Sheet 2 : Solutions (and more)

Problem 1

A point mass m moves under influence of potential $V = V(r)$ [no dependence on θ and φ]. Hence, by angular-momentum conservation, motion remains in a plane which we take to be the equatorial plane $\theta = \pi/2$. In that plane we use (r, φ) as (polar) coordinates the parametrise the position of the particle. Energy and angular-momentum will be preserved:

$$\frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + V(r) = E \quad \text{(energy)} \qquad (2.1.1)$$

$$m r^2 \dot{\varphi} = L \quad \text{(ang. momentum)} \qquad (2.1.2)$$

Iserting $\dot{\varphi} = L / m r^2$ into (2.1.1) gives

$$\frac{1}{2} m \dot{r}^2 + V_{eff}(r) = E \qquad (2.1.3)$$

where the "effective potential" $V_{eff}(r)$ is given by:

$$V_{eff}(r) = V(r) + \frac{L^2}{2mr^2}$$

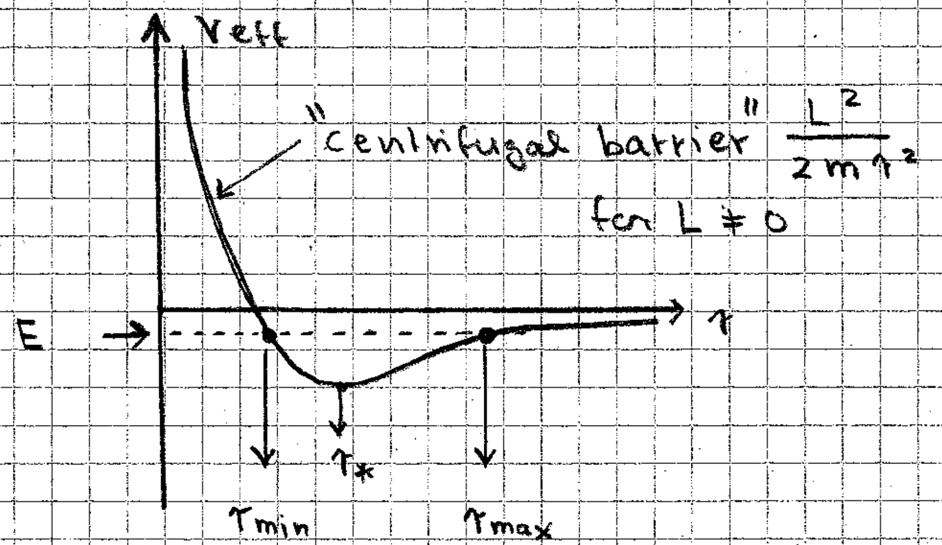
$$= -\frac{\alpha}{r} + \frac{L^2}{2mr^2} \, . \qquad (2.1.4)$$

Here we assumed $V(r)$ to be a "Kepler-Potential"

$$V(r) = -\frac{\alpha}{r} \, , \quad \alpha > 0 \qquad (2.1.5)$$

that leads to an attractive force

$$\vec{F}(\vec{x}) = -\vec{\nabla} V(r) = -\frac{\alpha}{r^3}\vec{x} \, . \qquad (2.1.6)$$



V_{eff} has minimum at $r = r_*$ with
$V'_{eff}(r = r_*) = 0 \iff r_* = L^2 / \alpha m$.

$$\Rightarrow V_{eff}(r = r_*) = -\frac{1}{2}\frac{\alpha^2 m}{L^2} \qquad (2.1.7)$$

From the graph of $V_{eff}(r)$ alone we can say that if E satisfies

$$-\frac{1}{2}\frac{\alpha^2 m}{L^2} < E < 0 \qquad (2.1.8)$$

then the system will oscillate between some minimal radius r_{min} and some maximal radius r_{max}. At $r = r_{min}$ and $r = r_{max}$ we have $\dot{r} = 0$, hence from (2.1.3) and (2.1.4) we get

$$V_{eff}(r = r_{min/max}) = E$$

$$\Leftrightarrow \quad -\frac{\alpha}{r} + \frac{L^2}{2mr^2} = E$$

$$\Leftrightarrow \quad r^2 + \frac{\alpha}{E}r - \frac{L^2}{2mE} = 0$$

$$\Leftrightarrow \quad r_{1/2} = -\frac{\alpha}{2E} \pm \left[\frac{\alpha^2}{4E^2} + \frac{L^2}{2mE}\right]^{1/2}$$

Hence

$$\left.\begin{aligned} r_{min} &= -\frac{\alpha}{2E} - \left[\frac{\alpha^2}{4E^2} + \frac{L^2}{2mE}\right]^{1/2} \\ r_{max} &= -\frac{\alpha}{2E} + \left[\frac{\alpha^2}{4E^2} + \frac{L^2}{2mE}\right]^{1/2} \end{aligned}\right\} (2.1.9)$$

$$\Rightarrow r_{min} + r_{max} = -\frac{\alpha}{E} \qquad (2.1.10)$$

Note that (2.1.8) ensures $\frac{\alpha^2}{4E^2} + \frac{L^2}{2mE} > 0$.

Before we proceed let us look at the special case $L = 0$ first. (2.1.2) implies $\dot{\varphi} = 0$ which turns (2.1.3) into

$$\frac{1}{2} m \dot{r}^2 - \frac{\alpha}{r} = E$$

$$\curvearrowright \quad \dot{r} = \pm \left[\frac{2}{m} \left(E + \frac{\alpha}{r} \right) \right]^{1/2}$$

$$\curvearrowright \quad t - t_0 = \pm \left[\frac{m}{2\alpha} \right]^{1/2} \int_{r_0}^{r} \frac{dr'}{\left[\frac{1}{r'} + \frac{E}{\alpha} \right]^{1/2}} \qquad (2.1.11)$$

If we release the particle at time $t_0 = 0$ with radial velocity $\dot{r}(t=0) = 0$ (free fall), at radius $r = r_0$, so that its energy is $E = -\alpha / r_0$, then the time it takes the particle to reach $r < r_0$ in free fall is

$$t(r) = \left(\frac{m}{2\alpha} \right)^{1/2} \int_{r}^{r_0} \frac{dr'}{\left(\frac{1}{r'} - \frac{1}{r_0} \right)^{1/2}}$$

$$= \left(\frac{m r_0^3}{2\alpha} \right)^{1/2} \int_{r/r_0}^{1} \frac{dx}{\left[\frac{1}{x} - 1 \right]^{1/2}} \qquad (2.1.12)$$

where we set $x := r/r_0$ and consider $x \in [0, 1]$.

Can you do the integral in (2.1.13)?

The integral in (2.1.12) is a messy expression in terms of square-roots and logarithms (do it with Wolfram-Alpha). Since we will encounter this integral again in connection with the proper-time of an observer falling into a black-hole, here is a trick to do it in parametric form: Recall the equation of a <u>Cycloid</u> (if you do not know what a cycloid is, look it up!):

$$\left.\begin{aligned} x &= a(\varphi - \sin\varphi) \\ y &= a(1-\cos\varphi) \end{aligned}\right\} \qquad (2.1.13)$$

$$\rightsquigarrow \quad \frac{dy}{dx} = \frac{\sin\varphi}{1-\cos\varphi} = \frac{(1-\cos^2\varphi)^{1/2}}{1-\cos\varphi}$$

$$= \left(\frac{1+\cos\varphi}{1-\cos\varphi}\right)^{1/2} = \left[\frac{2-y/a}{y/a}\right]^{1/2}$$

$$= \left(\frac{2a}{y} - 1\right)^{1/2} \qquad (2.1.14)$$

This is the differential equation for a cyloide of radius a, that is integrated by (2.1.13). If we compare this with our (2.1.12), which satisfies (for $\dot{r}<0$)

$$\left.\begin{aligned} \frac{dx}{dt} &= -\frac{1}{K}\left(\frac{1}{x}-1\right)^{1/2} \\ \text{where } K &= (m r_0^3/2\lambda)^{1/2} \end{aligned}\right\} \qquad (2.1.15)$$

then comparison of (2.1.14) and (2.1.15) makes it easy to guess (even derive!) the following parametric representation (we call the parameter λ) for our solution:

$$\left.\begin{aligned} r(\lambda) &= \frac{r_0}{2}\,(1-\cos(\lambda)) \\ t(\lambda) &= \frac{1}{2}\left[\frac{m r_0^3}{2\alpha}\right]^{1/2} (\lambda - \sin(\lambda) - \pi) \end{aligned}\right\} \quad (2.1.16)$$

for $\lambda \in [\pi, 2\pi]$, so that $t(\pi) = 0$ and $r(\pi) = r_0$. Setting $\lambda = \pi + \sigma$, $\sigma \in [0, \pi]$, get

$$\boxed{\begin{aligned} r(\sigma) &= \frac{r_0}{2}\,(1+\cos(\sigma)) \\ t(\sigma) &= \frac{1}{2}\left[\frac{m r_0^3}{2\alpha}\right]^{1/2} (\sigma + \sin(\sigma)) \end{aligned}} \quad (2.1.17)$$

Solution to radial infall with $r = r_0$ and $\dot{r} = 0$ at $t = 0$ where $\sigma \in [0, \pi]$ corresponding to $r = r_0$ and $r = 0$

The time for hitting the origin $r = 0$ is

$$T := t(\sigma = \pi) = \frac{\pi}{2} \cdot \left[\frac{m r_0^3}{2\alpha}\right]^{1/2} \quad (2.1.18)$$

We will see that an entirely similar formula holds in GR for the <u>eigentime</u> of a radially infalling observer!

One may check directly that (2.1.17) satisfies our differential equation for radial infall:

$$\dot{r} = \frac{dr/d\sigma}{dt/d\lambda} = \frac{-\frac{r_0}{2}\sin(\sigma)}{\frac{1}{2}[\,..\,]^{1/2}(1+\cos(\sigma))}$$

$$= -\left[\frac{2\alpha}{m r_0}\right]^{1/2}\left[\frac{1-\cos(\sigma)}{1+\cos(\sigma)}\right]^{1/2}$$

$$= -\left[\frac{2\alpha}{m r_0}\right]^{1/2}\left[\frac{2-2r/r_0}{2r/r_0}\right]^{1/2}$$

$$= -\left[\frac{2\alpha}{m r_0}\right]^{1/2}\left[\frac{1}{r/r_0}-1\right]^{1/2}$$

$$= -\left[\frac{2\alpha}{m}\right]^{1/2}\left[\frac{1}{r}-\frac{1}{r_0}\right]^{1/2} \qquad (2.1.19)$$

where we have used $\sin(\sigma) = +[1-\cos^2(\sigma)]^{1/2}$ for $\sigma \in [0,\pi]$ in the 2nd line.

After this rather long digression into the radial infall with $L=0$, we now return to the case $L \neq 0$.

If $L \neq 0$ we have $\dot{\varphi} = L / m r^2 \neq 0$ and we may replace Parameter t by φ, then

$$\frac{d}{dt} = \frac{L}{m r^2} \frac{d}{d\varphi} = \dot{\varphi} \frac{d}{d\varphi} \qquad (2.1.20)$$

and law of energy conservation becomes

$$\frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) = E - V$$

$$\frac{1}{2} m \left(\left(\frac{dr}{d\varphi} \right)^2 + r^2 \right) = \frac{E - V}{\dot{\varphi}^2}$$

$$= \frac{m^2 r^4}{L^2} (E - V)$$

$$\Leftrightarrow \quad \left[\frac{L}{r^2} \frac{dr}{d\varphi} \right]^2 = 2m(E - V) - \frac{L^2}{r^2}$$

Now set $u := \frac{1}{r}$

$$\Leftrightarrow \quad L^2 \left(\frac{du}{d\varphi} \right)^2 = 2m(E - V) - L u^2$$

This equation is valid for all $V = V(r)$. If we specialize for $V(r) = -\alpha / r$, we get (writing $u' = du / d\varphi$)

$$u'^2 = -u^2 + \frac{2m\alpha}{L^2} u + \frac{2mE}{L^2} \qquad (2.1.21)$$

A consequence of (2.1.21) is obtained by differentiation w.r.t. φ:

$$2\, u' u'' = -2\, u\, u' + \frac{2 m \alpha}{L^2}\, u' \qquad (2.1.22)$$

For $u' \neq 0$ (i.e. r not constant, i.e. no circular orbit) this is equivalent to

$$u'' + u = \frac{m\alpha}{L^2} \qquad (2.1.23)$$

The general solution of which is obtained by adding the general solution of homogeneous equation $u'' + u = 0$ (which is $u = K \cos(\varphi - \varphi_0)$) to a particular solution of inhomogeneous equation (e.g. $u = m\alpha / L^2 = \text{const}$):

$$u = \frac{m\alpha}{L^2} + K \cos(\varphi - \varphi_0) \qquad (2.1.24)$$

Since (2.1.23) is only a necessary but not a sufficient condition for (2.1.21), we have to re-insert (2.1.24) into (2.1.21). This leads to

$$K^2 \sin(\varphi-\varphi_0) = -\left[\frac{m\alpha}{L^2} + K\cos(\varphi-\varphi_0)\right]^2$$

$$+ \frac{2m\alpha}{L^2}\left[\frac{m\alpha}{L^2} + K\cos(\varphi-\varphi_0)\right] + \frac{2mE}{L^2}$$

$$= -\left(\frac{m\alpha}{L^2}\right)^2 - \cancel{2\frac{m\alpha}{L^2}K\cos(\varphi-\varphi_0)} - K^2\cos^2(\varphi-\varphi_0)$$

$$+ \frac{2m^2\alpha^2}{L^4} + \cancel{\frac{2m\alpha}{L^2}K\cos(\varphi-\varphi_0)} + \frac{2mE}{L^2}$$

$$\Leftrightarrow \quad K^2 = \frac{m^2\alpha^2}{L^4} + \frac{2mE}{L^2}$$

$$= \left(\frac{m\alpha}{L^2}\right)^2\left[1 + \frac{2EL^2}{m\alpha^2}\right] \qquad (2.1.25)$$

Inserted back into (2.1.24) we obtain the general solution to (2.1.21)

$$u(\varphi) = \frac{1}{p}\left(1 + \varepsilon\cos(\varphi-\varphi_0)\right)$$

or

$$\boxed{\begin{aligned} r(\varphi) &= \frac{p}{1+\varepsilon\cos(\varphi-\varphi_0)} \\ p &:= \frac{L^2}{m\alpha} \\ \varepsilon &:= \left[1 + \frac{2EL^2}{m\alpha^2}\right]^{1/2} \end{aligned}} \qquad (2.1.26)$$

For $E < 1$ there are the Kepler ellipses whose orbital parameters (so-called "Keplerian Parameters") are expressed in terms of E and L. There are

Semi-latus rectum p, excentricity ε, semi-mayor and semi-minor axis a and b, respectively. The latter two are

$$a = \frac{p}{1-\varepsilon^2} = -\frac{\alpha}{2E} \qquad (2.1.27)$$

$$b = (1-\varepsilon^2)^{1/2}\, a = \left(-\frac{2EL^2}{m\alpha^2}\right)^{1/2}\left(-\frac{\alpha}{2E}\right)$$

$$= \left[\frac{L^2}{2m|E|}\right]^{1/2} \qquad (2.1.28)$$

Note that

$$\left.\begin{aligned} r_{min} &= \frac{p}{1+\varepsilon} = a(1-\varepsilon) \\ r_{max} &= \frac{p}{1-\varepsilon} = a(1+\varepsilon) \end{aligned}\right\} \quad (2.1.29)$$

Using $a = -\frac{\alpha}{2E}$ and $\varepsilon = \left(1 + \frac{2EL^2}{m\alpha^2}\right)^{1/2}$ these are seen to be just the expressions (2.1.9).

Conversely, the physical quantities E and L can be obtained from the Keplerian parameters:

$$E = -\frac{\alpha}{2a} \qquad (2.1.30)$$

$$\left.\begin{aligned} L &= (m\alpha p)^{1/2} = [m\alpha a(1-\varepsilon^2)]^{1/2} \\ &= b\left(\frac{m\alpha}{a}\right)^{1/2} \end{aligned}\right\} (2.1.31)$$

since $p = \frac{b^2}{a}$

Circular orbits are only possible if a special relation between E and L holds that is given by $\varepsilon = 0$:

$$E = -\frac{m\alpha^2}{2L^2} \qquad (2.1.32)$$

The radius of the circle is then given by $(r = a = b = p)$

$$r = -\frac{\alpha}{2E} = \frac{L^2}{m\alpha} \qquad (2.1.33)$$

This coincides with the zeros of the right-hand side of (2.1.21):

$$\begin{aligned} u' = 0 \iff u &= \frac{m\alpha}{L^2}\left[1 \pm \underbrace{\left\{1 + \frac{2EL^2}{m\alpha^2}\right\}^{1/2}}_{=\varepsilon = 0}\right] \\ &= \frac{m\alpha}{L^2} \end{aligned} \qquad (2.1.34)$$

Hence the circular orbit indeed satisfies (2.1.21.) That needed to be checked independently, since (2.1.23) and hence (2.1.24) was deduced from (2.1.21) under the condition $u' \neq 0$ (no circular orbit). So we see that the $\varepsilon \to 0$ limit of (2.1.26) is indeed a solution.

We now focus on $\varepsilon \neq 0$ (and $E < 0$). Then we have proper ellipses with periastron (smallest distance from centre of force)

$$r_{min} = \frac{p}{1+\varepsilon} = \text{periastron distance} \qquad (2.1.35)$$

The periastron is periodic for $\varphi = \varphi_0 + 2\pi n$, $n \in \mathbb{N}$. Hence the φ-motion and r-motion have same period. This is a particular feature of the $1/r$-potential. It implies that the spatial orbits are closed. This ceases to be true for general potentials and can be seen as a result of the fact that for $1/r$-potentials there are additional conserved quantities.
Indeed, whereas the symmetries of time-translation and spatial rotations imply

via Noether's theorem the conservation of energy E and angular momentum $\vec{L}$ (together $1+3=4$ conserved quantities), there are three additional conserved quantities in the $V = V(r) = -\alpha/r$ case, that make up the so called "Laplace-Runge-Lenz" vector

$$\vec{A} := \vec{p} \times \vec{L} - m\alpha\,\vec{n} \qquad (2.1.36)$$

where $\vec{n} := \vec{x}/r$, $r = \|\vec{x}\|$, $m =$ mass, and $\vec{p} = m\dot{\vec{x}} =$ momentum. Indeed,

$$\dot{\vec{A}} = \dot{\vec{p}} \times \vec{L} + p \times \cancel{\dot{\vec{L}}}^{\,0} - \frac{m\alpha}{r}\left(\dot{\vec{x}} - \vec{n}(\vec{n}\cdot\dot{\vec{x}})\right)$$

$$= (-\vec{\nabla} V) \times \vec{L} - \frac{m\alpha}{r}\left(\dot{\vec{x}} - \vec{n}(\vec{n}\cdot\dot{\vec{x}})\right)$$

$$= -\frac{\alpha}{r^3}\,\vec{x} \times (\vec{x} \times m\dot{\vec{x}}) - \frac{m\alpha}{r}\left(\dot{\vec{x}} - \vec{n}(\vec{n}\cdot\dot{\vec{x}})\right)$$

$$= -\frac{m\alpha}{r}\left[\vec{n}(\vec{n}\cdot\dot{\vec{x}}) - \dot{\vec{x}} + \dot{\vec{x}} - n(\vec{n}\cdot\dot{\vec{x}})\right]$$

$$= \vec{0} \qquad (2.1.37)$$

Since $\vec{L} = \vec{x} \times \vec{p}$ have $\vec{n}\cdot\vec{L} = 0$ and therefore

$$\vec{A}\cdot\vec{L} = 0 \qquad (2.1.38)$$

which means that $\vec{A}$ lies within the orbital plane. Its magnitude is

$$\vec{A}^2 = \vec{p}^{\,2}\vec{L}^2 - \cancelto{0}{(\vec{p}\cdot\vec{L})^2} - 2m\alpha\,\underbrace{\vec{n}\cdot(\vec{p}\times\vec{L})}_{\underbrace{(\vec{n}\times\vec{p})}_{\frac{1}{r}\vec{L}}\cdot\vec{L}}$$
$$+ m^2\alpha^2$$

$$= 2m\frac{\vec{p}^{\,2}}{2m}\vec{L}^2 - 2m\frac{\alpha}{r}\vec{L}^2 + m^2\alpha^2$$

$$= 2m\vec{L}^2 E + m^2\alpha^2$$

$$= m^2\alpha^2\left(1 + \frac{2EL^2}{m\alpha^2}\right)$$

$$= m^2\alpha^2\varepsilon^2 \qquad (\text{comp.}\,(2.1.26))$$

$$\rightsquigarrow \qquad A := \|\vec{A}\| = m\alpha\varepsilon \qquad (2.1.39)$$

Even the orbit can be easily obtained from knowing the constancy of $\vec{A}$. Indeed, scalar multiplication of $\vec{A}$ with $\vec{X} = r\,\vec{n}$ gives (where $\varphi = \measuredangle(\vec{A},\vec{X})$)

$$\vec{A}\cdot\vec{X} = A\,r\cos\varphi = \vec{X}\cdot(\vec{p}\times\vec{L}) - m\alpha r$$

$$= \vec{L}^2 - m\alpha r$$

$$\rightsquigarrow \quad r\,(A\cos\varphi + m\alpha) = L^2$$

hence

$$r = r(\varphi) = \frac{L^2/m\alpha}{1 + \frac{A}{m\alpha}\cos\varphi} \qquad (2.1.40)$$

which is an ellipse with semi-latus rectum

$$p = \frac{L^2}{m\alpha} \qquad (2.1.41)$$

and eccentricity

$$\varepsilon = \frac{A}{m\alpha}. \qquad (2.1.41)$$

So even without using (2.1.26) to arrive at (2.1.39) can we derive (2.1.39) directly. Since $r = r_{min}$ for $\varphi = 0$ we infer that $\vec{A}$ is a vector in the orbital plane of magnitude $\varepsilon m\alpha$ that points from $\vec{x} = 0$ (the centre of force) to the point of closest approach, i.e. the periastron.

There is another interesting dynamical fact that can easily be derived from the vector $\vec{A}$. Writing (2.1.36) as

$$m\alpha\,\vec{n} = \vec{p}\times\vec{L} - \vec{A} \qquad (2.1.42)$$

and scalar-squaring it, we get using $\vec{p}\cdot\vec{L} = 0$

$$(m\alpha)^2 = p^2L^2 - 2\vec{A}\cdot(\vec{p}\times\vec{L}) + \vec{A}^2 \qquad (2.1.43)$$

Choosing the spatial basis $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ such that

$$\vec{L} = L\vec{e}_z \quad , \quad \vec{A} = A\vec{e}_x \qquad (2.1.44)$$

(as both are time independent), we have $\vec{A}\cdot(\vec{p}\times\vec{L}) = p_y AL$ and, since $\vec{p}\cdot\vec{L}=0$ so that $\vec{p} = p_x\vec{e}_x + p_y\vec{e}_y = m\dot{x}\vec{e}_x + m\dot{y}\vec{e}_y$:

$$\begin{aligned}(m\alpha)^2 &= (p_x^2 + p_y^2)L^2 - 2ALp_y + A^2 \\ &= L^2\left[p_x^2 + \left(p_y - \frac{A}{L}\right)^2\right] \\ &= L^2m^2\left[\dot{x}^2 + \left(\dot{y} - \frac{A}{Lm}\right)^2\right]\end{aligned}$$

$$\Rightarrow \quad \dot{x}^2 + \left(\dot{y} - \frac{A}{Lm}\right)^2 = \left(\frac{\alpha}{L}\right)^2 \qquad (2.1.45)$$

This means that the velocity vector $\vec{v} = (\dot{x}, \dot{y}, 0)$ always lies on a circle with radius (α/L) and centre $(0, A/Lm, 0)$. That is, it is off-centre in y-direction, i.e. the direction $\vec{L}\times\vec{A}$, by an amount A/Lm. In invariant terms we may say that the hodograph (the path the velocity vector describes in time) lies on a circle of radius (α/L) and centre $(\vec{L}\times\vec{A})/(L^2m)$.

After this long digression on the Kepler problem we now – finally! – come to its perturbation, i.e. to the actual task of problem 1. We start again from energy and angular-momentum conservation, which hold since the potential V only depends on r (not on t and not on θ and φ):

$$\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r) = E \qquad (2.1.46)$$

$$m r^2 \dot{\varphi} = L \qquad (2.1.47)$$

Solving (2.1.46) for $\dot{r}$:

$$\dot{r} = \pm \left[\frac{2}{m} (E - V) - r^2 \dot{\varphi}^2 \right]^{1/2} \qquad (2.1.48)$$

and replacing $\dot{\varphi} = \frac{L}{m r^2}$ and $\dot{r} = \dot{\varphi}\, r' = (L/m r^2)\, r'$, where $r' \doteq dr/d\varphi$, we get

$$\frac{L}{r^2} r' = \pm \left[2m (E - V) - \frac{L^2}{r^2} \right]^{1/2} \qquad (2.1.49)$$

In the φ-intervall in which r grows from its minimal value r_{min} to its maximal value r_{max} we choose the positive sign on the r.h.s. of (2.1.49).

Hence, in that intervall

$$d\varphi = \frac{\frac{L}{r^2}\, dr}{\left[2m(E-V) - L^2/r^2\right]^{1/2}} \qquad (2.1.50)$$

Integrating that from r_{min} to r_{max} gives us the angle φ it takes from r_{min} to r_{max}. Multiplying that by 2 gives us the angle φ it takes from r_{min} to r_{max} and back to r_{min}. That angle is 2π in the Keplerian case $V(r) = -\alpha/r$. Here, where $V(r) = -\alpha/r + \Delta V(r)$ is a small perturbation of the Keplerian case, the angle will be close but not quite 2π. We write $2\pi + \Delta\varphi$ for it. Hence

$$2\pi + \Delta\varphi = 2\int_{r_{min}}^{r_{max}} \frac{dr\; L/r^2}{\left[2m(E-V(r)) - L^2/r^2\right]^{1/2}}$$

$$= -2\, \frac{\partial}{\partial L}\bigg|_E \underbrace{\int_{r_{min}}^{r_{max}} dr \left[2m(E-V(r)) - \frac{L^2}{r^2}\right]^{1/2}}_{\text{considered as function of } L \text{ and } E.} \qquad (2.1.51)$$

$\Delta\varphi$ is the excess of azimuth per revolution, also called <u>periastron precession per revolution</u>.

$$\underbrace{\Delta\varphi = -2\,\frac{\partial}{\partial L}\bigg|_E \int_{r_{min}}^{r_{max}} dr \left[2m(E-V(r)) - \frac{L^2}{r^2}\right]^{1/2} - 2\pi}_{\text{"periastron precession per revolution"}} \qquad (2.1.52)$$

If $V(r)$ deviates only slightly from a Keplerian potential

$$\left.\begin{aligned} V(r) &= -\frac{\alpha}{r} + \Delta V(r) \\ |\Delta V(r)| &\ll \frac{\alpha}{r} \end{aligned}\right\} \qquad (2.1.53)$$

where the last inequality is meant to be valid for all r's of our orbits.

Expanding (2.1.52) in terms of ΔV gives in zeroth order for $V = -\alpha/r$ just a cancelation of the -2π (since $\Delta\varphi = 0$ for $V = -\alpha/r$) and then in leading (linear) order

$$\Delta\varphi = 2m\,\frac{\partial}{\partial L}\bigg|_E \int_{r_{min}}^{r_{max}} \frac{dr\,\Delta V(r)}{\left[2m(E+\frac{\alpha}{r}) - L^2/r^2\right]^{1/2}}$$

$$= 2m\,\frac{\partial}{\partial L}\bigg|_E \int_{\varphi}^{\varphi+\pi} \frac{(dr/d\varphi)\,\Delta V(r)\;d\varphi}{\left[2m(E+\frac{\alpha}{r}) - L^2/r^2\right]^{1/2}} \qquad (2.1.54)$$

Where the integrals are taken along the unperturbed solution orbits $r = r_*(\varphi)$ with potential $V = -\alpha/r$. Along them we have

$$r' = \frac{dr}{d\varphi} = \frac{r^2}{L}\left[2m\left(E + \frac{\alpha}{r}\right) - \frac{L^2}{r^2}\right]^{1/2} \tag{2.1.55}$$

so that

$$\Delta\varphi = 2m \left.\frac{\partial}{\partial L}\right|_E \int_{\varphi_0}^{\varphi_0+\pi} \frac{1}{L} r_*^2(\varphi)\,\Delta V(r_*(\varphi))\,d\varphi \tag{2.1.54}$$

For the unperturbed orbit we may take $\varphi_0 = 0$ and use $2\int_0^{\pi} \cdots = \int_0^{2\pi} \cdots$; then

$$\Delta\varphi = m \left.\frac{\partial}{\partial L}\right|_L \left\{ \frac{1}{L} \int_0^{2\pi} d\varphi\; r_*^2(\varphi)\,\Delta V(r_*(\varphi)) \right\} \tag{2.1.55}$$

In using this formula one has to be aware that $r_*(\varphi)$ depends on L, which has to be properly taken into account when the partial differentiation $\frac{\partial}{\partial L}$ is performed.

We consider perturbations of the form

$$\Delta_2 V(r) = \frac{\delta_2}{r^2} \tag{2.1.56}$$

$$\Delta_3 V(r) = \frac{\delta_3}{r^3} \tag{2.1.57}$$

The unperturbed orbit is

$$r_*(\varphi) = \frac{p}{1+\varepsilon\cos\varphi} \qquad (2.1.58)$$

When $p = p(E,L)$ and $\varepsilon = \varepsilon(E,L)$ as given in (2.1.26). For each of these perturbations we get from (2.1.55)

$$\Delta_2\varphi = m\frac{\partial}{\partial L}\left\{\frac{1}{L}\int_0^{2\pi} d\varphi\ \cancel{r_*^2(\varphi)}\ \frac{\delta_2}{\cancel{r_*^2(\varphi)}}\right\}$$

$$= 2\pi m\,\delta_2\left(-\frac{1}{L^2}\right)$$

$$= -2\pi\,\frac{\delta_2/\alpha}{a(1-\varepsilon^2)} \qquad (2.1.59)$$

Where we used (2.1.26) to re express L in terms of orbital parameters:

$$-\frac{1}{L^2} = -\frac{1}{pm\alpha} = -\frac{1}{m\alpha}\,\frac{1}{a(1-\varepsilon^2)} \qquad (2.1.60)$$

For $\Delta_3 V$ we get

$$\Delta_3\varphi = m\frac{\partial}{\partial L}\left\{\frac{1}{L}\int_0^{2\pi} d\varphi\ r_*^2(\varphi)\,\frac{\delta_3}{r_*^3(\varphi)}\right\}$$

$$= m\frac{\partial}{\partial L}\left\{\frac{1}{L}\int_0^{2\pi} d\varphi\ \delta_3\ \frac{1}{p}\,(1+\varepsilon\cos\varphi)\right\}$$

The integral over $\cos\varphi$ vanishes and

$$\int_0^{2\pi} d\varphi \; \delta_3 \frac{1}{p} = \frac{\delta_3}{p} \, 2\pi \qquad (2.1.61)$$

This we have to express in terms of E and L

$$\frac{\delta_3}{p} 2\pi = 2\pi \delta_3 \frac{m\alpha}{L^2} \qquad (2.1.62)$$

Hence

$$\Delta_3 \varphi = m \frac{\partial}{\partial L} \left(\frac{1}{L^3} \; 2\pi \delta_3 \, m\alpha \right)$$

$$= - 6\pi \delta_3 m^2 \alpha \; L^{-4} \qquad (2.1.63)$$

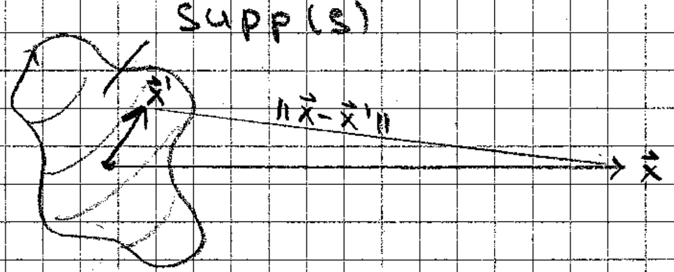
and now, after differentiation $\frac{\partial}{\partial L}$, we may reexpress this in terms of orbital parameters by substituting

$$\frac{1}{L^4} = \frac{1}{(p\, m\alpha)^2} = \frac{1}{m^2\alpha^2} \; \frac{1}{[a(1-\varepsilon^2)]^2}$$

$$\Rightarrow \Delta_3 \varphi = - 6\pi \delta_3 m^2 \alpha \frac{1}{m^2\alpha^2} \; \frac{1}{[a(1-\varepsilon^2)]^2}$$

$$= - 6\pi \; \frac{\delta_3 / \alpha}{[a(1-\varepsilon^2)]^2} \qquad (2.1.64)$$

Problem 2



We expand $\|\vec{x}-\vec{x}'\|^{-1}$ in powers of $\vec{x}'$ at $\vec{x}'=\vec{0}$. We set $\|\vec{x}\|=r$, then

$$\frac{1}{\|\vec{x}-\vec{x}'\|} = \frac{1}{r} + x'^a \frac{\partial}{\partial x'^a}\Big|_{\vec{x}'=0} \frac{1}{\|\vec{x}-\vec{x}'\|} + \frac{1}{2} x'^a x'^b \frac{\partial^2}{\partial x'^a \partial x'^b}\Big|_{\vec{x}'=0} \frac{1}{\|\vec{x}-\vec{x}'\|} + \dots \tag{2.2.1}$$

We have

$$\frac{\partial}{\partial x'^a}\left\{\frac{1}{\|\vec{x}-\vec{x}'\|}\right\} = \frac{x_a - x'_a}{\|\vec{x}-\vec{x}'\|^3}$$

$$\frac{\partial^2}{\partial x'^b \partial x'^a}\left\{\frac{1}{\|\vec{x}-\vec{x}'\|}\right\} = \frac{-\delta_{ab}}{\|\vec{x}-\vec{x}'\|^3} + 3\|\vec{x}-\vec{x}'\|^{-5}(x_b - x'_b)(x_a - x'_a) \tag{2.2.2}$$

Using $n_a := x_a / r$ we can write

$$\frac{\partial}{\partial x'^a}\Big|_{\vec{x}'=\vec{0}} \frac{1}{\|\vec{x}-\vec{x}'\|} = \frac{n_a}{r^2} \qquad (2.2.3)$$

$$\frac{\partial^2}{\partial x'^b \partial x'^a}\Big|_{\vec{x}'=\vec{0}} \frac{1}{\|\vec{x}-\vec{x}'\|} = \frac{-\delta_{ab} + 3 n_a n_b}{r^3} \qquad (2.2.4)$$

Therefore

$$\left.\begin{aligned} \frac{1}{\|\vec{x}-\vec{x}'\|} &= \frac{1}{r} + \frac{x'^a n_a}{r^2} \\ &\quad + \frac{1}{2} \frac{x'^a x'^b (3 n_a n_b - \delta_{ab})}{r^3} \\ &\quad + \text{Terms} \sim r^{-n},\ n \geqslant 4 \end{aligned}\right\} \quad (2.2.5)$$

Insert this into

$$\phi(t,\vec{x}) = -G \int \frac{\rho(t,\vec{x}')}{\|\vec{x}-\vec{x}'\|} \, d^3x' \qquad (2.2.6)$$

We get

$$\left.\begin{aligned} \phi(t,\vec{x}) = -G \Big[&\frac{M}{r} + \frac{n_a D^a}{r^2} \\ &+ \frac{1}{2} \frac{(n_a n_b - \frac{1}{3}\delta_{ab}) Q^{ab}}{r^3} \\ &+ \text{Terms} \sim r^{-n} \quad n \geqslant 4 \Big] \end{aligned}\right\} \quad (2.2.7)$$

Here the possible t-dependence resides in M, D^a and Q^{ab}, the $\vec{x}$-dependence in r and n_a.

$$M(t) := \int d^3x' \, \varrho(t, \vec{x}) \tag{2.2.8}$$

$$D^a(t) := \int d^3x' \, x'^a \, \varrho(t, \vec{x}') \tag{2.2.9}$$

$$Q^{ab}(t) := \int d^3x' \left(3x'^a x'^b - \delta^{ab} r'^2\right) \varrho(t, \vec{x}') \tag{2.2.10}$$

In getting the last (quadrupole) term we used

$$\int d^3x' \, \varrho(t, \vec{x}') \, x'^a x'^b \left(3 n_a n_b - \delta_{ab}\right)$$

$$= \int d^3x' \, \varrho(t, \vec{x}') \left(3x'^a x'^b - r'^2 \delta^{ab}\right) \left(n_a n_b - \tfrac{1}{3}\delta_{ab}\right) \tag{2.2.11}$$

where we could add the $r'^2 \delta^{ab}$ - Term since it is killed by the contraction with the traceless term $(n_a n_b - \frac{1}{3}\delta_{ab})$.

We can always choose the coordinate origin so that $D^a = 0$ at any given time. For suppose $D^a \neq 0$, then under translation $\vec{x} \to \tilde{\vec{x}} := \vec{x} + \vec{a}$ have

$$D^a \to \tilde{D}^a = D^a + \vec{a} M \tag{2.2.12}$$

Choosing $\vec{a} := -D^a/M$ sets $\tilde{D}^a$ to zero, provided $M \neq 0$.

Problem 3

We spezialize ϱ to be invariant under spatial rotations about a fixed axis, say the z (or 3'rd) axis. This means that

$$\varrho \circ R_\varphi = \varrho \qquad \forall \varphi \in [0, 2\pi) \tag{2.3.1}$$

where

$$R_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3.2}$$

For simplicity we assume ϱ to be time independent; otherwise everything we do and say applies at each moment in time.

Consider the definition of the dipol moment

$$\begin{aligned} \vec{D} &= \int \vec{x}\, \varrho(\vec{x})\, d^3x \\ &= \int \vec{x}\, \varrho(R_\varphi^{-1}\vec{x})\, d^3x \qquad (\text{by } (2.3.1)) \\ &= \int R_\varphi \vec{x}'\, \varrho(\vec{x}')\, d^3x' \\ &= R_\varphi \vec{D} \end{aligned} \tag{2.3.3}$$

Here we made a transformation of variables $\vec{x} \to \vec{x}' = R^{-1}_\varphi \vec{x}$ and used the unimodularity of R_φ, i.e. $d^3x' = d^3x$.

Hence $\vec{D}$ is invariant under all R_φ, i.e. under all rotation about z-axis, which implies that $\vec{D}$ points along z-direction: $\vec{D} = D\,\vec{e}_z$.

The very same argument applies to the quadrupole tensor

$$Q^{ab} = \int (3x^a x^b - \delta^{ab} r^2)\ \rho(\vec{x})\, d^3x$$

$$= \int (3x^a x^b - \delta^{ab} r)\ \rho(R_\varphi^{-1}\vec{x})\, d^3x$$

$$= R^a{}_c R^b{}_d \int (3x'^a x'^b - \delta^{ab} r'^2)\ \rho(\vec{x}')\, d^3x'$$

$$= R^a{}_c R^b{}_d\ Q^{cd} \qquad (2.3.4)$$

And this must be true for all $R^a{}_b$-matrices of the form (2.3.2) for all φ. For example, let $\varphi = 90°$, so that

$$R^a{}_b = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.3.5)$$

Then, in matrix-notation,

$$R\ Q\ R^T = Q \qquad (2.3.6)$$

implies

$$\begin{pmatrix} Q^{22} & -Q^{21} & Q^{23} \\ -Q^{12} & Q^{11} & -Q^{13} \\ Q^{32} & -Q^{31} & Q^{33} \end{pmatrix} = \begin{pmatrix} Q^{11} & Q^{12} & Q^{13} \\ Q^{21} & Q^{22} & Q^{23} \\ Q^{31} & Q^{32} & Q^{33} \end{pmatrix}$$

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From this we infer

$$Q^{11} = Q^{22} \qquad (2.3.8)$$

and by symmetry $Q^{ab} = Q^{ba}$ also

$$Q^{12} = -Q^{21} = -Q^{12} \Rightarrow Q^{12} = 0 \qquad (2.3.9)$$

In addition we also get

$$Q^{13} = Q^{23} \quad \text{and} \quad -Q^{13} = Q^{23} \qquad (2.3.10)$$

hence

$$Q^{13} = Q^{23} = Q^{31} = Q^{32} = 0 \qquad (2.3.11)$$

$$\Rightarrow Q = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -2Q \end{pmatrix} \qquad (2.3.12)$$

since $\delta_{ab} Q^{ab} = 0$

or

$$Q^{ab} = Q\,(\delta^{ab} - 3 n^a n^b) \qquad (2.3.13)$$

where $\vec{n} = \vec{e}_z$.

This is the general result: A quadrupole tensor that is invariant under rotations about $\vec{n}$-axis has the form (2.3.13). If we use $\vec{n}$ as the pole-axis of our Spherical polar coord. system we have

$$(n^1, n^2, n^3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \tag{2.3.14}$$

then

$$n_a n_b Q^{ab} = Q(\sin^2\theta - 2\cos^2\theta)$$

$$= Q(1 - 3\cos^2\theta) \tag{2.3.15}$$

The potential (2.2.7) now assumes the form in the frame where $\vec{D} = 0$

$$\phi(\vec{x}) = -G\frac{M}{r}\left[1 + J_2 \frac{R^2}{2r^2}(1 - 3\cos^2\theta)\right] \tag{2.3.16}$$

$$\text{with} \quad J_2 := \frac{Q}{MR^2} \tag{2.3.17}$$

Here the length R has been introduced in order to get J_2-parameter dimensionless. (2.3.16) does not depend on R. The characteristic physical parameters are M and $Q = Q^{11} = Q^{22} = -\frac{1}{2}Q^{33}$.

Problem 4

Let $$\varrho(\vec{x}) = \begin{cases} \varrho_0 = \text{const} & \text{for } \vec{x} \in B(a,b) \\ 0 & \text{otherwise} \end{cases} \tag{2.4.1}$$

where

$$B(a,b) = \left\{ \vec{x} \in \mathbb{R}^3 : \frac{x^2+y^2}{a^2} + \frac{z^2}{b} \leq 1 \right\} \tag{2.4.2}$$

The total mass is

$$M = \varrho_0 \int_{B(a,b)} dx\, dy\, dz = \varrho_0 \int_{\hat{B}(1)} a^2 b\, d\hat{x}\, d\hat{y}\, d\hat{z}$$

where $\hat{\vec{x}} = (\hat{x}, \hat{y}, \hat{z}) := \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{b}\right)$

and $\hat{B}(1) = \{ \hat{\vec{x}} \in \mathbb{R}^3 : \|\hat{\vec{x}}\| = 1 \}$.

$$\rightsquigarrow \quad M = \varrho_0\, a^2 b\, \frac{4\pi}{3} \tag{2.4.3}$$

In order to determine Q we calculate $Q^{33} = -2Q$

$$Q^{33} = \varrho_0 \int_{B(a,b)} (3z^2 - r^2)\, d^3x$$

$$= \varrho_0 \int_{B(a,b)} (2z^2 - x^2 - y^2)\, d^3x$$

$$= \varrho_0 \int_{\hat{B}(1)} \left[2 b^2 \hat{z}^2 - a^2 (\hat{x}^2 + \hat{y}^2) \right] a^2 b \, d^3 \hat{x}$$

$$= \underbrace{\varrho_0 a^2 b}_{\frac{3M}{4\pi}} \int_{\hat{B}(1)} (2 b^2 \hat{r}^2 \cos^2\theta - a^2 \hat{r}^2 \sin^2\theta) \, d^3 \hat{x}$$

φ - Integration

$$= \frac{3M}{4\pi} \, 2\pi \int_0^1 \hat{r}^4 d\hat{r} \int_{-1}^{+1} d(\cos\theta) \left[2 b^2 \cos^2\theta - a^2 \sin^2\theta \right]$$

$$= \frac{3M}{10} \int_{-1}^{+1} d(\cos\theta) \left[(2b^2 + a^2) \cos^2\theta - a^2 \right]$$

$$= \frac{3M}{10} \left[\frac{2}{3} (2b^2 + a^2) - 2a^2 \right]$$

$$= \frac{2}{5} M (b^2 - a^2) = Q^{33} \qquad (2.4.4)$$

Hence

$$Q = -\frac{1}{2} Q^{33} = \frac{M}{5} (a^2 - b^2)$$

$$\rightsquigarrow \frac{Q}{M} = \frac{a^2}{5} \left(1 - \frac{b^2}{a^2}\right) = \frac{a^2}{5} \varepsilon^2 \qquad (2.4.5)$$

since $\frac{b}{a} = (1 - \varepsilon^2)^{1/2}$.

If in the expression for J_2 we take for the characteristic length R the semi-major axis a, then

$$J_2 = \frac{Q}{MR^2} = \frac{\varepsilon^2}{5} \qquad (2.4.6)$$

or in terms of $f := 1 - \frac{b}{a}$, where $b/a = 1-f$

$$\varepsilon^2 = 1 - \frac{b^2}{a^2} = 1-(1-f)^2 = 2f - f^2$$

$$J_2 = \frac{2}{5} f(1-f/2) \cong \frac{2}{5} f \qquad (2.4.7)$$

The dimensionless quantity f is called the "ellipticity" or "flattening" of the ellipsoid. If $b < a$ the ellipsoid is "oblate" and $f > 0$ and $J_2 > 0$. If $b > a$ the ellipsoid is "prolate" and $f < 0$ and $J_2 < 0$. The Sun has the shape of an oblate ellipsoid with flattening being reported at

$$f = \begin{cases} 9 \cdot 10^{-6} & \text{(Wikipedia)} \\ 5 \cdot 10^{-5} & \text{(Smartconv.)} \end{cases} \qquad (2.4.8)$$

In fact, the sun's oblate ellipticity is known to vary, even over short times.

See, e.g.,

R.H. Dicke, J.R. Kuhn, K.G. Libbrecht
"The variable Oblateness of the Sun:
Measurements of 1984"
The Astrophysical Journal, Volume 311,
pages 1025-1030, year 1986.

They define "Oblateness" by

$$\frac{\Delta r}{r} = \frac{r_{eq} - r_p}{r} \qquad (2.4.9)$$

where r_{eq} = equatorial radius = our a,
and r_p = polar radius = our b
and r = some mean radius like
$\frac{1}{2}(r_{eq} + r_p) = \frac{1}{2}(a+b)$. Hence their $\frac{\Delta r}{r}$ is

$$\frac{\Delta r}{r} = \frac{a-b}{\frac{1}{2}(a+b)} = 2\,\frac{1-\frac{b}{a}}{1+\frac{b}{a}}$$

$$= 2\,\frac{\left(1-\frac{b}{a}\right)}{2+\left(\frac{b}{a}-1\right)} = \frac{1-\frac{b}{a}}{1-\frac{1}{2}\left(1-\frac{b}{a}\right)}$$

$$= \left(1-\frac{b}{a}\right)\left[1+\frac{1}{2}\left(1-\frac{b}{a}\right)\right] + O\left[\left(1-\frac{b}{a}\right)^3\right]$$

$$= f\left(1+\frac{1}{2}f^2\right) + O(f^2)$$

$$\cong f + O(f^2) \qquad (2.4.10)$$

We just need the linear order.

Dicke et. al. report values that correspond to values of f in the range

$$5,8 \cdot 10^{-6} \leq f \leq 4,3 \cdot 10^{-5} \qquad (2.4.11)$$

In 1967 an even higher value had been reported in

R. H. Dicke and H. M. Goldenberg
"Solar Oblateness and General Relativity"
Physical Review Letters, Volume 18,
Number 9, pages 313-316, year 1967.

They reported

$$f = (5,0 \pm 0,7) \times 10^{-5} \qquad (2.4.12)$$

We will see in the next problem why there higher values were once thought to be dangerous for GR.

Problem 5

We consider the motion of a test mass in the potential (2.3.16):

$$\phi(r,\theta) = -G\frac{M}{r}\left[1 + J_2\frac{R^2}{2r^2}\left(1-3\cos^2\theta\right)\right] \quad (2.5.1)$$

It is invariant under rotations about the z-axis (it does not depend on φ). In addition, it has an additional discrete symmetry under reflections at the $z=0$ plane, i.e. the xy-plane. In polar coordinates (r,θ,φ) this reflection is

$$R: (r,\theta,\varphi) \mapsto (r',\theta',\varphi') = (r, \pi-\theta, \varphi). \quad (2.5.2)$$

This implies the following: Let $\vec{x}(t)$ be a solution to

$$\ddot{\vec{x}}(t) = -\vec{\nabla}\,\phi(\vec{x}(t)) \quad (2.5.3)$$

then $\vec{x}'(t) := R\,\vec{x}(t)$ is again a solution to the same equation.

Now consider a solution $\vec{x}(t)$ whose initial condition is such that $z(0)=0$ and $\dot{z}(0)=0$, i.e. at $t=0$ the orbit is in, and tangential to, the xy-plane.

Suppose the orbit $\vec{x}(t)$ does not run entirely within the xy-plane. Then the reflected orbit $\vec{x}'(t) = R\vec{x}(t)$ is different from $\vec{x}(t)$. But it has the <u>same</u> initial conditions (the xy-plane is pointwise fixed by R). This contradicts the Theorem of Picard-Lindelöf on the uniqueness of solutions to ordinary differential equations (provided some local Lipschitz condition is satisfied). Therefore $\vec{x}(t)$ cannot leave the xy-plane if initially it starts within, and tangential to, that plane.

For motions within the equatorial plane (i.e. the xy-plane) we can just take $V|_{\theta=\pi/2}$:

$$\left.\begin{aligned} V\Big|_{\theta=\frac{\pi}{2}}(r) &= -G\frac{M}{r}\left[1 + J_2\frac{R^2}{2r^2}\right] \\ &= -\frac{\alpha}{r} + \frac{\delta_3}{r^3} \end{aligned}\right\} \quad (2.5.4)$$

$$\left.\begin{aligned} \alpha &= GM, \\ \delta_3 &= -GM\tfrac{1}{2}J_2R^2 = -\frac{GQ}{2} \end{aligned}\right\} \quad (2.5.5)$$

Using the result (2.1.64) of Problem 1, we infer a contribution of the quadrupole moment to the periastron shift per revolution of

$$(\Delta\varphi)_{quad} = -6\pi \frac{\delta 3/\alpha}{[a(1-\varepsilon^2)]^2}$$

$$= 3\pi J_2 \left[\frac{R}{a(1-\varepsilon^2)}\right]^2 \qquad (2.5.6)$$

This we wish to apply to the orbit of Mercury. Its orbital parameters are

a = semi-major axis of Mercury orbit

$= 57,909,050$ km $\cong 58 \cdot 10^6$ km

ε = eccentricity of Mercury orbit

$= 0.205630 \cong 0.206$

For R we take the sun's equatorial radius

R = equatorial radius of sun

$= 6.96 \cdot 10^5$ km

Hence

$$\left[\frac{R}{a(1-\varepsilon^2)}\right] = 1.57 \cdot 10^{-4}$$

$$\Rightarrow (\Delta\varphi)_{Quad} = 2\pi \cdot J_2 \cdot \frac{3}{2}\left[\frac{R}{a(1-\varepsilon^2)}\right]^2$$

$$= 2\pi \cdot J_2 \cdot 2.36 \cdot 10^{-4} \qquad (2.5.7)$$

This is the perihel-shift in arc measure per revolution. In order to convert this into arc seconds per century (100 years), we have to multiply this by

$$\underbrace{\frac{360}{2\pi}}_{\text{converts arc measure to degrees}} \times \underbrace{3600}_{\text{converts degrees to arc second}} \times \underbrace{\frac{100\,y}{T_M}}_{\text{converts per Mercurys orbital period }(T_M)\text{ to per century}} = X \qquad (2.5.8)$$

Recall: an arc second, denoted by $1''$, is the $\frac{1}{60}$ th part of an arc minute, which in turn is the $\frac{1}{60}$ th part of a single degree. Hence $1^\circ = 3600''$.

Further

$$T_M = \text{Sidereal period of Mercury}$$
$$= 87.9691 \text{ d (days)}$$
$$= 0.240846 \text{ y (years)} \qquad (2.5.9)$$

$$\Rightarrow \frac{100\,y}{T_M} = 415.20 \qquad (2.5.10)$$

Multiplication of (2.5.7) by the factor X from (2.5.8) yields $(\Delta\varphi)_{Quad}$ in units of (arc-seconds)/100 y :

$$(\Delta\varphi)^{Mercury}_{Quad} = [\,]_2 \cdot 2.36 \cdot 10^{-4}$$
$$\times (360)^2 \cdot 10 \cdot 415.20\,]'' / 100\,y$$

$$= [\,]_2 \cdot 1.27 \cdot 10^5\,]'' / 100\,y \qquad (2.5.11)$$

In order for this to account for the "anomalous" part of Mercury's perihelion shift (i.e. that part which cannot be accounted for by the perturbation of the other planets, most importantly Venus, Earth, and Jupiter), which is 43''/100 y, We would need

$$J_2^* = 3.4 \cdot 10^{-4} \qquad (2.5.12)$$

If we use a homogeneous mass distribution model for the sun, we have (2.4.7), i.e. $J_2 = (2/5) f$ and hence

$$f^* = \frac{5}{2} J_2^* = 8.5 \cdot 10^{-4} \qquad (2.5.13)$$

This is well above the realistic values.

<u>But</u>: If a value of $f = 5 \cdot 10^{-5}$ as reported by Dicke and Goldenberg in 1967 (2.4.12) had been confirmed, it would have accounted for $5 \cdot 10^{-5} / 8.5 \cdot 10^{-4} \cong 0.06$, i.e. 6% of the anomalous perihel shift, which is measured much better than the 1% level. Therefore it would have turned Einsteins prediction into a quite severe problem for GR.
For this reason Dicke and Goldenberg proposed a scalar-tensor theory, the scalar being responsible for a smaller predicted value.

The problem of an accurate and reliable determination of the sun's quadrupole moment J_2 was settled only quite recently. The problem was that no reliable estimates for the sun's interior mass distribution existed. That changed with the advent of "helioseismology".

The following paper contains the modern value for J_2, based on that method:

Frank P. Pijpers
"Helioseismic determination of the solar gravitational quadrupole moment".
Monthly Notices of the Royal Astronomical Society, Volume 297, pages L76-L80, year 1998.

They obtained

$$J_2 = (2.18 \pm 0.06) \cdot 10^{-7} \qquad (2.5.14)$$

and that value only leads to

$$(\Delta\varphi)^{\text{Mercury}}_{\text{Quad}} = (2.77 \cdot 10^{-2})'' / 100\,y$$

$$= 6.4 \cdot 10^{-4} \, (\Delta\varphi)^{\text{Mercury}}_{\text{Measured}}$$