Sheet 11: Solutions

Problem 1

$$\left.\begin{aligned} R_{0101} &= -R_{2323} = \frac{r_s}{r^3} \\ R_{0202} &= R_{0303} \\ &= -R_{1212} \\ &= -R_{1313} = -\frac{1}{2}\frac{r_s}{r^3} \end{aligned}\right\} \quad (11.1.1)$$

Lorentz-Transformation in radial direction

$$\{\Lambda^\alpha{}_\beta\} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & 0 & 0 \\ \sinh(\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (11.1.2)$$

$$\theta^\alpha \mapsto \hat{\theta}^\alpha := \Lambda^\alpha{}_\beta \theta^\beta \quad (11.1.3)$$

$$R = R_{\alpha\beta\mu\nu}\, \theta^\alpha \otimes \theta^\beta \otimes \theta^\mu \otimes \theta^\nu$$

$$\rightarrow \hat{R} := \Lambda^{\otimes 4}(R) = R_{\alpha\beta\mu\nu}\, \hat{\theta}^\alpha \otimes \cdots \otimes \hat{\theta}^\nu$$

$$= \hat{R}_{\alpha\beta\mu\nu}\, \theta^\alpha \otimes \cdots \otimes \theta^\nu$$

where $\hat{R}_{\alpha\beta\mu\nu} = R_{\lambda\sigma\kappa\rho} \Lambda^\lambda{}_\alpha \Lambda^\sigma{}_\beta \Lambda^\kappa{}_\mu \Lambda^\rho{}_\nu$

(11.1.4)

Have

$$\left.\begin{aligned} \Lambda^{\mu}{}_{0} &= \cosh(\alpha)\,\delta^{\mu}{}_{0} + \sinh(\alpha)\,\delta^{\mu}{}_{1} \\ \Lambda^{\mu}{}_{1} &= \sin(\alpha)\,\delta^{\mu}{}_{0} + \cosh(\alpha)\,\delta^{\mu}{}_{1} \\ \Lambda^{\mu}{}_{2} &= \delta^{\mu}{}_{2} \\ \Lambda^{\mu}{}_{3} &= \delta^{\mu}{}_{3} \end{aligned}\right\} \quad (11.1.5)$$

This gives

$$\hat{R}_{0101} = R_{\alpha\beta\mu\nu}\,\Lambda^{\alpha}{}_{0}\Lambda^{\beta}{}_{1}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{1}$$

$$= R_{0101}\left(\cosh^2(\alpha) - \sinh^2(\alpha)\right)^2$$

$$= R_{0101} \quad (11.1.7)$$

$$\hat{R}_{2323} = R_{\alpha\beta\mu\nu}\,\Lambda^{\alpha}{}_{2}\Lambda^{\beta}{}_{3}\Lambda^{\mu}{}_{2}\Lambda^{\nu}{}_{3}$$

$$= R_{2323} \quad (11.1.8)$$

$$\hat{R}_{1212} = R_{\alpha\beta\mu\nu}\,\Lambda^{\alpha}{}_{1}\Lambda^{\beta}{}_{2}\Lambda^{\mu}{}_{1}\Lambda^{\nu}{}_{2}$$

$$= R_{\alpha 2\mu 2}\,\Lambda^{\alpha}{}_{1}\Lambda^{\mu}{}_{1}$$

$$= \sinh^2(\alpha)\underbrace{R_{0202}}_{=-R_{1212}} + \cosh^2(\alpha)\,R_{1212}$$

$$+\,2\underbrace{R_{0212}}_{=0}\sinh(\alpha)\cosh(\alpha)$$

$$= R_{1212} \quad (11.1.9)$$

$$\hat{R}_{1313} = \text{same as for } R_{1212}$$

$$= R_{1313} \qquad (11.1.10)$$

$$\hat{R}_{0202} = R_{\alpha\beta\mu\nu}\,\Lambda^{\alpha}{}_{0}\,\Lambda^{\beta}{}_{2}\,\Lambda^{\mu}{}_{0}\,\Lambda^{\nu}{}_{2}$$

$$= R_{\alpha 2\mu 2}\,\Lambda^{\alpha}{}_{0}\,\Lambda^{\mu}{}_{0}$$

$$= R_{0202}\cosh^2(\alpha)$$

$$+ \underbrace{R_{1212}}_{=-R_{0202}}\sinh^2(\alpha)$$

$$+ 2\sinh(\alpha)\cosh(\alpha)\underbrace{R_{0212}}_{=0}$$

$$= R_{0202} \qquad (11.1.11)$$

$$\hat{R}_{0303} = \text{same as for } R_{0202}$$

$$= R_{0303} \qquad (11.1.12)$$

Showing so far that for the index combinations (0101), (0202), (0303), (1212), (1313), and (2323) the components are as before. But there will also be no new non-zero components, e.g.

E11.4

$$\hat{R}_{0212} = R_{\alpha\beta\mu\nu} \Lambda^{\alpha}{}_{0} \Lambda^{\beta}{}_{2} \Lambda^{\mu}{}_{1} \Lambda^{\nu}{}_{2}$$

$$= R_{\alpha 2 \mu 2} \Lambda^{\alpha}{}_{0} \Lambda^{\mu}{}_{1}$$

$$= \underbrace{R_{1212}}_{= -R_{0202}} \sinh(\alpha)\cosh(\alpha)$$

$$+ R_{0202} \sinh(\alpha)\cosh(\alpha)$$

$$+ \cancel{R_{0212}}^{\,0} \cosh^2(\alpha)$$

$$+ \cancel{R_{1202}}^{\,0} \cosh^2(\alpha)$$

$$= 0 \qquad (11.1.13)$$

Problem 2

From

$$\frac{d^2 n^a}{d\tau^2} = c^2 R^a{}_{00b}\, n^b \qquad (11.2.1)$$

and (11.1.1)

$$R_{0101} = -2 R_{0202} = -2 R_{0303}$$

$$= r_s / r^3$$

hence

$$R^1{}_{001} = -2 R^2{}_{002} = -2 R^3{}_{003}$$

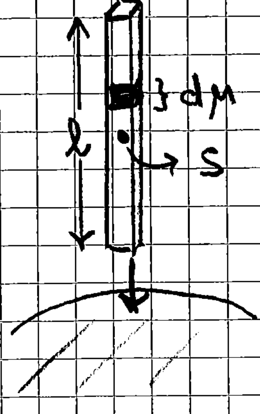
$$= r_s / r^3 \qquad (11.2.2)$$

we immediately get

$$\frac{d^2 n^1}{d\tau^2} = c^2 \frac{r_s}{r^3} n^1 \qquad (11.2.3)$$

$$\frac{d^2 n^{2,3}}{d\tau^2} = -\frac{c^2}{2} \frac{r_s}{r^3} n^{2,3} \qquad (11.2.4)$$

Consider an elastic rod falling, end first, radially in the grav.-field. A mass-element $d\mu$ at height h above the centre point suffers outward force of

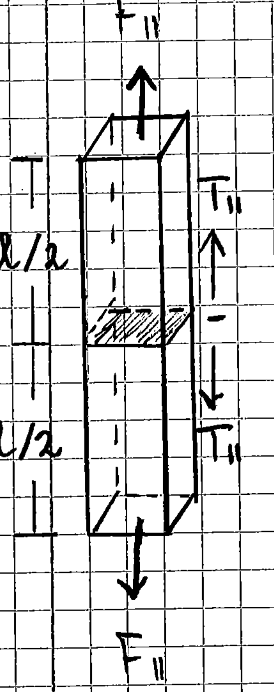


according to (11.2.3)

$$dF_{\parallel} = \left(c^2 \; r_s \; \frac{h}{r^3}\right) d\mu \qquad (11.2.5)$$

that tries to seperate it from the centre S. For a homogeneous rod with cross section $q = b^2$ of square shape (side length = b) we get as total outward pulling force (for $h \ll r$)

$$F_{\parallel} = c^2 \; \frac{r_s}{r^3} \int_0^{\ell/2} q \, \varrho \, h \; dh$$



$$= c^2 \; \frac{r_s}{r^3} \; q \, \varrho \; \frac{\ell^2}{8} \qquad (11.2.6)$$

The tension on the cross-section at the mid-point is

$$\boxed{\begin{aligned} T_{\parallel} &:= \frac{F_{\parallel}}{q} \\ &= \frac{1}{8} \cdot \varrho c^2 \left(\frac{\ell}{r_s}\right)^2 \left(\frac{r_s}{r}\right)^3 \end{aligned}} \qquad (11.2.7)$$

(longitudinal stress)

The transversall stress is $-\frac{1}{2}$ that (pressure).

If we write $r_s = 2GM/c^2$ in (11.2.7) we get

$$T_{11} = \frac{1}{8}\, \rho c^2\, \ell^2 \left(\frac{c^2}{2GM}\right)^2 \left(\frac{r_s}{r}\right)^3$$

$$= \frac{1}{32}\, \rho \ell^2\, \frac{c^6}{G^2 M^2} \left(\frac{r_s}{r}\right)^3$$

$$= \rho \ell^2 \cdot \underbrace{\frac{1}{32} \cdot \frac{c^6}{G^2 (10^4 M_\odot)^2}}_{12.7\ s^{-2}} \cdot \left(\frac{10^4 M_\odot}{M}\right)^2 \left(\frac{r_s}{r}\right)^3 \qquad (11.2.8)$$

Hence

$$T_{11}\,[N \cdot m^{-2}] = 12.7 \times \rho\,[kg \cdot m^{-3}] \times \ell^2\,[m] \times (10^4 M_\odot / M)^2 \times (r_s/r)^3$$

$$= 1.27 \times 10^4 \times \rho\,[g \cdot cm^{-3}] \times \ell^2\,[m] \times (10^4 M_\odot / M)^2 \times (r_s/r)^3$$

(11.2.9)

Problem 3

Part 1 Neutron star

$$\left.\begin{aligned} M &= 1.5\ M_\odot \\ r_s &= 4.5\ \mathrm{km} \\ r &= R = 10\ \mathrm{km} \end{aligned}\right\} \quad (11.3.1)$$

$\Rightarrow (r_s/r) = 0.45$

Rod

$$\left.\begin{aligned} \rho &= 7.85\ \mathrm{g \cdot cm^{-3}} \\ &= 7.85 \cdot 10^{3}\ \mathrm{kg \cdot m^{-3}} \\ l &= 25\ \mathrm{m} \\ \sigma_{max} &= 2.1 \cdot 10^{3}\ \mathrm{N/mm^2} \\ &= 2.1\ 10^{9}\ \mathrm{N \cdot m^{-2}} \end{aligned}\right\} \quad (11.3.2)$$

$$T_{\parallel}[\mathrm{N \cdot m^{-2}}] = 12.7 \times 7.85 \cdot 10^{3} \cdot (25)^2 \cdot \left(\frac{10^4}{1.5}\right)^2 (0.45)^3$$

$$= 2.5\ 10^{14} \gg \sigma_{max} \quad (11.3.3)$$

Since this is quadratic in l, the length l_{max} at which this would equal would be the n'th part of $l = 25$ m, where

$$n = (T_{\parallel}/\sigma_{max})^{1/2} = 346.7 \quad (11.3.4)$$

Hence that piece larger than

$$\frac{25\ m}{346.7} = l_{max} = 7.2\ cm \qquad (11.3.5)$$

reaches the surface of the neutron star. The rod breaks into at least 347 pieces.

Part 2 Human body

$$\left.\begin{array}{l} l = 1.8\ m \\ \rho = 1\ g \cdot cm^{-3} \end{array}\right\} (11.3.6)$$

From (11.2.9)

$$T_{\parallel}[N \cdot m^{-2}] = 1.27 \times 10^{4} \cdot (1.8)^{2} \left(\frac{10^{4} M_{\odot}}{M}\right)^{2} \left(\frac{1 s}{\tau}\right)^{3}$$

$$= 4.1 \times 10^{4} \left(\frac{10^{4} M_{\odot}}{M}\right)^{2} \left(\frac{1 s}{\tau}\right)^{3} \qquad (11.3.7)$$

If a mass of 100 Kg hangs on your feet while your hands hold on to a high bar, a force of 10^{3} N will be distributed over your cross sectional area of $(0.33\ m)^{2} = 0.1\ m^{2}$ giving rise to a tension at the "pain limit" of $10^{4}\ N\ m^{-2}$

$$E = 10^{4}\ N m^{-2} = 10^{4}\ Kg\ m^{-1} s^{-2} = 10^{5}\ g \cdot cm^{-1} s^{-2} \qquad (11.3.8)$$

Hence

$$\frac{I_{11}}{E} = 4.1 \cdot \left(\frac{10^4 M_\odot}{M}\right)^2 \left(\frac{r_s}{r}\right)^3 \qquad (11.3.9)$$

Or for $r = r_s$, i.e. jumping through the horizon,

$$\frac{I_{11}}{E} = 4.1 \left(\frac{M}{10^4 M_\odot}\right)^{-2} \qquad (11.3.10)$$

$\rightarrow$ relatively "painless" for $M \gg 10^4 M_\odot$

For the galactic Black Hole have

$$M = 4.1 \times 10^6 M_\odot \qquad (11.3.11)$$

$$\rightsquigarrow \quad \frac{I_{11}}{E} = 2.4 \; 10^{-5} \qquad (11.3.12)$$

Like a mass of $\cong 10^{-2}$ grammes hanging on your feet; ie, you will feel <u>nothing</u>!

Problem 4

$$\left.\begin{aligned} g = &\left(1-\frac{r_s}{r}\right) c\,dt \otimes c\,dt \\ &- \left(1-\frac{r_s}{r}\right)^{-1} dr \otimes dr \\ &- r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi\right) \end{aligned}\right\} \quad (11.4.1)$$

Any timelike curve $(t(\tau), r(\tau), \theta(\tau), \varphi(\tau))$ satisfies

$$\left(1-\frac{r_s}{r}\right) c^2 \dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-1} \dot{r}^2$$
$$- r^2 \left(\dot{\theta}^2 + \sin^2(\theta)\, \dot{\varphi}^2\right) = c^2$$

or

$$\dot{r}^2 = \left(\frac{r_s}{r} - 1\right)\left[c^2 + r^2\left(\dot{\theta}^2 + \sin^2(\theta)\,\dot{\varphi}^2\right)\right]$$
$$+ \left(\frac{r_s}{r} - 1\right)^2 c^2 \dot{t}^2$$

$$> \left(\frac{r_s}{r} - 1\right) c^2 \qquad (11.4.2)$$

for $r < r_s$

If the curve has entered the region $r < r_s$ so that $\dot{r} < 0$ for r just below, then

$$-\dot{r} > \left(\frac{r_s}{r}-1\right)^2 c$$

$$\Rightarrow \quad -\frac{dr}{\left(\frac{r_s}{r}-1\right)^{1/2}} > c\, d\tau \qquad (11.4.3)$$

for all future τ. Hence the time until $r(\tau)$ reaches $r=0$ is bounded above

$$\int d\tau = \tau < -\frac{1}{c}\int_{r=r_s}^{r=0} \frac{dr'}{\left(\frac{r_s}{r'}-1\right)^{1/2}}$$

$$\text{But} \int_{r_s}^{0} \frac{dr'}{\left(\frac{r_s}{r'}-1\right)^{1/2}} = -r_s\int_0^1 \frac{dx}{\left(\frac{1}{x}-1\right)^{1/2}}$$

$$= -r_s \int_0^1 \frac{x\,dx}{(x-x^2)^{1/2}} \qquad x = \frac{r'}{r_s}$$

$$= -r_s \int_0^1 \frac{x\,dx}{\left[\frac{1}{4}-\left(x-\frac{1}{2}\right)^2\right]^{1/2}}$$

$$= -2r_s \int_0^1 \frac{x\,dx}{\left[1-(2x-1)^2\right]^{1/2}} \qquad 2x-1=y$$

$$= -\frac{r_s}{2}\int_{-1}^{+1} \frac{(y+1)\,dy}{(1-y^2)^{1/2}} = -\frac{r_s}{2}\int_{-1}^{+1} \frac{db}{(1-y^2)^{1/2}}$$

$$= -\frac{r_s}{2}\sin^{-1}(y)\Big|_{-1}^{1} = -\frac{\pi}{2} r_s \qquad (11.4.4)$$

Hence

$$\tau < \frac{\pi}{2}\left(\frac{r_s}{c}\right) \qquad (11.4.5)$$

The upper bound $\frac{\pi}{2}(r_s/c)$ is what we found for the radial geodesic in Lecture 19, Equation (19.54) for $R = r_s$.

For the galactic black hole with

$$\left.\begin{aligned} M &= 4.1 \times 10^{6}\, M_\odot \\ \Rightarrow r_s &= 1.2 \times 10^{10}\ \mathrm{m} \end{aligned}\right\} (11.4.6)$$

have

$$\tau < 63.8\ \mathrm{s} \qquad (11.4.7)$$

So your remaining lifetime is about a minute before you get "spaghettified". Boosting your spaceship and all other measures only shorten your lifetime. Timelike geodesics are the longest among all timelike curves connecting the same points.

Problem 5

$$g = \left(1 - \frac{r_s}{r}\right) c\,dt \otimes c\,dt - \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr - r^2\, d\Omega^2$$

$$= \left(1 - \frac{r_s}{r}\right) \left\{ c\,dt \otimes c\,dt - \underbrace{\frac{dr \otimes dr}{\left(1 - \frac{r_s}{r}\right)^2}} \right\} - r^2\, d^2\Omega. \qquad (11.5.1)$$

define r_* so that this equals $dr_* \otimes dr_*$

$$\Rightarrow \quad dr_* = \frac{dr}{1 - \frac{r_s}{r}} = \frac{r\,dr}{r - r_s} = r_s \frac{x\,dx}{x - 1} \qquad (11.5.2)$$

where $x := r / r_s$

$$\frac{x\,dx}{x-1} = \left(1 + \frac{1}{x-1}\right) dx = d\left(x + \ln|x-1|\right)$$

Hence

$$r_* = r_s \left(\frac{r}{r_s} + \ln \left| \frac{r}{r_s} - 1 \right| \right)$$

$$= r + r_s \ln \left| \frac{r}{r_s} - 1 \right| \qquad (11.5.3)$$

$$g = \left(1 - \frac{r_s}{r}\right) \left(c\,dt \otimes c\,dt - dr_* \otimes dr_* \right)$$

$$- r^2 d^2\Omega \qquad (11.5.4)$$

For

$$\left.\begin{aligned} u &:= ct - r_*(r) \\ v &:= ct + r_*(r) \end{aligned}\right\} \quad (11.5.5)$$

have

$$du \otimes dv = (dct - dr_*) \otimes (dct + dr_*)$$

$$= dct \otimes dct - dr_* \otimes dr_*$$

$$+ dct \wedge dr_* \qquad (11.5.6)$$

Hence, if we write

$$du \vee dv := \frac{1}{2} (du \otimes dv + dv \otimes du) \qquad (11.5.7)$$

for the symmetrized tensor product, then

$$du \vee dv = dct \otimes dct - dr_* \otimes dr_* \qquad (11.5.8)$$

and

$$g = \left(1 - \frac{r_s}{r}\right) du \vee dv - r^2 d\Omega. \qquad (11.5.9)$$

where $r = r(u, v)$.

Note that for the Jacobian for the transformation

$$(ct, r) \to (u(ct, r), v(ct, r))$$

$$\begin{pmatrix} u_{,ct} & u_{,r} \\ v_{,ct} & v_{,r} \end{pmatrix} = \begin{pmatrix} 1 & -r_*' \\ 1 & +r_*' \end{pmatrix} \qquad (11.5.10)$$

the determinant of which is

$$2 r_*' = 2 \left(1 - \frac{r_s}{r}\right)^{-1} \qquad (11.5.11)$$

Radial lightlike geodesics obey

$$\left(1 - \frac{r_s}{r}\right) c^2 \dot{t} = \tilde{E} = \text{const} \qquad (11.5.12)$$

$$\left(1 - \frac{r_s}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 = 0 \qquad (11.5.13)$$

E11.17

We eliminate $\dot{t}$ in the second equation through the first, i.e.

$$\dot{t} = (\tilde{E}/c^2)\left(1-\frac{r_s}{r}\right)^{-1} \qquad (11.5.14)$$

then

$$\left(1-\frac{r_s}{r}\right)c^2\left(\frac{\tilde{E}}{c^2}\right)^2\left(1-\frac{r_s}{r}\right)^{-2} - \left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2 = 0$$

$$\Rightarrow \qquad \dot{r} = \pm \tilde{E}/c \qquad (11.5.15)$$

So r as function of an affine parameter is simply

$$r(\lambda) = \pm(\tilde{E}/c)\lambda + r_0 \qquad (11.5.16)$$

where $r(\lambda=0) = r_0$. In order to get r as function of t we look at the quotient

$$\frac{\dot{r}}{\dot{t}} = \frac{dr}{dt} = \pm c\left(1-\frac{r_s}{r}\right) \qquad (11.5.16)$$

which is equivalent to

$$\left(1-\frac{r_s}{r}\right)^{-1}\frac{dr}{dt} = \pm c \qquad (11.5.17)$$

The left hand side is clearly just dr^*/dt, i.e. (11.5.17) is equivalent to

$$\frac{dr_*}{dt} = \pm c \qquad (11.5.18)$$

$$\left.\begin{array}{l} \Rightarrow \quad r_*(t) = \pm c\,t + r_0^* \\ \text{where } r_0^* = r_*(t=0) \end{array}\right\} \qquad (11.5.19)$$

For an outgoing null ray we take the upper sign and (11.5.19) is equivalent to

$$u = ct - r_*(t) = -r_0^* = k \qquad (11.5.20)$$

Hence the curve $u = k = \text{const.}$ is a radial lightlike autoparallel that intersects the hypersurface $t = 0$ at $r_*(t=0) = -k$, i.e. at

$$r(t=0) = r_*^{-1}(-k) \qquad (11.5.20)$$

Likewise the curve $v = k = \text{const.}$ corresponds to

$$v = ct + r_*(t) = r_0^* = k \qquad (11.5.21)$$

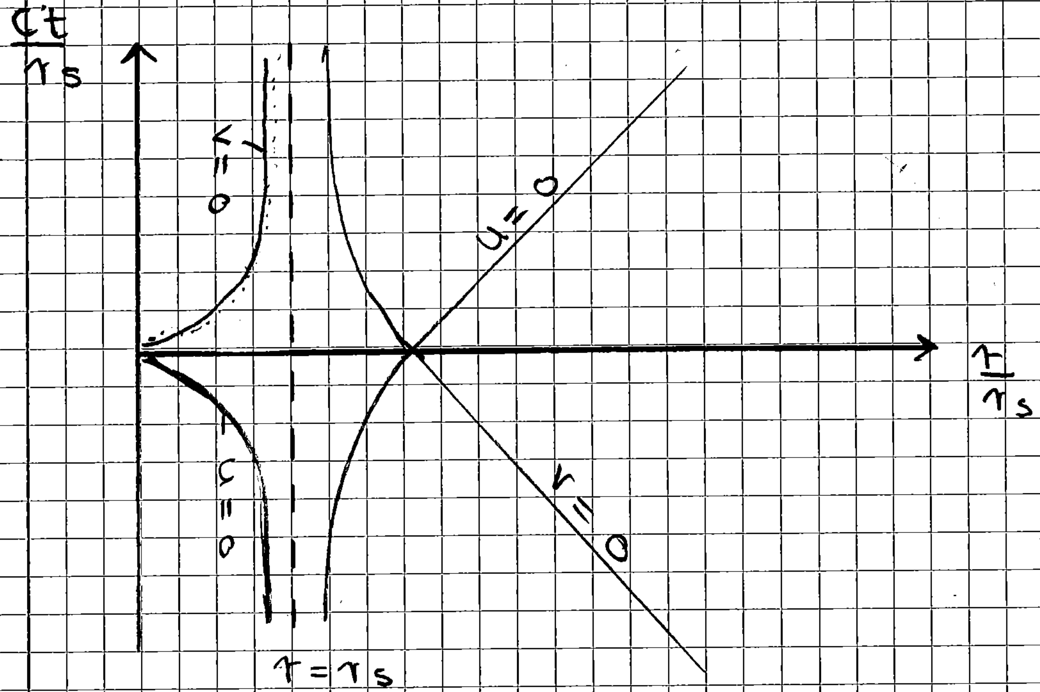
i.e. an ingoing null ray intersecting $t = 0$ at

$$r(t=0) = r_*^{-1}(k). \qquad (11.5.22)$$

E 11.19

The curves $u = 0$ and $v = 0$ obey

$$\frac{ct}{r_s} = \pm \left(\frac{r}{r_s} + \ln \left| \frac{r}{r_s} - 1 \right| \right) \quad (11.5.23)$$



(11.5.24)

Finally we mention the possibility to replace t by either u or v, but to retain r, i.e. to make the coordinate transformations (EFC = Eddington-Finkelstein coordinates).

$$\left.\begin{array}{l} (ct, r, \theta, \varphi) \longrightarrow (u, r, \theta, \varphi) \\ \text{called "outgoing EFC"} \end{array}\right\} \quad (11.5.25)$$

$$\left.\begin{array}{l} (ct, r, \theta, \varphi) \longrightarrow (v, r, \theta, \varphi) \\ \text{called "ingoing" EFC} \end{array}\right\} \quad (11.5.26)$$

E11.20

For the <u>outgoing EFC</u> the metric is

$$g = \left(1 - \frac{r_s}{r}\right)\left(du + r_*' \, dr\right)^{\otimes 2}$$
$$- \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr - r^2 d^2\Omega$$

$$= \left(1 - \frac{r_s}{r}\right) du \otimes du$$
$$+ du \otimes dr + dr \otimes du$$
$$- r^2 d^2\Omega \qquad (11.5.27)$$

and for the <u>ingoing EFC</u>

$$g = \left(1 - \frac{r_s}{r}\right)\left(dv - r_*' \, dr\right)^{\otimes 2}$$
$$- \left(1 - \frac{r_s}{r}\right)^{-1} dr \otimes dr - r^2 d^2\Omega$$

$$= \left(1 - \frac{r_s}{r}\right) dv \otimes dv$$
$$- dv \otimes dr - dr \otimes dv$$
$$- r^2 d^2\Omega \qquad (11.5.28)$$

Clearly the metric coefficients are regular $\forall\, r > 0$. The value of $\det\{g_{\alpha\beta}\}$ is

$$\det\{g_{\alpha\beta}\} = - r^4 \sin^2(\theta) . \qquad (11.5.29)$$

Note that the vector field $\frac{\partial}{\partial r}$ in the outgoing EFC is not identical with $\frac{\partial}{\partial r}$ in ingoing EFC

$$\left(\frac{\partial}{\partial r}\right)_{\substack{u,\theta,\varphi \\ const}} \neq \left(\frac{\partial}{\partial r}\right)_{\substack{v,\theta,\varphi \\ const.}} \qquad (11.5.30)$$

However, both are lightlike:

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0 \qquad (11.5.31)$$

For fixed (θ, φ) it points along the lines of constant u in the first, and constant v in the second case.

Moreover

$$g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right)$$

$$= \left(1 - \frac{r_s}{r}\right) \qquad (11.5.32)$$

For outgoing EFC, any other lightlike vector in Span$\{\partial_u, \partial_r\}$ is

$$X = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial r} \qquad (11.5.33)$$

with $g(X, X) = 0 \iff$

$$\left.\begin{aligned} a^2\left(1 - \frac{r_s}{r}\right) + 2ab = 0 \end{aligned}\right\} \quad (11.5.34)$$

which except for $a = 0$, i.e. $X \sim \frac{\partial}{\partial r}$, has the further solution for $a \neq 0$:

$$b = -\frac{a}{2}\left(1 - \frac{r_s}{r}\right). \qquad (11.5.35)$$

So the lightlike directions are

$$\left.\begin{array}{l} \mathrm{Span}\left\{\frac{\partial}{\partial r}\right\} \\ \mathrm{Span}\left\{\frac{\partial}{\partial u} - \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\frac{\partial}{\partial r}\right\} \end{array}\right\} \qquad (11.5.36)$$

For ingoing EFC we set

$$X = a\frac{\partial}{\partial v} + b\frac{\partial}{\partial r} \qquad (11.5.37)$$

and have

$$\left.\begin{array}{l} g(X,X) = 0 \iff \\ a^2\left(1 - \frac{r_s}{r}\right) - 2ab = 0 \end{array}\right\} \qquad (11.5.38)$$

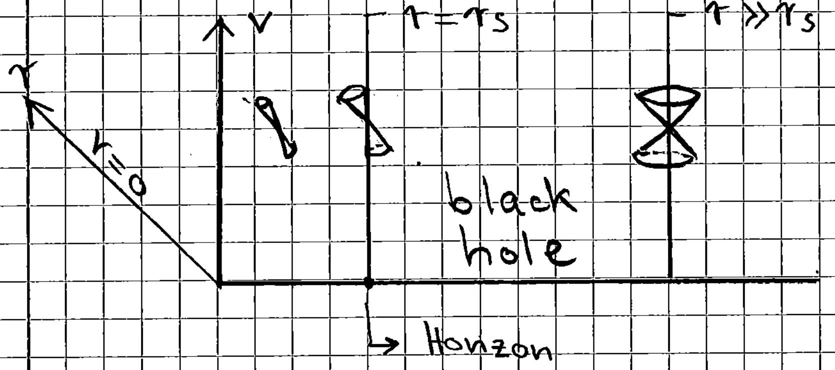
which again except for $a = 0$, i.e. $X \sim \frac{\partial}{\partial r}$ has the further solution for $a \neq 0$:

$$b = \frac{1}{2}\left(1 - \frac{r_s}{r}\right) \qquad (11.5.39)$$

Hence for ingoing EFC the lightlike directions are

$$\left.\begin{array}{l} \mathrm{Span}\left\{ \frac{\partial}{\partial r} \right\} \\ \mathrm{Span}\left\{ \frac{\partial}{\partial v} + \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\frac{\partial}{\partial r} \right\} \end{array}\right\} \quad (11.5.40)$$

Here are the drawings of the (u, r) and (v, r) plane, where we begin with the latter:

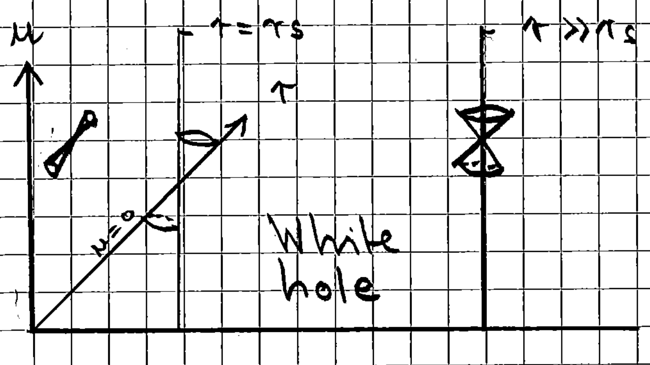


Vertical lines = constant r

-45° - lines = constant v (11.5.41)

Light cone: One side always \ at -45°

other side from / to \ as $r \searrow 0$



Vertical lines = constant v

$+45^\circ$ - lines = constant u (11.5.42)

Light cone: One side always / at $+45^\circ$

other side from \ to / as $r \searrow 0$

Problem 6

Given a metric of the form

$$g = \phi \, c\,dt \otimes c\,dt - \phi^{-1} \, dr \otimes dr - r^2 \left(d\theta \otimes d\theta + \sin(\theta) \, d\varphi \otimes d\varphi \right) \quad (11.6.1)$$

We can introduce a new time coordinate

$$cT(r,t) := ct + f(r) \quad (11.6.2)$$

so that

$$c\,dt = c\,dT - f' \, dr \quad (11.6.3)$$

then (11.6.1) becomes

$$g = \phi \left(c\,dT - f' \, dr \right) \otimes \left(c\,dT - f' \, dr \right) - \phi^{-1} \, dr \otimes dr - r^2 \, d^2\Omega$$

$$= \phi \, c\,dT \otimes c\,dT - \phi f' \left(c\,dT \otimes dr + dr \otimes c\,dT \right) - \left(\phi^{-1} - f'^2 \phi \right) dr \otimes dr - r^2 \, d^2\Omega \quad (11.6.4)$$

For this to be the spatially flat metric on $dT = 0$ slices we ought to have

$$\phi^{-1} - f'^2 \phi = 1$$

or $$f'^2 = \phi^{-2} - \phi^{-1} \tag{11.6.5}$$

Now, in our case

$$\phi = 1 - \frac{r_s}{r} \tag{11.6.6}$$

hence

$$\phi^{-2} - \phi^{-1} = \frac{1}{\left(1 - \frac{r_s}{r}\right)^2} - \frac{1}{1 - \frac{r_s}{r}}$$

$$= \frac{r_s / r}{\left(1 - \frac{r_s}{r}\right)^2} = \frac{r / r_s}{\left(\frac{r}{r_s} - 1\right)^2} \tag{11.6.7}$$

$$\Rightarrow \quad f' = \frac{\sqrt{r / r_s}}{\frac{r}{r_s} - 1} \tag{11.6.8}$$

where we have chosen the positive root.

$$\Rightarrow \quad f(r) = r_s \cdot \left\{ 2 \left(\frac{r}{r_s}\right)^{1/2} + \ln \frac{\sqrt{r/r_s} - 1}{\sqrt{r/r_s} + 1} \right\}$$

(11.6.9)

Inserting (11.6.8) into (11.6.4) and

$$\phi\, f' = \left(1 - \frac{r_s}{r}\right) \frac{\sqrt{r_s / r}}{1 - \frac{r_s}{r}}$$

$$= \sqrt{r_s / r} \qquad (11.6.10)$$

gives

$$g = \left(1 - \frac{r_s}{r}\right) c\, dT \otimes c\, dT$$

$$- \left(\frac{r_s}{r}\right)^{1/2} \left(c\, dT \otimes dr + dr \otimes c\, dT\right)$$

$$- \left(dr \otimes dr + r^2 \left(d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi\right)\right) \qquad (11.6.11)$$

The matrix of coefficients is

$$\begin{pmatrix} \left(1 - \frac{r_s}{r}\right) & -\left(\frac{r_s}{r}\right)^{1/2} & 0 & 0 \\ -\left(\frac{r_s}{r}\right)^{1/2} & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2(\theta) \end{pmatrix} \qquad (11.6.12)$$

the determinant of which is

$$\det\{g_{\alpha\beta}\} = -r^4 \sin^2\theta \qquad (11.6.13)$$

Radial geodesics obey (Lecture 19)

$$\left(1-\frac{r_s}{r}\right) c^2 \dot{t} = \tilde{E} \qquad (11.6.14)$$

$$\left(1-\frac{r_s}{r}\right) c^2 \dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-1} \dot{r}^2 = c^2 \qquad (11.6.15)$$

$$\rightsquigarrow \quad \dot{t} = (\tilde{E}/c^2)\left(1-\frac{r_s}{r}\right)^{-1}$$

$$\left(1-\frac{r_s}{r}\right) c^2 \left(\frac{\tilde{E}}{c^2}\right)^2 \left(1-\frac{r_s}{r}\right)^{-2}$$

$$- \left(1-\frac{r_s}{r}\right)^{-1} \dot{r}^2 = c^2$$

$$\Rightarrow \quad \dot{r}^2 + c^2\left(1-\frac{r_s}{r}\right) = \frac{\tilde{E}^2}{c^2} \qquad (11.6.16)$$

Choose $\tilde{E}$ such that $\dot{r} = 0$ for $r = \infty$, i.e.

$$\tilde{E} = c^2$$

then

$$\dot{r}^2 = c^2 \frac{r_s}{r}, \quad \dot{r} = -c\left(\frac{r_s}{r}\right)^{1/2} \qquad (11.6.17)$$

$$\dot{t} = \left(1-\frac{r_s}{r}\right)^{-1} \qquad (11.6.18)$$

where we have chosen the negative root for $\dot{r}$ to get infalling geodesics.

Hence the four-velocity of the radially infalling geodesics is

$$\left.\begin{aligned} u &= \dot{t}\,\frac{\partial}{\partial t} + \dot{r}\,\frac{\partial}{\partial r} \\ &= \left(1-\frac{r_s}{r}\right)^{-1}\frac{\partial}{\partial t} - c\left(\frac{r_s}{r}\right)^{1/2}\frac{\partial}{\partial r} \\ &= u^\alpha \frac{\partial}{\partial x^\alpha} \end{aligned}\right\} \quad (11.6.19)$$

$$\left.\begin{aligned} u^{ct} &= c\,(1 - r_s/r)^{-1} \\ u^r &= -c\,(r_s/r)^{1/2} \end{aligned}\right\} \quad (11.6.20)$$

The corresponding 1-form components for

$$u^\downarrow = g(u,\cdot) = u_\alpha\, dx^\alpha$$

are

$$\left.\begin{aligned} u_{ct} &= g_{ct\,ct}\, u^{ct} = c \\ u_r &= g_{rr}\, u^r = c\left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{r_s}{r}\right)^{1/2} \end{aligned}\right\} \quad (11.6.21)$$

$$\left.\begin{aligned} u^\downarrow &= c\left(c\,dt + \frac{(r_s/r)^{1/2}}{1-r_s/r}\,dr\right) \\ &= c^2\, dT \end{aligned}\right\} \quad (11.6.22)$$

$\uparrow$ Gullstrand-Painlevé time

If we compare (11.6.20) with the Newtonian case, where

$$\frac{1}{2} m \dot{r}^2 - \frac{GMm}{r} = E = 0 \quad (\dot{r} = 0 \text{ for } r \to \infty)$$

$$\sim \dot{r} = -(2GM/r)^{1/2} = -c\,(r_s/r)^{1/2} \qquad (11.6.23)$$

We see that $r(\tau)$ in the relativistic case is the same function as $r(t)$ in the Newtonian case.

We now express $u = u^{ct} \frac{\partial}{\partial ct} + u^r \frac{\partial}{\partial r}$ in terms of $\partial/\partial T$ and $\partial/\partial r$. Quite generally, for a coordinate change of the form

$$(t, r) \mapsto (\tilde{t}(t,r), \tilde{r}(t,r)) \qquad (11.6.24)$$

we have

$$u = u^{ct} \frac{\partial}{\partial ct} + u^r \frac{\partial}{\partial r}$$

$$= u^{ct} \left(\frac{\partial c\tilde{t}}{\partial ct} \frac{\partial}{\partial c\tilde{t}} + \frac{\partial \tilde{r}}{\partial ct} \frac{\partial}{\partial \tilde{r}} \right)$$

$$+ u^r \left(\frac{\partial c\tilde{t}}{\partial r} \frac{\partial}{\partial c\tilde{t}} + \frac{\partial \tilde{r}}{\partial r} \frac{\partial}{\partial \tilde{r}} \right) \qquad (11.6.25)$$

Here $\tilde{t} = T$, $\tilde{r} = r$

so that $\frac{\partial \tilde{t}}{\partial t} = 1$, $\frac{\partial c\tilde{t}}{\partial r} = f'$, $\qquad (11.6.26)$

hence

$$u = u^{ct} \frac{\partial}{\partial cT} + u^r \left(f' \frac{\partial}{\partial cT} + \frac{\partial}{\partial \tilde{r}} \right)$$

$$= (u^{ct} + f' u^r) \frac{\partial}{\partial cT} + u^r \frac{\partial}{\partial \tilde{r}} \quad (11.6.27)$$

$$\Rightarrow u^{cT} = u^{ct} + f' u^r$$
$$= c(1 - r_s/r)^{-1} + \frac{\sqrt{r_s/r}}{1 - r_s/r} \left(-c \left(\frac{r_s}{r} \right)^{1/2} \right)$$
$$= c \quad (11.6.28)$$

$$u^{\tilde{r}} = -c\,(r_s/r)^{1/2} \quad (11.6.29)$$

or calling $\tilde{r}$ again r:

$$\left.\begin{aligned} u &= \frac{\partial}{\partial T} - c\left(\frac{r_s}{r}\right)^{1/2} \frac{\partial}{\partial r} \\ &= \dot{T} \frac{\partial}{\partial T} - \dot{r} \frac{\partial}{\partial r} \end{aligned}\right\} (11.6.30)$$

This says that $\dot{T} = 1$ along the geodesic, i.e. that T is just the eigentime.

Problem 7

$$\left.\begin{aligned} ct(\lambda, \xi) &= \xi \sinh(\lambda) \\ x(\lambda, \xi) &= \xi \cosh(\lambda) \end{aligned}\right\} \quad (11.7.1)$$

$$\frac{\partial ct}{\partial \lambda} = \xi \cosh(\lambda) \quad (11.7.2a)$$

$$\frac{\partial ct}{\partial \xi} = \sinh(\lambda) \quad (11.7.2b)$$

$$\frac{\partial x}{\partial \lambda} = \xi \sinh(\lambda) \quad (11.7.2c)$$

$$\frac{\partial x}{\partial \xi} = \cosh(\lambda) \quad (11.7.2d)$$

$$\det \begin{pmatrix} \frac{\partial ct}{\partial \lambda} & \frac{\partial ct}{\partial \xi} \\ \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial \xi} \end{pmatrix}$$

$$= \det \begin{pmatrix} \xi \cosh(\lambda) & \sinh(\lambda) \\ \xi \sinh(\lambda) & \cosh(\lambda) \end{pmatrix} = \xi \quad (11.7.3)$$

$$d(ct) = \xi \cosh(\lambda)\, d\lambda + \sinh(\lambda)\, d\xi \quad (11.7.4a)$$

$$dx = \xi \sinh(\lambda)\, d\lambda + \cosh(\lambda)\, d\xi \quad (11.7.4b)$$

$$\eta = d(ct) \otimes d(ct) - dx \otimes dx$$
$$= (\xi \cosh(\lambda) d\lambda + \sinh(\lambda) d\xi)^{\otimes 2}$$
$$- (\xi \sinh(\lambda) d\lambda + \cosh(\lambda) d\xi)^{\otimes 2}$$

$$= \xi^2 \, d\lambda \otimes d\lambda - d\xi \otimes d\xi \qquad (11.7.5)$$

$$K = x \frac{\partial}{\partial(ct)} + ct \frac{\partial}{\partial x} \qquad (11.7.6)$$

Integral curve $\lambda \longmapsto (ct(\lambda), x(\lambda))$
satisfies

$$c\dot{t} = x \quad , \quad \dot{x} = ct \qquad (11.7.7)$$

$$\rightsquigarrow \quad c\ddot{t} = ct \, , \quad \ddot{x} = x \qquad (11.7.8)$$

$$\rightsquigarrow \quad ct(\lambda) = A e^{\lambda} + B e^{-\lambda} \qquad (11.7.9a)$$

$$x(\lambda) = A e^{\lambda} - B e^{-\lambda} \qquad (11.7.9b)$$

Initial condition $ct(\lambda = 0) = 0 \rightarrow B = -A$

$$\rightsquigarrow \left. \begin{aligned} ct(\lambda) &= \xi \sinh(\lambda) \\ x(\lambda) &= \xi \cosh(\lambda) \end{aligned} \right\} \qquad (11.7.10)$$

where $\xi = 2A$.

Proper length along integral curve

$$ds = (c^2 \dot{t}^2 - \dot{x}^2)^{1/2} d\lambda = (x^2 - (ct)^2)^{1/2} d\lambda = \xi \, d\lambda$$
$$(11.7.11)$$

↝ Eigenzeit

$$d\tau = \frac{1}{c}\, ds = \frac{\xi}{c}\, d\lambda \qquad (11.7.12)$$

or $\quad \tau = (\xi/c)\,\lambda$

if $\tau = 0$ for $\lambda = 0$. $\qquad (11.7.13)$

Along the curve have

$$\frac{d\,ct}{d\tau} = \frac{c}{\xi}\frac{d}{d\lambda}\, ct = \frac{c}{\xi}\, c\dot{t}$$

$$= \frac{c}{\xi}\, x(\xi) = c\cosh(\lambda) \qquad (11.7.14)$$

$$\frac{d^2\, ct}{d\tau^2} = \frac{c}{\xi}\frac{d}{d\lambda}\, c\cosh(\lambda) = \frac{c^2}{\xi}\sinh(\lambda) \qquad (11.7.15)$$

$$\frac{dx}{d\tau} = \frac{c}{\xi}\frac{d}{d\lambda}\, x = \frac{c}{\xi}\,\dot{x}$$

$$= \frac{c}{\xi}\, ct(\lambda) = c\sinh(\lambda) \qquad (11.7.16$$

$$\frac{d^2 x}{d\tau^2} = \frac{c}{\xi}\frac{d}{d\lambda}\, c\sinh(\lambda) = \frac{c^2}{\xi}\cosh(\lambda) \qquad (11.7.17)$$

$$\Rightarrow \frac{d^2}{d\tau^2}\left(ct(\tau), x(\tau)\right)$$

$$= \frac{c^2}{\xi}\left(\sinh(\lambda), \cosh(\lambda)\right) \qquad (11.7.18)$$

The Mink.-Norm of that is c^2/ξ. $\qquad (11.7.19)$

Note

$$\frac{\partial}{\partial \lambda} = c\dot{t}\,\frac{\partial}{\partial ct} + \dot{x}\,\frac{\partial}{\partial x}$$

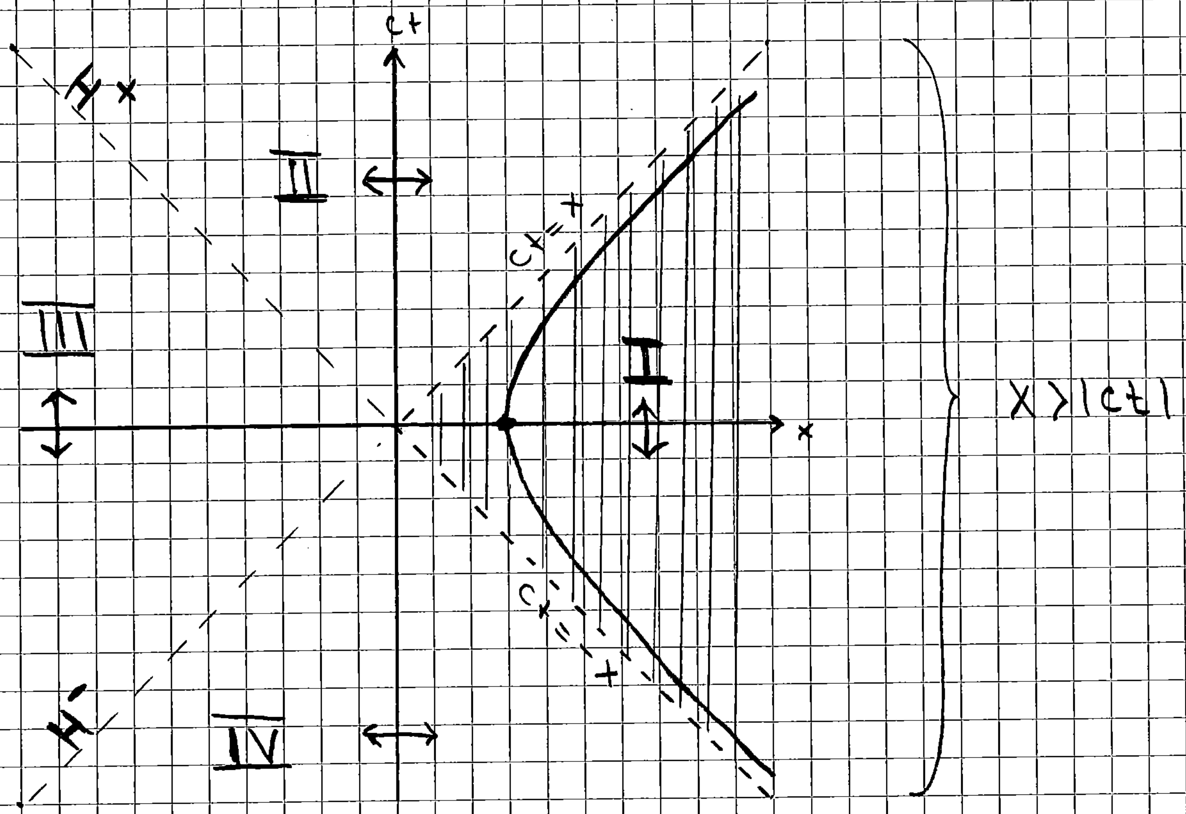
$$= x\,\frac{\partial}{\partial ct} + ct\,\frac{\partial}{\partial x} = K \qquad (11.7.20)$$

$\frac{\partial}{\partial \xi}$ = Vector field with integral lines

$$\xi \mapsto \left(ct(\lambda, \xi),\ x(\lambda, \xi)\right) \qquad (11.7.21)$$

$$\left.\begin{array}{l} \frac{\partial}{\partial \lambda} \perp \frac{\partial}{\partial \xi} \quad \text{from} \\ \\ g = \xi^2\, d\lambda \otimes d\lambda - d\xi \otimes d\xi \end{array}\right\} (11.7.22)$$

E 11.35



The constantly accelerated observer on the hyperbola $\xi = \text{const}$ is confined to region I : $x > |ct|$. He can receive signals from

$$\mathrm{I} \cup \mathrm{IV} = \{(ct, x) : \; ct - x < 0\} \tag{11.7.23}$$

and can send signals to

$$\mathrm{I} \cup \mathrm{II} = \{(ct, x) : \; ct + x > 0\} \tag{11.7.24}$$

He cannot receive any signals from

$$\overline{\mathrm{II} \cup \mathrm{III}} = \{ (ct, x) : ct - x \geq 0 \} \tag{11.7.25}$$

and he cannot send any signals to

$$\overline{\mathrm{III} \cup \mathrm{IV}} = \{ (ct, x) : ct + x \leq 0 \}$$

The causal complement is the region

$$\overline{\mathrm{III}} = (\overline{\mathrm{II} \cup \mathrm{III}}) \cap (\overline{\mathrm{III} \cup \mathrm{IV}}) \tag{11.7.26}$$

of events from which the constantly accelerated observer can neither receive nor to which he can send signals.

The boundary

$$\begin{aligned} \partial (\mathrm{II} - \mathrm{III}) &:= \overline{\mathrm{II} \cup \mathrm{III}} - (\mathrm{II} \cup \mathrm{III})^{\circ} \\ &= \{ (ct, x) : ct = x \} \end{aligned} \tag{11.7.27}$$

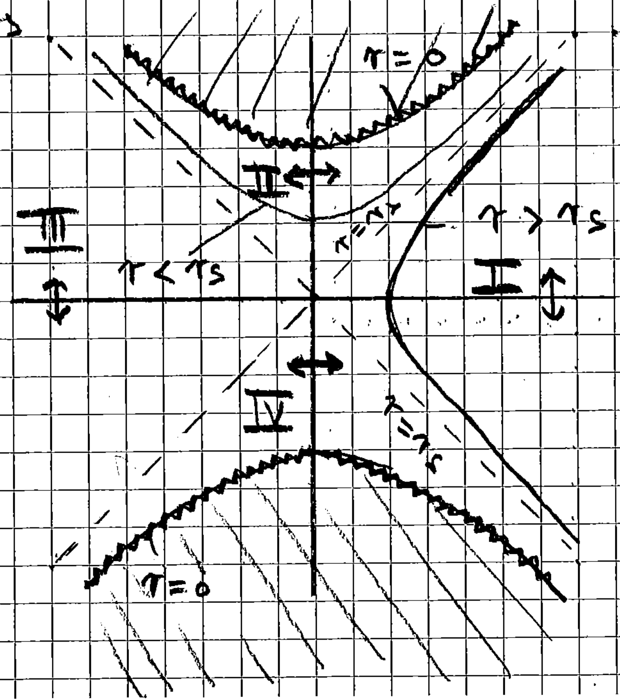
of the region from which the observer cannot receive any signals is called his <u>future event horizon</u> H^+

The boundary

$$\partial(\overline{\text{III}} \cup \overline{\text{IV}}) = (\overline{\overline{\text{III}} \cup \overline{\text{IV}}}) - (\overline{\text{III}} \cup \overline{\text{IV}})^{\circ} \qquad (11.7.28)$$

of the region to which the observer cannot send any signal is called his <u>past event horizon</u> H^-

The Killing vector field $K = \frac{\partial}{\partial \lambda}$ (11.7.20) is somewhat analogous to the Killing vector field $\partial/\partial t$ of staticity in the exterior Schwarzschild geometry. Worldlines of constant ξ here correspond to worldlines of constant r there; here $\xi \searrow 0$ corresponds to $r \searrow r_s$ there. The maximal analytic extension of exterior Schwarzschild is the Kruskal manifold that has a structure as follows.



$\overline{\text{I}}$ = ext. Schw. /1
$\overline{\text{II}}$ = black hole
$\overline{\text{III}}$ = ext Schw. /2
$\overline{\text{IV}}$ = white hole