Sheet 12: Solutions

Problem 1

In Lecture 20 we derived the following exact expression for the binding energy of a homogeneous, spherically symmetric star made out of incompressible matter

$$\frac{-\Delta E_{Bind}}{Mc^2} = \frac{3}{2}\left(\frac{R}{r_s}\right)^{3/2}\left[\sin^{-1}\left(\frac{r_s}{r}\right)^{1/2} - \left(\frac{r_s}{R}\right)^{1/2}\left(1-\frac{r_s}{R}\right)^{1/2}\right] - 1$$

$$= \frac{3}{10}\frac{r_s}{R} + \frac{9}{56}\left(\frac{r_s}{R}\right)^2 + \cdots$$

Here R = area-radius of star

M = gravitational mass of star

$r_s = 2GM/c^2$

The Newtonian value is just the first term – $3r_s/(10R)$ – in the expansion.

For a neutron star

$$R = 10\ km$$

$$M = 1.5\ M_\odot = 3 \cdot 10^{30}\ Kg$$

$$r_s = 4{,}45 \cdot 10^3\ m = 4{,}45\ Km$$

Hence $\frac{R}{r_s} = 2{,}25$

$$\frac{r_s}{R} = 4{,}45 \cdot 10^{-1}$$

$$\Rightarrow \frac{-\Delta E_{Bind}}{Mc^2} \cong 0.178976$$

The Newtonian value is

$$-\left.\frac{\Delta E_{Bind}}{Mc^2}\right|_{Newton} \frac{3}{10}\ 4.45 \cdot 10^{-1}$$

$$= 0.1335$$

So, from the Newtonian point of view, we must increase the Newtonian value by

$$\frac{0.178976 - 0.1335}{0.1335} \times 100 = 34\ \%$$

to get the GR-value. Even if we took the 2nd order approximation we would still underestimate by 8.25 %.

Problem 2

$$\left.\begin{aligned} g = {} & e^{2a}\, c\,dt \otimes c\,dt \\ & - e^{2b}\, dr \otimes dr \\ & - r^2 (d\theta \otimes d\theta + \sin^2(\theta)\, d\varphi \otimes d\varphi) \end{aligned}\right\} \quad (12.2.1)$$

with

$$e^{2a} = e^{-2b} = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \qquad (12.2.2)$$

The curvature components for a metric of the form (12.2.1) are listed in Lecture 17, eqns. (17.28). Since here $\dot{b} = 0$ and $a' = -b'$, the only non-vanishing components are:

$$\begin{aligned} R_{0101} &= -e^{-2b}(-b'' + 2b'^2) \\ &= -\frac{1}{2}\left(e^{-2b}\right)'' \\ &= -\frac{1}{2}\left(-\frac{4m}{r^3} + 6\frac{q^2}{r^4}\right) = \left(\frac{2m}{r^3} - 3\frac{q^2}{r^4}\right) \end{aligned} \qquad (12.2.3)$$

$$\begin{aligned} R_{0202} = R_{0303} &= -R_{1212} = -R_{1313} \\ &= \frac{b'}{r} e^{-2b} = -\frac{1}{2r}\left(e^{-2b}\right)' \\ &= -\frac{1}{2}\left(\frac{2m}{r^3} - \frac{2q^2}{r^4}\right) = \left(-\frac{m}{r^3} + \frac{q^2}{r^4}\right) \end{aligned} \qquad (12.2.4)$$

$$R_{2323} = -\frac{1}{r^2}\left(1 - e^{-2b}\right)$$

$$= -\frac{2m}{r^3} + \frac{q^2}{r^4} \qquad (12.2.5)$$

The Kretschmann-scalar is

$$K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$$

$$= 4\left(R^2_{0101} + R^2_{0202} + R^2_{0303} + R^2_{1212} + R^2_{1313} + R^2_{2323}\right)$$

$$= 4\left(R^2_{0101} + 4R^2_{0202} + R^2_{2323}\right) \qquad (12.2.6)$$

(since $R_{0202} = R_{0303} = -R_{1212} = -R_{1313}$)

$$K = 4\left\{\left(\frac{2m}{r^3} - 3\frac{q^2}{r^4}\right)^2 + 4\left(\frac{m}{r^3} - \frac{q}{r^4}\right)^2 + \left(\frac{2m}{r^3} - \frac{q^2}{r^4}\right)^2\right\}$$

$$= 4\left\{12\left(\frac{m}{r^3}\right)^2 - 24\frac{mq^2}{r^7} + 14\left(\frac{q^2}{r^4}\right)^2\right\}$$

$$= 48\left\{\left(\frac{m}{r^3}\right)^2 - 2\frac{mq^2}{r^7} + \frac{7}{6}\left(\frac{q^2}{r^4}\right)^2\right\}$$

$$= 48\left(\frac{m}{r^3}\right)^2\left\{1 - 2\frac{q}{m}\frac{q}{r} + \frac{7}{6}\left(\frac{q}{m}\right)^2\left(\frac{q}{r}\right)^2\right\}$$

(12.2.7)

The Kretschmann scalar has been computed for Kerr-Newmann by

Richard Conn Henry:

"Kretschmann scalar for a Kerr-Newman Black-Hole"

The Astrophysical Journal, 535: 350-353 May 2020.

The result is:

$$K = \frac{8}{(r^2 + a^2\cos^2\theta)^6}$$

$$\times\Big[6m^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)$$

$$- 12mq^2r(r^4 - 10a^2r^2\cos^2\theta + 5a^4\cos^4\theta) \qquad (12.2.8)$$

$$+ q^4(7r^4 - 34a^2r^2\cos^2\theta + 7a^4\cos^4\theta)\Big]$$

Which for $a = 0$ just gives our expression (12.2.7).

Problem 3

m, q and a are related to the physical quantities mass M, charge Q and angular momentum J as follows:

$$m = \frac{GM}{c^2} \qquad (12.3.1)$$

$$q = \left(\frac{G}{4\pi\varepsilon_0}\right)^{1/2} \frac{1}{c^2} Q$$

$$= \left(\frac{G}{c^2}\right)^{1/2} \left(\frac{\mu_0}{4\pi}\right)^{1/2} Q \qquad (12.3.2)$$

$$a = J / Mc \qquad (12.3.3)$$

Where we used $\mu_0 \varepsilon_0 = c^{-2}$ in the expression for q.

We recall a few constants:

- Fine structure

$$\alpha = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{\hbar c} = \frac{\mu_0}{4\pi} \frac{e^2 c}{\hbar} = (137.036)^{-1} \qquad (12.3.4)$$

- Planck mass and length

$$m_p = (\hbar c / G)^{1/2} = 2.2 \times 10^{-8}\ \mathrm{kg} \qquad (12.3.5)$$

$$l_p = (\hbar G / c^3) = 1.6 \times 10^{-35}\ \mathrm{m} \qquad (12.3.6)$$

- Reduced Compton-wavelength of electron

$$\lambdabar_e = \frac{\hbar}{m_e c} = 3.86 \times 10^{-13}\ \mathrm{m} \quad (12.3.7)$$

- Classical electron radius

$$r_e = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{m_e c^2}$$

$$= \frac{\alpha \hbar c}{m_e c^2} = \lambdabar_e \alpha \quad (12.3.8)$$

The ratio q/m can be written as

$$\frac{q}{m} = \left(\frac{G}{4\pi\varepsilon_0}\right)^{1/2} \frac{1}{c^2} Q \Big/ \frac{GM}{c^2}$$

$$= \left(\frac{Q^2}{4\pi\varepsilon_0}\right)^{1/2} \frac{1}{\sqrt{G}} \frac{1}{M}$$

$$= \left(\frac{e^2}{4\pi\varepsilon_0}\right)^{1/2} \frac{|Q|}{e} \frac{1}{\sqrt{G}} \frac{1}{M}$$

$$= \sqrt{\alpha} \left(\frac{\hbar c}{G}\right)^{1/2} \frac{|Q|}{e} \frac{1}{M}$$

$$= \sqrt{\alpha}\ \frac{|Q|/e}{M/m_p}$$

$$= \left(\frac{m_p}{M}\right) \frac{|Q|}{e} \sqrt{\alpha} \quad (12.3.9)$$

So q/m is $\sqrt{\alpha} = \frac{1}{11.7}$ times the quotient of Q in units of e and M in units of m_p. For the electron $|Q|/e = 1$ and

$$\frac{m_e}{m_p} = \frac{9.1 \times 10^{-31}\ \text{kg}}{2.2 \times 10^{-8}\ \text{kg}} = 4.1 \times 10^{-23} \qquad (12.3.10)$$

Hence

$$\left(\frac{q}{m}\right)_{\text{electron}} \cong 2 \times 10^{21} \quad ! \qquad (12.3.11)$$

<u>Similarly for the ratio a/m</u>

$$\frac{a}{m} = \frac{J}{Mc}\,\frac{c^2}{GM} = \frac{J}{\hbar}\,\frac{\hbar c}{GM^2}$$

$$= \frac{J}{\hbar} \cdot \left(\frac{m_p}{M}\right)^2 \qquad (12.3.12)$$

For the electron $J = \frac{1}{2}\hbar$ and $M = m_e$ with m_e/m_p as above:

$$\left(\frac{a}{m}\right)_{\text{electron}} = \frac{1}{2}\ 2.9 \times 10^{44} \quad !\ ! \qquad (12.3.13)$$

Generally

$$\frac{\sqrt{a^2+q^2}}{m} = \left(\frac{m_p}{M}\right)\left\{\left(\frac{J}{\hbar}\,\frac{m_p}{M}\right)^2 + \left(\frac{|Q|}{e}\right)^2 \alpha\right\}^{1/2} \qquad (12.3.14)$$

Problem 4

Assume a homogeneous sphericall ball of radius R. Its moment of inertia is

$$I = \frac{2}{5} M R^2 \tag{12.4.1}$$

its angular momentum

$$J = I\omega = \frac{2}{5} M R^2 \omega = \frac{2}{5} M R^2 \frac{2\pi}{T} \tag{12.4.2}$$

T = period.

Hence, using $r_s = 2GM/c^2$,

$$\frac{a}{m} = \frac{J}{Mc}\,\frac{c^2}{GM} = \frac{Jc}{GM^2} = \frac{2}{5} M R^2 \omega \,\frac{c}{GM^2}$$

$$= \frac{4}{5}\left(\frac{R\omega}{c}\right) \Big/ \left(\frac{r_s}{R}\right) \tag{12.4.3}$$

SR-effect $\longleftrightarrow$ GR-effect

Which one wins?

E 12.10

	Earth	Jupiter	Sun
$R\,[m]$	6.378×10^{6}	7×10^{7}	6.96×10^{8}
$\omega\,[s^{-1}]$	7.3×10^{-5}	1.76×10^{-4}	$\approx 3 \times 10^{-6}$
$r_s\,[m]$	8.87×10^{-3}	2.82	2.95×10^{3}
$\frac{\omega R}{c}$	1.55×10^{-6}	4.1×10^{-5}	6.96×10^{-6}
r_s / R	$1.4\ 10^{-9}$	3.26×10^{-8}	4.2×10^{-6}
a/m	8.86×10^{2}	1×10^{3}	1.33

(12.5.5)

Only the sun comes close to $a/m = 1$, perhaps even below 1 since the mean angular velocity has likely been over-estimated.

Problem 5

Heisenberg - microscope heuristics:

"In order to localise a system with resolution better than Δl need photon of wavelength

$$\lambda / 2 < l \qquad (12.5.1)$$

and hence

$$\frac{1}{\lambda} > \frac{1}{2l}$$

$$E = h \frac{c}{\lambda} > \frac{hc}{2l} \text{ ."} \qquad (12.5.2)$$

For that to be smaller than $2Mc^2$ (in order not to produce particle-anti-particle pairs) must have

$$2Mc^2 > E > \frac{hc}{2l} \qquad (12.5.3)$$

$$\text{or} \quad l > \frac{h}{4Mc} = \frac{\pi}{2} \frac{\hbar}{Mc}$$

$$= \frac{\pi}{2} \lambda\!\!\!^{-} \qquad (12.5.4)$$

So, roughly and heuristically, the reduced Compton wavelength $\lambda\!\!\!^{-}$ is the distance below which a single "system" cannot be localised without "destroying" the "system" in the sense of leaving the "one-system-sector". In order to speak of the horizon of a single black hole, one should clearly stay below the pair-production threshold, i.e.

$$r_S = \frac{2GM}{c^2} > \frac{\pi}{2} \lambda = \frac{\pi}{2} \frac{\hbar}{Mc} \qquad (12.5.5)$$

$$\Leftrightarrow \qquad M^2 > \frac{\pi}{4} \frac{\hbar c}{G}$$

$$\text{or} \quad M > \frac{\sqrt{\pi}}{2} m_P \cong m_P \qquad (12.5.6)$$

$$m_P = \left(\frac{\hbar c}{G} \right)^{1/2} = 2.2 \times 10^{-8} \text{ Kg}$$

$$= 1.2 \times 10^{19} \text{ GeV}/c^2$$

$\Rightarrow$ Black holes of mass below the Planck mass have Compton wavelength larger than horizon size. In that range, the notion of horizon makes no sense of a classically localisable geometric entity.

Another way to phrase this is to say that a photon that localises a volume l should not produce a black hole with $r_s > l$, i.e. we want

$$r_s = 2\frac{GM}{c^2} < l \qquad (12.5.7)$$

where

$$M = \frac{E}{c^2} = \frac{h\nu}{c^2} = \frac{h}{c\lambda} \qquad (12.5.8)$$

with $\lambda = 2l$ (12.5.9)

(half a wavelength fits into l)

$$2\frac{G}{c^2}\,\frac{h}{c\,2l} < l$$

$$\Leftrightarrow \quad l > \left(\frac{Gh}{c^3}\right)^{1/2} = \sqrt{2\pi}\left(\frac{G\hbar}{c^3}\right)^{1/2}$$

$$= \sqrt{2\pi}\; l_P \qquad (12.5.10)$$

where $l_P = 1.6 \times 10^{-35}$ m

$\Rightarrow$ Resolutions below the Planck length need photons the energy of which will turn the measured volume into a black hole, thus preventing the attempt for this resolution.

Finally, note the following interesting relation between the Schwarzschild radius and the Compton wavelength

$$r_s \cdot \lambda\!\!\!^{-} = 2 \frac{GM}{c^2} \frac{\hbar}{Mc} \qquad (12.5.11)$$

$$= 2 \frac{G\hbar}{c^3} = 2\, l_p^2$$

or

$$l_p = 2^{-1/2} \sqrt{r_s \cdot \lambda\!\!\!^{-}} \qquad (12.5.12)$$

$\Rightarrow$ The Planck length is (up to a factor $2^{-1/2}$) the geometric mean of the Schwarzschild radius and the (reduced) Compton wavelength of any mass.

Problem 6

A spherical shell of rest mass M_0 and (homogeneously distributed) charge Q has total energy

$$E = M_0 c^2 + \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R} - G\frac{M_g^2}{2R} \qquad (12.6.1)$$

Here M_g denotes the gravitating mass. What mass should this be? Here we invoke the following principle:

$\|$ All energies gravitate according to $E = Mc^2$. $\|$

Hence we set

$$M_g = E/c^2 \qquad (12.6.2)$$

and obtain from (12.6.1) a quadratic equation for E, or simply $M := E/c^2$:

$$G\frac{M^2}{2R} + Mc^2 - \left(M_0 c^2 + \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{2R}\right) = 0$$

$$M^2 + 2M\frac{Rc^2}{G} - \frac{2R}{G}\left(M_0 c^2 + \frac{Q^2}{8\pi\varepsilon_0 R}\right) = 0$$

$$\Rightarrow M = -\frac{Rc^2}{G} + \left\{\left(\frac{Rc^2}{G}\right)^2 + \frac{2R}{G}\left(M_0 c^2 + \frac{Q^2}{8\pi\varepsilon_0 R}\right)\right\}^{1/2} \qquad (12.6.3)$$

(only positive root makes sense here)

$$M(R) =$$

$$\frac{Rc^2}{G}\left\{\left[1+\frac{2G}{c^2R}\left(M_0+\frac{Q^2}{8\pi\varepsilon_0 c^2 R}\right)\right]^{1/2}-1\right\}$$

$$= \frac{Rc^2}{G}\left\{\left[1+\frac{r_S}{R}\left(1+\frac{r_Q}{R}\right)\right]^{1/2}-1\right\} \qquad (12.6.4)$$

$$\left.\begin{aligned} \text{where } r_S &= \frac{2GM_0}{c^2} \\ r_Q &= \frac{Q^2}{8\pi\varepsilon_0 M_0 c^2} \end{aligned}\right\} (12.6.5)$$

The limit $R \to 0$ exists and is given by

$$M(0) = \frac{c^2}{G}\left(r_S r_Q\right)^{1/2}$$

$$= \frac{\cancel{c^2}}{G}\left(\frac{2G\cancel{M_0}}{\cancel{c^2}} \; \frac{Q^2}{8\pi\varepsilon_0 \cancel{M_0 c^2}}\right)^{1/2}$$

$$= \frac{|Q|}{\sqrt{4\pi\varepsilon_0}} \; \frac{1}{\sqrt{G}}$$

$$= \frac{|Q|}{e} \sqrt{2\hbar c/G}$$

$$= \sqrt{2}\,\frac{|Q|}{e}\, m_P \qquad (12.6.6)$$

Comparison with (12.3.9) shows that mass and charge are such that

$$q/m = 1 \qquad ! \qquad (12.6.7)$$

This is of course a huge mass; for $|Q| = e$ about 21 orders of magnitude larger than the electron mass (cf. (12.3.10)).

If we expand (12.6.4) for finite R in a power series in G we get

$$\left.\begin{aligned} M(R) &= a - \frac{a^2 x}{2} + \frac{a^3 x^2}{2} - \frac{5\, a^4 x^3}{8} + \cdots \\ a &= M_0 + \frac{Q^2}{8\pi\varepsilon_0 c^2 R} \\ x &= \frac{G}{Rc^2} \end{aligned}\right\} \quad (12.6.8)$$

Which diverges at any order if $R \searrow 0$.