$$

$$= e_a \frac{d^2 n^a}{ds^2} \underset{(10.1.14)}{=} R(e_0, n^a e_a)\, e_0$$

$$= e_b R^b{}_{0 0 a}\, n^a$$

$$= e_a R^a{}_{0 0 b}\, n^b \qquad (10.2.3)$$

$$\Rightarrow \quad \frac{d^2 n^a}{ds^2} = R^a{}_{0 0 b}\, n^b \qquad (10.2.4)$$

Relative acceleration of neighbouring geodesics in orthonormal frame components

Problem 3

$$g = \left(1 + \frac{2\phi(\vec{x})}{c^2}\right) c\,dt \otimes c\,dt$$

$$- \left(1 - \frac{2\phi(\vec{x})}{c^2}\right) d\vec{x} \otimes d\vec{x} \qquad (10.3.1)$$

$$= (\eta_{\alpha\beta} + h_{\alpha\beta})\, dx^\alpha \otimes dx^\beta \qquad (10.3.2)$$

where

$$h_{\alpha\beta} = \delta_{\alpha\beta} \frac{2\phi}{c^2} \qquad (10.3.3)$$

To linear order in $h_{\alpha\beta}$ have from (10.3-4) in terms of Kulkarni-Nomizu prod.:

$$R_{\alpha\beta\mu\nu} = -\frac{1}{2}\left(\partial^2 \owedge h\right)_{\alpha\beta\mu\nu} \qquad (10.3.4)$$

$$= -\frac{1}{2}\left(\partial_\alpha \partial_\mu h_{\beta\nu} + \partial_\beta \partial_\nu h_{\alpha\mu} - \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\beta \partial_\mu h_{\alpha\nu}\right) \qquad (10.3.5)$$

$$= \frac{1}{c^2}\left(\delta_{\alpha\nu}\, \phi_{,\beta\mu} + \delta_{\beta\mu}\, \phi_{,\alpha\nu} - \delta_{\beta\nu}\, \phi_{,\alpha\mu} - \delta_{\alpha\mu}\, \phi_{,\beta\nu}\right) \qquad (10.3.6)$$

For $\beta = \mu = 0$ and since $\phi_{,0} = 0$, have

$$R_{a00b} = \frac{1}{c^2}\,\phi_{,ab} \qquad (10.3.7)$$

These are the coordinate components which differ from the orthonormal components with respect to

$$\left.\begin{aligned}
\theta^0 &= \left(1 + \frac{2\phi}{c^2}\right)^{1/2} c\,dt \\
&\overset{(1)}{=} \left(1 + \frac{\phi}{c^2}\right) c\,dt \\
\theta^a &= \left(1 - \frac{2\phi}{c^2}\right)^{1/2} dx^a \\
&\overset{(1)}{=} \left(1 - \frac{\phi}{c^2}\right) dx^a
\end{aligned}\right\} \quad (10.3.8)$$

by term $\sim \left(\frac{\phi}{c^2}\right)$. But R_{a00b} is already of order ϕ/c^2, so in linear approximation coordinate components and orthonormal-frame components coincide.

$$\Rightarrow \quad \frac{d^2 n^a}{ds^2} = R^a{}_{00b}\, n^b$$

$$= -\frac{1}{c^2}\,\phi_{,ab}\, n^b \qquad (10.3.9)$$

(raising index a)

Writing $ds/c = d\tau$ = proper time, this is

$$\frac{d^2 n^a}{d\tau^2} = -\phi_{,ab}\, n^b \qquad (10.3.10)$$

This is precisely the Newtonian equation for tidal acceleration: Let $\vec{z}(t,\vec{\sigma})$ be a 3-parameter family of solutions for

$$\ddot{\vec{z}}(t,\vec{\sigma}) = -\vec{\nabla}\phi(\vec{z}(t,\vec{\sigma})) \qquad (10.3.11)$$

Then for $\vec{\sigma} = \vec{\sigma} + \varepsilon\vec{v}$

$$\ddot{\vec{z}}(t,\vec{\sigma}+\varepsilon\vec{v}) = -\vec{\nabla}\phi(\vec{z}(t,\vec{\sigma}+\varepsilon\vec{v}))$$

Taking the $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ of that and writing

$$\vec{n} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \vec{z}(t,\vec{\sigma}+\varepsilon\vec{v})$$
$$= \left(\vec{v}\cdot\frac{\partial}{\partial\vec{\sigma}}\right)\vec{z}(t,\vec{\sigma}), \qquad (10.3.12)$$

We get

$$\ddot{\vec{n}} = -(\vec{n}\cdot\vec{\nabla})\vec{\nabla}\phi \qquad (10.3.13)$$

or
$$\frac{d^2 n^a}{dt^2} = -\phi_{,ab}\, n^b. \qquad (10.3.14)$$

<u>In GR, tidal forces measure Curvature</u>

Problem 4

$$g = c\,dt \otimes c\,dt$$
$$- (1 - h_+(z - ct))\, dx \otimes dx$$
$$- (1 + h_+(z - ct))\, dy \otimes dy$$
$$- dz \otimes dz. \qquad (10.4.1)$$

Again we work in linear approximation so that components of curvature tensor with respect to $c\,dt, dx, dy, dz$ are with respect to

$$\left.\begin{aligned} \theta^0 &= c\,dt \\ \theta^1 &= (1 - h_+)^{1/2}\, dx = \left(1 - \frac{h_+}{2}\right) dx \\ \theta^2 &= (1 + h_+)^{1/2}\, dy = \left(1 + \frac{h_+}{2}\right) dy \\ \theta^3 &= dz \end{aligned}\right\} \qquad (10.4.2)$$

are identical (to linear order in h_+)

We have

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad (10.4.3)$$

with only $h_{11} = -h_{22} = h_+$ non zero both depending on $(z - ct)$.

$$R_{\alpha\beta\mu\nu} = -\frac{1}{2}\left(\partial^2_{\alpha\mu} h_{\beta\nu} + \partial^2_{\beta\nu} h_{\alpha\mu} - \partial^2_{\alpha\nu} h_{\beta\mu} - \partial^2_{\beta\mu} h_{\alpha\nu}\right) \qquad (10.4.7)$$

The components can be listed according to the entries of a symmetric 6×6 - matrix with $\frac{1}{2}\, 6 \cdot 7 = 21$ components (the 1-fold redundancy due to the 1st. Bianchi Identity is unimportant):

	01	02	03	12	13	23
01	R_{0101}	R_{0102}	R_{0103}	R_{0112}	R_{0113}	R_{0123}
02	R_{0201}	R_{0202}	R_{0203}	R_{0212}	R_{0213}	R_{0223}
03	R_{0301}	R_{0302}	R_{0303}	R_{0312}	R_{0313}	R_{0323}
12	R_{1201}	R_{1202}	R_{1203}	R_{1212}	R_{1213}	R_{1223}
13	R_{1301}	R_{1302}	R_{1303}	R_{1312}	R_{1313}	R_{1323}
23	R_{2301}	R_{2302}	R_{2303}	R_{2312}	R_{2313}	R_{2323}

$$R_{0101} = -\frac{1}{2}\left(\partial^2_{00} h_{11} + \partial^2_{11} h_{00} - \partial^2_{01} h_{10} - \partial^2_{10} h_{01}\right)$$
$$= -\frac{1}{2}\ddot{h}_+ \qquad (10.4.5a)$$

$$R_{0102} = -\frac{1}{2}\left(\partial^2_{00} h_{12} + \partial^2_{12} h_{00} - \partial^2_{02} h_{10} - \partial^2_{10} h_{02}\right)$$
$$= 0 \qquad (10.4.5b)$$

$$R_{0103} = -\frac{1}{2}\left(\partial^2_{00} h_{13} + \partial^2_{13} h_{00} - \partial^2_{03} h_{10} - \partial^2_{10} h_{03}\right)$$
$$= 0 \qquad (10.4.5c)$$

$$R_{0112} = -\frac{1}{2}\left(\partial^2_{01} h_{12} + \partial^2_{12} h_{01} - \partial^2_{02} h_{11} - \partial^2_{11} h_{02}\right)$$
$$= 0 \qquad (10.4.5d)$$

$$R_{0113} = -\frac{1}{2}\left(\partial^2_{01} h_{13} + \partial^2_{13} h_{01} - \partial^2_{03} h_{11} - \partial^2_{11} h_{03}\right)$$
$$= \frac{1}{2}\,\partial^2_{03} h_+ = -\frac{1}{2}\, h''_+ \qquad (10.4.5e)$$

$$R_{0123} = -\frac{1}{2}\left(\partial^2_{02} h_{13} + \partial^2_{13} h_{02} - \partial^2_{03} h_{12} - \partial^2_{12} h_{03}\right)$$
$$= 0 \qquad (10.4.5f)$$

$$R_{0202} = -\frac{1}{2}\left(\partial^2_{00} h_{22} + \partial^2_{22} h_{00} - \partial^2_{02} h_{20} - \partial^2_{20} h_{02}\right)$$
$$= +\frac{1}{2}\, h''_+ \qquad (10.4.5g)$$

$$R_{0203} = -\frac{1}{2}\left(\partial^2_{00} h_{23} + \partial^2_{23} h_{00} - \partial^2_{03} h_{20} - \partial^2_{20} h_{03}\right)$$
$$= 0 \qquad (10.4.5h)$$

$$R_{0212} = -\frac{1}{2}\left(\partial^2_{01} h_{22} + \partial^2_{22} h_{01} - \partial^2_{02} h_{21} - \partial^2_{21} h_{02}\right)$$
$$= 0 \qquad (10.4.5i)$$

$$R_{0213} = -\frac{1}{2}\left(\partial^2_{01} h_{23} + \partial^2_{23} h_{01} - \partial^2_{03} h_{21} - \partial^2_{21} h_{03}\right)$$
$$= 0 \qquad (10.4.5j)$$

$$R_{0223} = -\frac{1}{2}\left(\partial^2_{02} h_{23} + \partial^2_{23} h_{02} - \partial^2_{03} h_{22} - \partial^2_{22} h_{03}\right)$$
$$= \frac{1}{2}\,\partial^2_{03} h_{22} = \frac{1}{2}\, h''_+ \qquad (10.4.5k)$$

$$R_{0303} = -\frac{1}{2}\left(\partial^2_{00} h_{33} + \partial^2_{33} h_{00} - \partial^2_{03} h_{30} - \partial^2_{30} h_{03}\right)$$
$$= 0 \qquad (10.4.5l)$$

$$R_{0312} = -\frac{1}{2}\left(\partial^2_{01} h_{32} + \partial^2_{32} h_{01} - \partial^2_{02} h_{31} - \partial^2_{31} h_{02}\right)$$
$$= 0 \qquad (10.4.5m)$$

$$R_{0313} = -\frac{1}{2}\left(\partial^2_{01} h_{33} + \partial^2_{33} h_{01} - \partial^2_{03} h_{31} - \partial^2_{31} h_{03}\right)$$
$$= 0 \qquad (10.4.5n)$$

$$R_{0323} = -\frac{1}{2}\left(\partial^2_{02} h_{33} + \partial^2_{33} h_{02} - \partial^2_{03} h_{32} - \partial^2_{32} h_{03}\right)$$
$$= 0 \qquad (10.4.5o)$$

$$R_{1212} = -\frac{1}{2}\left(\partial^2_{11} h_{22} + \partial_{22} h_{11} - \partial^2_{12} h_{21} - \partial^2_{21} h_{12}\right)$$
$$= 0 \qquad (10.4.5p)$$

$$R_{1213} = -\frac{1}{2}\left(\partial^2_{11} h_{23} + \partial^2_{23} h_{11} - \partial^2_{13} h_{21} - \partial^2_{21} h_{13}\right)$$
$$= 0 \qquad (10.4.5q)$$

$$R_{1223} = -\frac{1}{2}\left(\partial^2_{12} h_{23} + \partial^2_{23} h_{12} - \partial_{13} h_{22} - \partial_{22} h_{13}\right)$$
$$= 0 \qquad (10.4.5r)$$

$$R_{1313} = -\frac{1}{2}\left(\partial^2_{11} h_{33} + \partial^2_{33} h_{11} - \partial^2_{13} h_{31} - \partial^2_{31} h_{13}\right)$$
$$= -\frac{1}{2}\partial^2_{33} h_{11} = -\frac{1}{2} h''_+ \qquad (10.4.5s)$$

$$R_{1323} = -\frac{1}{2}\left(\partial^2_{12} h_{33} + \partial^2_{33} h_{12} - \partial^2_{13} h_{32} - \partial^2_{32} h_{13}\right)$$
$$= 0 \qquad (10.4.5t)$$

$$R_{2323} = -\frac{1}{2}\left(\partial^2_{22} h_{33} + \partial^2_{33} h_{22} - \partial^2_{23} h_{32} - \partial^2_{32} h_{23}\right)$$
$$= -\frac{1}{2}\partial^2_{33} h_{22} = +\frac{1}{2} h''_+ \qquad (10.4.5u)$$

Hence the non-vanishing components of the Riemann-tensor are:

$$\frac{1}{2} h_+'' = - R_{0101} \quad (10.4.6a)$$

$$= - R_{0113} \quad (10.4.6b)$$

$$= + R_{0202} \quad (10.4.6c)$$

$$= + R_{0223} \quad (10.4.6d)$$

$$= - R_{1313} \quad (10.4.6e)$$

$$= + R_{2323} \quad (10.4.6f)$$

Note that h_+' denotes the derivative of h_+ with respect to its argument $z-ct$. Hence

$$h_+' = -\frac{1}{c} \frac{\partial h_+}{\partial t} \quad (10.4.7)$$

and

$$\frac{1}{2} h_+'' = \frac{1}{2c^2} \frac{\partial^2 h_+}{\partial t^2} \quad (10.4.8)$$

Of the R_{a0ob} only $R_{0101} = -R_{0202}$ do not vanish, that is, tidal "forces" occur only in directions perpendicular to the direction of propagation of the GW-wave with opposite signs in x and y-direction:

$$\frac{d^2 n^a}{ds^2} = R^a{}_{00b}\, n^b$$

$$\frac{d^2 n^1}{ds^2} = R^1{}_{001}\, n^1 = - R_{1001}\, n^1$$

$$= R_{0101}\, n^1$$

$$= - \frac{1}{2c^2}\, \partial^2 h_+ / \partial t^2 \qquad (10.4.9)$$

or
$$\frac{d^2 n^1}{d\tau^2} = - \frac{1}{2} \frac{\partial^2 h_+}{\partial t^2}\, n^1 \qquad (10.4.10)$$

and
$$\frac{d^2 n^2}{d\tau^2} = + \frac{1}{2} \frac{\partial^2 h_+}{\partial t^2}\, n^2 \qquad (10.4.11)$$

where again

$$d\tau = ds / c \qquad (10.4.12)$$

is the eigentime along the geodesic.

Problem 5

$$g = e^{2a}\, dx^0 \otimes dx^0$$
$$- e^{2b}\, dr \otimes dr$$
$$- e^{2c} R^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi \right)$$

$$= \theta^0 \otimes \theta^0 - \sum_{a=1}^{3} \theta^a \otimes \theta^a \qquad (10.5.1)$$

$$\theta^0 = e^a\, dx^0 \qquad (10.5.2a)$$

$$\theta^1 = e^b\, dr \qquad (10.5.2b)$$

$$\theta^2 = e^c R\, d\theta \qquad (10.5.2c)$$

$$\theta^3 = e^c R \sin\theta\, d\varphi \qquad (10.5.2d)$$

a, b, and c are functions of x^0 and r, R is a constant with physical dimension of length.

We use the Cartan structure equation to determine the connection 1-forms $\omega^\alpha{}_\beta$ and curvature 2-forms $\Omega^\alpha{}_\beta$

Determination of the connection via

$$d\theta^\alpha = -\omega^\alpha{}_\beta \wedge \theta^\beta \qquad (10.5.3)$$

$\alpha = 0)$

$$d\theta^0 = (\dot{a}\, dx^0 + a'\, dr) \wedge dx^0\, e^a$$

$$= -e^{-b} a'\, \theta^0 \wedge \theta^1$$

$$= -\omega^0{}_1 \wedge \theta^1 - \omega^0{}_2 \wedge \theta^2 - \omega^0{}_3 \wedge \theta^3$$

$$\Rightarrow \quad \omega^0{}_1 = e^{-b} a'\, \theta^0 + \text{Terms} \sim \theta^1 \qquad (10.5.4a)$$

$$\omega^0{}_2 = \text{Terms} \sim \theta^2 \qquad (10.5.4b)$$

$$\omega^0{}_3 = \text{Terms} \sim \theta^3 \qquad (10.5.4c)$$

$\alpha = 1)$

$$d\theta^1 = (\dot{b}\, dx^0 + a'\, dr) \wedge dr\, e^b$$

$$= -e^{-a} \dot{b}\, \theta^1 \wedge \theta^0$$

$$= -\omega^1{}_0 \wedge \theta^0 - \omega^1{}_2 \wedge \theta^2 - \omega^1{}_3 \wedge \theta^3$$

$$\Rightarrow \quad \omega^1{}_0 = e^{-a} \dot{b}\, \theta^1 + \text{Terms} \sim \theta^0 \qquad (10.5.5a)$$

$$\omega^1{}_2 = \text{Terms} \sim \theta^2 \qquad (10.5.5b)$$

$$\omega^1{}_3 = \text{Terms} \sim \theta^3 \qquad (10.5.5c)$$

$\alpha = 2)$

$$d\theta^2 = (\dot{c}\, dx^0 + c'\, dr) R\, d\theta\, e^c$$
$$= -\dot{c}\, e^{-a}\, \theta^2 \wedge \theta^0 - c'\, e^{-b}\, \theta^2 \wedge \theta^1$$
$$= -\omega^2{}_0 \wedge \theta^0 - \omega^2{}_1 \wedge \theta^1 - \omega^2{}_3 \wedge \theta^3$$

$$\Rightarrow \quad \omega^2{}_0 = \dot{c}\, e^{-a}\, \theta^2 + \text{Terms} \sim \theta^0 \qquad (10.5.6a)$$
$$\omega^2{}_1 = c'\, e^{-b}\, \theta^2 + \text{Terms} \sim \theta^1 \qquad (10.5.6b)$$
$$\omega^2{}_3 = \text{Terms} \sim \theta^3 \qquad (10.5.6c)$$

$\alpha = 3)$

$$d\theta^3 = (\dot{c}\, dx^0 + c'\, dr) \wedge e^c R \sin\theta\, d\varphi$$
$$+ e^c R \cos\theta\, d\theta \wedge d\varphi$$
$$= -\dot{c}\, e^{-a}\, \theta^3 \wedge \theta^0 - c'\, e^{-b}\, \theta^3 \wedge \theta^1$$
$$- e^{-c} \frac{1}{R} \cot\theta\, \theta^3 \wedge \theta^2$$
$$= -\omega^3{}_0 \wedge \theta^0 - \omega^3{}_1 \wedge \theta^1 - \omega^3{}_2 \wedge \theta^2$$

$$\Rightarrow \quad \omega^3{}_0 = \dot{c}\, e^{-a}\, \theta^3 + \text{Terms} \sim \theta^0 \qquad (10.5.7a)$$
$$\omega^3{}_1 = c'\, e^{-b}\, \theta^3 + \text{Terms} \sim \theta^1 \qquad (10.5.7b)$$
$$\omega^3{}_2 = e^{-c} \frac{\cot\theta}{R}\, \theta^3 + \text{Terms} \sim \theta^2 \qquad (10.5.7c)$$

From the equations (10.5.4-7) we can infer all $\omega^\alpha{}_\beta$ by using the metricity conditions (no summation over α or β):

$$\omega^\alpha{}_\beta = -\varepsilon_\alpha \varepsilon_\beta \, \omega^\beta{}_\alpha \tag{10.5.8}$$

where $\varepsilon_\alpha = \eta_{\alpha\alpha}$, $\varepsilon_\beta = \eta_{\beta\beta}$. This means

$$\left.\begin{aligned} \omega^0{}_a &= \omega^a{}_0 \\ \text{and} \quad \omega^a{}_b &= -\omega^b{}_a \end{aligned}\right\} (10.5.9)$$

Hence we conclude

from (10.5.4a-5a):

$$\omega^0{}_1 = \omega^1{}_0 = e^{-b} a' \, \theta^0 + e^{-a} \dot{b} \, \theta^1 \tag{10.5.10a}$$

from (10.5.4b, 6a):

$$\omega^0{}_2 = \omega^2{}_0 = \dot{c} \, e^{-a} \theta^2 \tag{10.5.10b}$$

from (10.5.4c, 7a)

$$\omega^0{}_3 = \omega^3{}_0 = \dot{c} \, e^{-a} \theta^3 \tag{10.5.10c}$$

from (10.5.5b, 6b):

$$\omega^1{}_2 = -\omega^2{}_1 = -c' e^{-b} \theta^2 \tag{10.5.10d}$$

from (10.5.5c, 7b):

$$\omega^1{}_3 = -\omega^3{}_1 = -c' e^{-b} \theta^3 \qquad (10.5.10e)$$

from (10.5.6c, 7c):

$$\omega^2{}_3 = -\omega^3{}_2 = -e^{-c} \frac{\cot\theta}{R} \theta^3 \qquad (10.5.10f)$$

We list them once more, also to express them not in terms of θ^λ but dx^λ, which facilitates the calculation of curvature

$$\left.\begin{aligned} \omega^0{}_1 = \omega^1{}_0 &= e^{-b} a' \theta^0 + e^{-a} \dot{b}\, \theta^1 \\ &= e^{(a-b)} a' dx^0 + e^{(b-a)} \dot{b}\, dr \end{aligned}\right\} (10.5.11a)$$

$$\left.\begin{aligned} \omega^0{}_2 = \omega^2{}_0 &= \dot{c}\, e^{-a} \theta^2 \\ &= \dot{c} R\, e^{(c-a)} d\theta \end{aligned}\right\} (10.5.11b)$$

$$\left.\begin{aligned} \omega^0{}_3 = \omega^3{}_0 &= \dot{c}\, e^{-a} \theta^3 \\ &= \dot{c} R\, e^{(c-a)} \sin\theta\, d\varphi \end{aligned}\right\} (10.5.11c)$$

$$\left.\begin{aligned} \omega^1{}_2 = -\omega^2{}_1 &= -c' e^{-b} \theta^2 \\ &= -c' R\, e^{(c-b)} d\theta \end{aligned}\right\} (10.5.11d)$$

$$\left.\begin{aligned} \omega^1{}_3 = -\omega^3{}_1 &= -c' e^{-b}\, \theta^3 \\ &= -c' R\, e^{(c-b)} \sin\theta\, d\varphi \end{aligned}\right\} \quad (10.5.11e)$$

$$\left.\begin{aligned} \omega^2{}_3 = -\omega^3{}_2 &= -e^{-c}\, \frac{\cot\theta}{R}\, \theta^3 \\ &= -\cos\theta\, d\varphi \end{aligned}\right\} \quad (10.5.11f)$$

Determination of Curvature via

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta \qquad (10.5.12)$$

0 1)

$$d\omega^0{}_1 = \left(a'' + a'(a' - b')\right) e^{(a-b)}\, dr \wedge dx^0$$
$$+ \left(\ddot{b} + \dot{b}(\dot{b} - \dot{a})\right) e^{(b-a)}\, dx^0 \wedge dr$$

$$= -e^{-2b}\left[a'' + a'^2 - a'b'\right] \theta^0 \wedge \theta^1$$
$$+ e^{-2a}\left[\ddot{b} + \dot{b}^2 - \dot{a}\dot{b}\right] \theta^0 \wedge \theta^1$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_1 = \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1$$
$$= 0$$

(since $\omega^0{}_2$ and $\omega^2{}_1 \sim \theta^2$, and $\omega^0{}_3$ and $\omega^3{}_1 \sim \theta^3$).

$$\Rightarrow \Omega^0{}_1 = \Big[e^{-2a} (\ddot{b} + \dot{b}^2 - \dot{a}\dot{b})$$

$$- e^{-2b} (a'' + a'^2 - a'b') \Big] \theta^0 \wedge \theta^1 \quad (10.5.13a)$$

02)

$$d\omega^0{}_2 = (\ddot{c} + \dot{c}(\dot{c} - \dot{a}))\, e^{(c-a)} R\, dx^0 \wedge d\theta$$

$$+ (\dot{c}' + \dot{c}(c' - a'))\, e^{(c-a)} R\, dr \wedge d\theta$$

$$= e^{-2a} [\ddot{c} + \dot{c}(\dot{c} - \dot{a})] \theta^0 \wedge \theta^2$$

$$+ e^{-(a+b)} [\dot{c}' + \dot{c}c' - a'\dot{c}] \theta^1 \wedge \theta^2$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_2 = \omega^0{}_1 \wedge \omega^1{}_2 + \omega^0{}_3 \wedge \omega^3{}_2$$

$$= \omega^0{}_1 \wedge \omega^1{}_2 \quad (\text{since } \omega^0{}_3 \text{ and } \omega^3{}_2 \sim \theta^3)$$

$$= (e^{-b} a' \theta^0 + e^{-a} \dot{b} \theta^1)(-c' e^{-b} \theta^2)$$

$$= - e^{-2b} a'c' \theta^0 \wedge \theta^2 - e^{-(a+b)} \dot{b} c' \theta^1 \wedge \theta^2$$

$$\Rightarrow \Omega^0{}_2 = \Big\{ e^{-2a} (\ddot{c} + \dot{c}^2 - \dot{a}\dot{c}) - e^{-2b} a'c' \Big\} \theta^0 \wedge \theta^2$$

$$+ e^{-(a+b)} (\dot{c}' + \dot{c}c' - a'\dot{c} - \dot{b}c') \theta^1 \wedge \theta^2$$

(10.5.13b)

03)

$$d\omega^0{}_3 = (\ddot{c} + \dot{c}(\dot{c}-\dot{a}))\, e^{(c-a)} R\sin\theta\, dx^0 \wedge d\varphi$$
$$+ (\dot{c}' + \dot{c}(c'-a'))\, e^{(c-a)} R\sin\theta\, dr \wedge d\varphi$$
$$+ \dot{c} R\, e^{(c-a)} \cos\theta\, d\theta \wedge d\varphi$$

$$= [\ddot{c} + \dot{c}^2 - \dot{a}\dot{c}]\, e^{-2a}\, \theta^0 \wedge \theta^3$$
$$+ [\dot{c}' + \dot{c}c' - a'\dot{c}]\, e^{-(a+b)}\, \theta^1 \wedge \theta^3$$
$$+ \dot{c}\, \frac{\cot\theta}{R}\, e^{-(a+c)}\, \theta^2 \wedge \theta^3$$

$$\omega^0{}_\lambda \wedge \omega^\lambda{}_3 = \omega^0{}_1 \wedge \omega^1{}_3 + \omega^0{}_2 \wedge \omega^2{}_3$$
$$= (e^{-b} a' \theta^0 + e^{-a} \dot{b}\, \theta^1)(-c' e^{-b} \theta^3)$$
$$+ \dot{c}\, e^{-a} \theta^2 \left(-e^{-c} \frac{\cot\theta}{R} \theta^3\right)$$
$$= -e^{-2b} a'c'\, \theta^0 \wedge \theta^3 - e^{-(a+b)} \dot{b} c'\, \theta^1 \wedge \theta^3$$
$$- e^{-(a+c)} \dot{c}\, \frac{\cot\theta}{R}\, \theta^2 \wedge \theta^3$$

$$\Rightarrow \Omega^0{}_3 = \left\{ e^{-2a}(\ddot{c} + \dot{c}^2 - \dot{a}\dot{c}) - e^{-2b} a'c' \right\} \theta^0 \wedge \theta^3$$
$$e^{-(a+b)} (\dot{c}' + \dot{c}c' - a'\dot{c} - \dot{b}c')\, \theta^1 \wedge \theta^3$$

(10.5.13c)

12)

$$d\omega^1{}_2 = -(\dot{c}' + c'(\dot{c}-\dot{b}))\,R\,e^{(c-b)}\,dx^0\wedge d\theta$$
$$-(c'' + c'(c'-b'))\,R\,e^{(c-b)}\,dr\wedge d\theta$$

$$= -(\dot{c}' + c'\dot{c} - c'\dot{b})\,e^{-(a+b)}\,\theta^0\wedge\theta^2$$
$$-(c''+c'^2-c'b')\,e^{-2b}\,\theta^1\wedge\theta^2$$

$$\omega^1{}_\lambda\wedge\omega^\lambda{}_2 = \omega^1{}_0\wedge\omega^0{}_2 + \omega^1{}_3\wedge\omega^3{}_2$$
$$= \omega^1{}_0\wedge\omega^0{}_2 \quad (\text{since } \omega^1{}_3 \text{ and } \omega^3{}_2 \sim \theta^3)$$
$$= (e^{-b}a'\,\theta^0 + e^{-a}\dot{b}\,\theta^1)\wedge \dot{c}\,e^{-a}\,\theta^2$$
$$= e^{-(a+b)}\,a'\dot{c}\,\theta^0\wedge\theta^2 + e^{-2a}\,\dot{b}\dot{c}\,\theta^1\wedge\theta^2$$

$$\Rightarrow \Omega^1{}_2 = (a'\dot{c} - \dot{c}' - c'\dot{c} - c'\dot{b})\,e^{-(a+b)}\,\theta^0\wedge\theta^2$$
$$+\left[e^{-2a}\,\dot{b}\dot{c} + e^{-2b}(c'b'-c''-c'^2)\right]\theta^1\wedge\theta^2$$

(10.5.13d)

13)

$$d\omega^1{}_3 = -(\dot{c}' + c'(\dot{c}-\dot{b}))\,R\,e^{(c-b)}\sin\theta\,dx^0\wedge d\varphi$$
$$-(c''+c'(c'-b'))\,R\,e^{(c-b)}\sin\theta\,dr\wedge d\varphi$$
$$-c'\,e^{(c-b)}\,R\cos\theta\,d\theta\wedge d\varphi$$

$$= -(\dot{c}' + c'\dot{c} - \dot{b}c')\,e^{-(a+b)}\,\theta^0\wedge\theta^3$$
$$-\,(c'' + c'(c'-b'))\,e^{-2b}\,\theta^1\wedge\theta^3$$
$$-\,c'\,e^{-(b+c)}\,\frac{\cot\theta}{R}\,\theta^2\wedge\theta^3$$

$$\omega^1{}_\lambda\wedge\omega^\lambda{}_3 = \omega^1{}_0\wedge\omega^0{}_3 + \omega^1{}_2\wedge\omega^2{}_3$$
$$= (e^{-b}a'\theta^0 + e^{-a}\dot{b}\theta^1)\,\dot{c}\,e^{-a}\,\theta^3$$
$$+(-c'e^{-b}\theta^2)\left(-e^{-c}\frac{\cot\theta}{R}\theta^3\right)$$
$$= a'\dot{c}\,e^{-(a+b)}\,\theta^0\wedge\theta^3 + e^{-2a}\dot{b}\dot{c}\,\theta^1\wedge\theta^3$$
$$+\,c'\,e^{-(b+c)}\,\frac{\cot\theta}{R}\,\theta^2\wedge\theta^3$$

$$\Rightarrow \Omega^1{}_3 = (a'\dot{c} - \dot{c}' - c'\dot{c} + \dot{b}c')\,e^{-(a+b)}\,\theta^0\wedge\theta^3$$
$$+\left(e^{-2a}\dot{b}\dot{c} + e^{-2b}(b'c' - c'' - c'^2)\right)\theta^1\wedge\theta^3$$

(10.5.13e)

23)

$$d\omega^2{}_3 = \sin\theta\, d\theta\wedge d\varphi$$
$$= e^{-2c}R^{-2}\,\theta^2\wedge\theta^3$$

$$\omega^2{}_1 \wedge \omega^1{}_3 = \omega^2{}_0 \wedge \omega^0{}_3 + \omega^3{}_1 \wedge \omega^1{}_2$$

$$= (\dot{c}\, e^{-a}\, \theta^2) \wedge (\dot{c}\, e^{-a}\, \theta^3)$$

$$+ (c'\, e^{-b}\, \theta^2) \wedge (-c' e^{-b}\, \theta^3)$$

$$= (e^{-2a}\, \dot{c}^2 - e^{-2b}\, c'^2)\, \theta^2 \wedge \theta^3$$

$$\Rightarrow \Omega^2{}_3 = (e^{-2a}\, \dot{c}^2 - e^{-2b}\, c'^2 + e^{-2c}\, R^{-2})\, \theta^2 \wedge \theta^3 \qquad (10.5.13f)$$

Using all the expressions (10.5.13 a-f)
for

$$\Omega^\alpha{}_\beta = \frac{1}{2} R^\alpha{}_{\beta\mu\nu}\, \theta^\mu \wedge \theta^\nu \qquad (10.5.14)$$

we can read-off the components of the Riemann Curvature Tensor w.r.t. the orthonormal frame θ^α. We shall write them in a slightly redundant fashion, listing separately, e.g., $R^0{}_{313}$ and $R^1{}_{303}$ for later convenience. The following list generalises our list (17.28) of Lecture 17 for $e^c = r$ ...

$$\left.\begin{aligned} R^0{}_{101} &= R_{0101} \\ &= e^{-2a}(\ddot{b}+\dot{b}^2-\dot{a}\dot{b}) - e^{-2b}(a''+a'^2-a'b') \end{aligned}\right\} \quad (10.5.15a)$$

$$\left.\begin{aligned} R^0{}_{202} &= R_{0202} \\ &= e^{-2a}(\ddot{c}+\dot{c}^2-\dot{a}\dot{c}) - e^{-2b}a'c' \end{aligned}\right\} \quad (10.5.15b)$$

$$\left.\begin{aligned} R^0{}_{212} &= R_{0212} \\ &= e^{-(a+b)}(\dot{c}'+\dot{c}c'-a'\dot{c}-\dot{b}c') \end{aligned}\right\} \quad (10.5.15c)$$

$$\left.\begin{aligned} R^0{}_{303} &= R_{0303} \\ &= e^{-2a}(\ddot{c}+\dot{c}^2-\dot{a}\dot{c}) - e^{-2b}a'c' \end{aligned}\right\} \quad (10.5.15d)$$

$$\left.\begin{aligned} R^0{}_{313} &= R_{0313} \\ &= e^{-(a+b)}(\dot{c}'+\dot{c}c'-a'\dot{c}-\dot{b}c') \end{aligned}\right\} \quad (10.5.15e)$$

$$\left.\begin{aligned} R^1{}_{212} &= -R_{1212} \\ &= e^{-2a}\dot{b}\dot{c} + e^{-2b}(c'b'-c''-c'^2) \end{aligned}\right\} \quad (10.5.15f)$$

$$\left.\begin{aligned} R^1{}_{202} &= -R_{1202} \\ &= e^{-(a+b)}(a'\dot{c}-\dot{c}'-c'\dot{c}-\dot{b}c') \end{aligned}\right\} \quad (10.5.15g)$$

$$\left.\begin{aligned} R^{1}{}_{313} &= -R_{1313} \\ &= e^{-2a}\,\dot{b}\dot{c} + e^{-2b}\,(b'c' - c'' - c'^{2}) \end{aligned}\right\} \quad (10.5.15h)$$

$$\left.\begin{aligned} R^{1}{}_{303} &= -R_{1303} \\ &= e^{-(a+b)}\,(a'\dot{c} - \dot{c}' - c'\dot{c} + \dot{b}c') \end{aligned}\right\} \quad (10.5.15i)$$

$$\left.\begin{aligned} R^{2}{}_{323} &= -R_{2323} \\ &= e^{-2a}\,\dot{c}^{2} - e^{-2b}\,c'^{2} + e^{-2c}\,R^{-2} \end{aligned}\right\} \quad (10.5.15j)$$

One can check that for

$$R e^{c} = r \iff r = R \ln(c) \qquad (10.5.16)$$

(10.5.15 a-j) turn into (17.28 a-j) presented in Lecture 17.

From the covariant components $R_{\alpha\beta\mu\nu}$ the Einstein Tensor - components can be calculated as follows

$$G_{00} = R_{00} - \frac{1}{2}\eta_{00}(R_{00} - R_{11} - R_{22} - R_{33})$$

$$= \frac{1}{2}(R_{00} + R_{11} + R_{22} + R_{33})$$

$$= \frac{1}{2}(R^{\lambda}{}_{0\lambda 0} + R^{\lambda}{}_{1\lambda 1} + R^{\lambda}{}_{2\lambda 2} + R^{\lambda}{}_{3\lambda 3}) \qquad (10.5.17)$$

The sum in brackets is

$$\begin{aligned}
&-R_{1\cancel{0}\cancel{1}0} - R_{\cancel{2}\cancel{0}20} - R_{\cancel{3}\cancel{0}30}\\
&+R_{0\cancel{1}\cancel{0}1} - R_{2121} - R_{3131}\\
&+R_{0\cancel{2}\cancel{0}2} - R_{1212} - R_{3232}\\
&+R_{0\cancel{3}\cancel{0}3} - R_{1313} - R_{2323}
\end{aligned}$$

$$= -2(R_{1212} + R_{1313} + R_{2323})$$

Hence

$$G_{00} = -R_{1212} - R_{1313} - R_{2323}$$

$$= -2R_{1212} - R_{2323} \qquad (10.5.18)$$

Since $R_{1212} = R_{1313}$

Similarly

$$G_{11} = R_{11} + \frac{1}{2}\eta_{11}\left(R_{00} - R_{11} - R_{22} - R_{33}\right)$$

$$= \frac{1}{2}\left(R_{00} + R_{11} - R_{22} - R_{33}\right)$$

$$= \frac{1}{2}\left(R^{\lambda}{}_{0\lambda 0} + R^{\lambda}{}_{1\lambda 1} + R^{\lambda}{}_{2\lambda 2} - R^{\lambda}{}_{3\lambda 3}\right) \quad (10.5.19)$$

The bracket is

$$\begin{aligned}
&- \cancel{R_{1010}} - R_{2020} - R_{3030} \\
&+ \cancel{R_{0101}} - \cancel{R_{2121}} - \cancel{R_{3131}} \\
&- R_{0202} + \cancel{R_{1212}} + R_{3232} \\
&- R_{0303} + \cancel{R_{1313}} + R_{2323}
\end{aligned}$$

$$= -2\left(R_{0202} + R_{0303} - R_{2323}\right)$$

Hence

$$G_{11} = R_{2323} - 2R_{0202} \quad (10.5.20)$$

since $R_{0202} = R_{0303}$

$$G_{22} = R_{22} - \frac{1}{2}\eta_{22}\left(R_{00} - R_{11} - R_{22} - R_{33}\right)$$

$$= \frac{1}{2}\left(R_{00} + R_{22} - R_{11} - R_{33}\right)$$

$$= \frac{1}{2}\left(R^{\lambda}{}_{0\lambda 0} + R^{\lambda}{}_{2\lambda 2} - R^{\lambda}{}_{1\lambda 1} - R^{\lambda}{}_{3\lambda 3}\right) \qquad (10.5.21)$$

The term in brackets is

$$\begin{aligned}
&- R_{1010} - \cancel{R_{2020}} - R_{3030} \\
&+ \cancel{R_{0202}} - \cancel{R_{1212}} - \cancel{R_{3232}} \\
&- R_{0101} + \cancel{R_{2121}} + R_{3131} \\
&- R_{0303} + R_{1313} + \cancel{R_{2323}}
\end{aligned}$$

$$= 2\left(-R_{0101} - R_{0303} + R_{1313}\right)$$

Hence

$$G_{22} = G_{33}$$

$$= R_{1313} - R_{0101} - R_{0303} \qquad (10.5.22)$$

E 10.31

$$G_{01} = R_{01} = R^{\lambda}{}_{0\lambda 1}$$

$$= -R_{2021} - R_{3031}$$

$$= -2R_{0212} \qquad (10.5.23)$$

since $R_{2021} = R_{3031} = R_{0212}$.

Clearly, by spherical symmetry

$$G_{02} = G_{03} = G_{12} = G_{13} = 0. \qquad (10.5.24)$$

as can also be checked from the formulae above; e.g.,

$$G_{02} = R^{\lambda}{}_{0\lambda 2} = -R_{1012} - R_{3032} = 0$$

since both curvature components vanish.