Sheet 3: Solutions

Problem 1

$$T^{\mu\nu} = \rho u^\mu u^\nu + (-\eta^{\mu\nu} + u^\mu u^\nu/c^2)\, p \tag{3.1.1}$$

η = Mink. Metric.

An observer comoving with a fluid moves with 4-velocity u. Using the frame

$$\{e_0 = u/c,\ e_1, e_2, e_3\} \tag{3.1.2}$$

where

$$\eta(e_\mu, e_\nu) = \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$

We get, using that

$$P^\mu{}_\nu = \delta^\mu{}_\nu - \frac{u^\mu u_\nu}{c^2} \tag{3.1.3}$$

are the components of the projection map into the orthogonal complement of u

$$T^{00} = \rho c^2 \tag{3.1.4}$$

$$T^{0a} = 0 \tag{3.1.5}$$

$$T^{ab} = -\eta^{ab} p = \mathrm{diag}(p,p,p) = \delta^{ab} p \tag{3.1.6}$$

under spatial rotations

$$e_a \to e'_a = R^b{}_a\, e_b \qquad (3.1.7)$$

with $\{R^a{}_b\} \in SO(3)$, i.e.

$$R^a{}_c R^b{}_d\, \delta^{cd} = \delta^{ab} \qquad (3.1.8)$$

$T^{ab} = \delta^{ab} p$ is invariant. (3.1.9)

Shear forces correspond to off diagonal elements in the spatial part of $T^{\mu\nu}$, ie to non-zero entries T^{ab}, for $a \neq b$ which are zero here, and heat conduction corresponds to non-zero elements T^{0a}. Note that these criteria only apply to the components of T with respect to the local rest frame of the fluid, with spatial vectors e_1, e_2, e_3 being mutually orthogonal.

Problem 2

$$T^{\mu\nu} = \varrho\, u^\mu u^\nu + (-\eta^{\mu\nu} + u^\mu u^\nu / c^2)\, p$$
$$= (\varrho + p/c^2)\, u^\mu u^\nu - \eta^{\mu\nu} p \qquad (3.2.1)$$

$$\nabla_\mu T^{\mu\nu} = u^\nu\, \nabla_\mu \big((\varrho + p/c^2)\, u^\mu \big)$$
$$+ (\varrho + p/c^2) \underbrace{u^\mu \nabla_\mu u^\nu}_{\dot{u}^\nu} - \nabla^\nu p \qquad (3.2.2)$$

This we decompose parallel and perpendicular to u. The first term is already $\parallel u$; the second term is already $\perp u$ since $\dot{u} \perp u$. This follows from $\eta(u,u) = c^2 \rightarrow$
$u^\nu \nabla_\nu (\eta(u,u)) = 0 = 2\eta(u,\dot{u})$.
The third term, $\nabla^\nu p$, has component

$$\frac{u^\nu u_\mu}{c^2} \nabla^\mu p = \frac{1}{c^2} u^\nu \dot{p} \qquad (3.2.3)$$

parallel to u and component

$$\nabla^\nu p - \frac{1}{c^2} u^\nu \dot{p} = \left(\eta^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p \qquad (3.2.4)$$

perpendicular to u.

Hence $\nabla_\mu T^{\mu\nu} = 0$ if and only if the component $\parallel$ and $\perp$ to u vanish individually:

$$\nabla_\mu \left[(\rho + p/c^2) u^\mu \right] - \frac{1}{c^2} \dot{p} = 0 \qquad (3.2.5)$$

and

$$\dot{u}^\nu (\rho + p/c^2) + \left(-\eta^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p = 0 \qquad (3.2.6)$$

The first condition, (3.2.5), can be written as

$$\nabla_\mu (\rho u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu + \cancel{\frac{1}{c^2} u^\mu \nabla_\mu p} - \cancel{\frac{1}{c^2} \dot{p}}$$

$$= \nabla_\mu (\rho u^\mu) + \frac{p}{c^2} \nabla_\mu u^\mu = 0 \qquad (3.2.7)$$

(3.2.6) and (3.2.7) are just eqns. (3a, b) on the problem sheet. Note that

$$\left(-\eta^{\mu\nu} + \frac{u^\mu u^\nu}{c^2}\right) \nabla_\mu p \qquad (3.2.8)$$

is just the ν'th component of the η-orthogonal projection of $\nabla^\nu p$ perpendicular to u (i.e. in the fluids local rest frame). (3.2.6) is the relativistic Euler equation, (3.2.7) replaces rest-mass conservation.

The non-conservation of rest mass is due to the fact that if the fluid is compressible, any work ΔE that is done on the fluid elements by pressure induced compression will enhance its rest mass by $\Delta E/c^2$. More precisely, from (3.2.7) have

$$\nabla(\rho u^\mu) = -\frac{p}{c^2}(\nabla_\mu u^\mu) \neq 0 \Leftrightarrow p \neq 0 \text{ and } \nabla_\mu u^\mu \neq 0.$$

What does $\nabla_\mu u^\mu \neq 0$ imply? To see this, express the 4-velocity u^μ in terms of the 3-velocity $\vec{v}$ as usual: If $\gamma := (1 - v^2/c^2)^{-1/2}$, then
$u = \gamma(c, \vec{v})$ and hence

$$\nabla_\mu u^\mu = \frac{\partial}{\partial x^0}(\gamma c) + \vec{\nabla}\cdot(\gamma\vec{v})$$

$$= \left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\gamma + \gamma\,\vec{\nabla}\vec{v}$$

$$= \gamma^3 \frac{\vec{v}}{c^2}\cdot\underbrace{\left(\frac{\partial}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right)}_{=:\,\vec{a}\ =\ \text{comoving acceleration}} + \gamma\,\vec{\nabla}\cdot\vec{v}$$

$$= \gamma^3 \frac{\vec{v}\cdot\vec{a}}{c^2} + \gamma\,\vec{\nabla}\cdot\vec{v} \qquad (3.2.9)$$

In the rest frame of the fluid, where $\vec{v} = \vec{0}$, have $\partial_\mu u^\mu = \vec{\nabla}\cdot\vec{v}$ hence $\partial_\mu u^\mu \neq 0$ is equivalent to the "rest-frame compressibility". The other factor $\sim c^{-2}$ just comes from the convective derivative, $(\partial/\partial t + \vec{v}\cdot\vec{\nabla})$, of γ, which had to appear since the notion of "comoving volume" occupied by fluid elements depends on γ.

Problem 3

Note: An equation between distributions is valid iff both sides give the same result if applied to test-functions.

Let the space of test-functions be the set of C^∞-functions $f: M \to \mathbb{R}$ of compact support, i.e.

$$C_0^\infty(M, \mathbb{R}) \qquad (3.3.1)$$

Now

$$\int d^4x \, f(x) \, \nabla_\mu j^\mu(x)$$

$$\equiv -\int d^4x \, (\nabla_\mu f)(x) \, j^\mu(x)$$

$$= -e \int d^4x \, (\nabla_\mu f)(x) \int d\tau \, \delta^{(4)}(x - z(\tau)) \, \dot{z}^\mu(\tau)$$

$$= -e \int d\tau \, \dot{z}^\mu(\tau) \, \nabla_\mu f(z(\tau))$$

$$= -e \int d\tau \, \frac{d}{d\tau} f(z(\tau)) = -e \, f(z(\tau)) \Big|_{-\infty}^{+\infty}$$

$$= 0 \qquad (3.3.2)$$

since $z(\tau)$ is timelike and hence leaves any compact domain at both ends.

Further

$$\int d^4x \; f(x) \; \nabla_\mu T^{\mu\nu}(x)$$

$$\equiv - \int d^4x \, (\nabla_\mu f)(x) \; T^{\mu\nu}(x)$$

$$= - m \int d^4x \; (\nabla_\mu f)(x) \int d\tau \; \delta^{(4)}(x - z(\tau)) \; \dot{z}^\mu(\tau) \, \dot{z}^\nu(\tau)$$

$$= - m \int d\tau \; \dot{z}^\mu(\tau) \; \nabla_\mu f(z(\tau)) \; \dot{z}^\nu(\tau)$$

$$= - m \int d\tau \left(\frac{d}{d\tau} f(z(\tau)) \right) \dot{z}^\nu(\tau)$$

$$= \underbrace{- m \, f(z(\tau)) \, \dot{z}^\nu(\tau) \Big|_{-\infty}^{+\infty}}_{= 0 \text{ since } f \text{ has comp. support}} + m \int d\tau \; f(z(\tau)) \, \ddot{z}^\nu(\tau)$$

$$= m \int d\tau \; f(z(\tau)) \; \ddot{z}^\nu(\tau) \qquad (3.3.3)$$

If this is to vanish for all $f \in C_0^\infty(M, \mathbb{R})$ we must have

$$\ddot{z}^\nu(\tau) = 0 \quad \forall \, \tau . \qquad (3.3.4)$$

Problem 4

$T : V \to V$ is symmetric wrt. η:

$$\eta(Tv, w) = \eta(v, Tw) \qquad (3.4.1)$$

$\forall\, v, w \in V$. Let v be eigenvector

$$Tv = \lambda v \quad , \lambda \neq 0 \qquad (3.4.2)$$

$$\Rightarrow \quad \eta(Tv, w) = \lambda \eta(v, w) = \eta(v, Tw). \qquad (3.4.3)$$

Hence, if $w \in \{v\}^{\perp}$ then $Tw \in \{v\}^{\perp}$

If v a lightlike eigenvector then $v \in \{v\}^{\perp}$, hence $\operatorname{Span}\{v\}$ is an invariant one-dimensional subspace in the $(n-1)$-dimensional space, which means that T acts linearly on the $(n-2)$-dimensional quotient space

$$\left.\begin{aligned} \tilde{V} &:= \{v\}^{\perp} / \operatorname{Span}\{v\} \\ \tilde{T} &: \tilde{V} \to \tilde{V} \\ \text{by} \quad \tilde{T}([w]) &:= [T(w)] \end{aligned}\right\} \qquad (3.4.4)$$

i.e. the image of the equivalence class of $w \in \{v\}^{\perp}$ is the equivalence class of <u>any</u> representative w of $[w]$.

On $\tilde{V}$ η defines a negative definite inner product

$$\left.\begin{aligned} &\tilde{\eta} : \tilde{V} \times \tilde{V} \longrightarrow \mathbb{R} \\ &\tilde{\eta}([u],[w]) := \eta(u,w) \end{aligned}\right\} \quad (3.4.5)$$

for any $u, w \in \{v\}^{\perp}$, with respect to which $\tilde{T}$ is symmetric.
(We could have defined $-\tilde{\eta}$ to make this inner product positive definite.)

$$\left.\begin{aligned} &\tilde{\eta}(\tilde{T}[u],[w]) = \tilde{\eta}([u],\tilde{T}[w]) \\ &\forall \; [u],[w] \in \tilde{V}. \end{aligned}\right\} \quad (3.4.6)$$

If $\{e_0, \ldots, e_n\}$ diagonalizes η, i.e.

$$\eta(e_a, e_b) = \operatorname{diag}(1, -1, \ldots, -1)$$

and T then e_1 is a timelike eigenvector of T. Conversally, if v is a timelike eigenvector of T

$$Tv = \lambda v$$

where w.l.o.g. $\eta(v,v) = 1$, then $-\eta|_{\{v\}^{\perp}}$ is positive definite and we know that there is a orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ of $\{v\}^{\perp}$ diagonalising $T|_{\{v\}^{\perp}}$.

Then $\{e_0 = v, e_1, \dots, e_{n-1}\}$ diagonalises T.

Hence, if T admits a timelike eigenvector there is an orthonormal basis $\{e_0, \dots, e_{n-1}\}$ so that

$$T(e_a, e_b) = \mathrm{diag}(\lambda_0, \lambda_1, \dots \lambda_{n-1})$$
$$= T^{ab} = \eta^{ac}\eta^{bd} T_{bd} \quad (3.4.7)$$

This means there exists a rest frame in which the energy-current density ($\sim$ momentum density) vanishes.
Hence there is always an observer "catching up" with the flow of energy, which is only possible if energy cannot flow at or above the velocity of light.

For a plane electromagnetic wave in, say, x-direction with amplitude $\sim \exp(ik_\mu x^\mu)$, $k^\mu = (\omega, \omega, 0, 0)$ we expect $T^{\mu\nu} \sim k^\mu k^\nu$ hence

$$T^{\mu\nu} = \begin{pmatrix} \omega^2 & \omega^2 & 0 & 0 \\ \omega^2 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

E3.10b

$$T^{\mu}{}_{\nu} = T^{\mu\lambda}\eta_{\lambda\nu} =$$

$$= \begin{pmatrix} \omega^2 & -\omega^2 & 0 & 0 \\ \omega^2 & -\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim k^{\mu} k_{\nu}$$

This is a rank $= 1$ map with kernel $= \{k\}^{\perp}$ $= \operatorname{Span}\{k, e_2, e_3\}$. It is not diagonalisable.

Problem 5

First some preliminary stuff: Let

$$V_+ := \{ v \in V : \eta(v,v) > 0 \} \qquad (3.5.1)$$

be the space of timelike vectors. It has two connected components $V_+^{\uparrow}$ and $V_+^{\downarrow}$, called the future and past cone. These may be characterised by a fiducial timelike vector in one of these components, say $n \in V_+^{\uparrow}$, the future timelike cone, in which case n is said to be "future pointing". Any other timelike vector $v \in V_+$ is said to be in the future component $V_+^{\uparrow}$, or to be "future pointing" iff $\eta(n,v) > 0$. Hence

$$\left.\begin{aligned} V_+^{\uparrow} &= \{ v \in V_+ : \eta(n,v) > 0 \} \\ V_+^{\downarrow} &= \{ v \in V_+ : \eta(n,v) < 0 \}, \end{aligned}\right\} \qquad (3.5.2)$$

Now, there is a surjective continuous map

$$\left.\begin{aligned} f : V_+ &\longrightarrow \mathbb{R} - \{0\} \\ v &\longmapsto \eta(n,v) \end{aligned}\right\} \qquad (3.5.3)$$

Since continuous maps of connected sets are connected and since the image of V_+ under f has two connected components, V_+ must also consist of at least two connected components. But each of the sets $V_+^{\uparrow}$ and $V_+^{\downarrow}$ are connected. The proof is thus: Let V_1 and V_2 be in, say, $V_+^{\uparrow}$; then the straight line segment connecting them $\lambda \longmapsto V(\lambda) := \lambda V_1 + (1-\lambda) V_2$ lies entirely within the set of timelike vectors, i.e.

$$V(\lambda) \in V_+ \qquad \forall \lambda \in [0,1] \tag{3.5.4}$$

and also

$$\eta(n, V(\lambda)) = \lambda \underbrace{\eta(n, V_1)}_{>0} + (1-\lambda) \underbrace{\eta(n, V_2)}_{>0} > 0.$$

Hence

$$V(\lambda) \in V_+^{\uparrow} \qquad \forall \lambda \in [0,1] \tag{3.5.5}$$

This shows that V_+ has precisely two connected components,

$$\left.\begin{aligned} &V_+ = V_+^{\uparrow} \cup V_+^{\downarrow} \\ &\text{where } V_+^{\uparrow} \cap V_+^{\downarrow} = \phi \end{aligned}\right\} \tag{3.5.6}$$

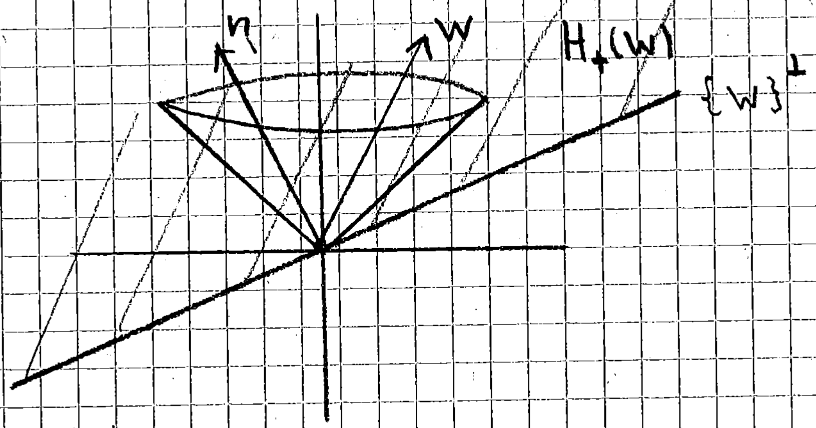
If $W \in V_+^{\uparrow}$ define

$$H_+(W) = \{ V \in V : \eta(V, W) > 0 \} \subset V \tag{3.5.7}$$

This is the open "half-space" above the hyperplane

$$\{w\}^\perp := \{v \in V : \eta(v,w) = 0\}$$

where "above" to that part of the two connected components of $V - \{w\}^\perp$ containing w (i.e into which w points).



Note that for any $w \in V_+^\uparrow$ we obviously have

$$\overline{V_+^\uparrow} \subset H_+(w) \qquad (3.5.8)$$

and that for any $v \notin \overline{V_+^\uparrow}$ there is some $w \in V_+^\uparrow$ so that $v \notin H_+(w)$. Hence

$$\bigcap_{w \in V_+^\uparrow} H_+(w) = \overline{V_+^\uparrow} \qquad (3.5.9)$$

Now we turn to the energy conditions: The weak and strong conditions say that T and $T - \frac{1}{2}\,\mathrm{trace}(T)\,\mathrm{id}_V$, respectively, map each each $v \in V_+^\uparrow$ to $H_+(v)$ and – by linearity – then also each $v \in V_+^\downarrow$ to $H_-(v)$.

As for the condition of energy-dominance, it consists of two parts

1. $$v \in V_+ \Rightarrow \eta(Tv, Tv) > 0$$
$$\Rightarrow Tv \in V_+ \qquad (3.5.10)$$

and

2.) $$v \in V_+ \Rightarrow \eta(v, Tv) > 0$$
$$\Rightarrow Tv \in H_+(v) \qquad (3.5.11)$$

Hence energy dominance is equivalent to

$$v \in V_+ \Rightarrow Tv \in V_+ \cap H_+(v) \qquad (3.5.12)$$

which in turn is equivalent to

$$v \in V_+^{\uparrow} \Rightarrow Tv \in V_+^{\uparrow} \qquad (3.5.13)$$

which clearly (by linearity) is equivalent to

$$v \in V_+^{\downarrow} \Rightarrow Tv \in V_+^{\downarrow}. \qquad (3.5.14)$$

So the images of $V_+^{\uparrow}$ and $V_+^{\downarrow}$ under T are contained in $V_+^{\uparrow}$ and $V_+^{\downarrow}$, respectively and the same holds for the topological closures $\overline{V}_+^{\uparrow}$ and $\overline{V}_+^{\downarrow}$ by continuity.

E3.15

Let's apply the energy condition to an ideal fluid. We write

$$\eta(V, W) = V \cdot W \qquad (3.5.15)$$

and observe that by linearity we may restrict to normalised V, i.e. $\eta(V,V)=1$. Then we have the following relations

$$TV = \left(\rho + \frac{p}{c^2}\right)(u \cdot V)\, u - pV \qquad (3.5.16)$$

$$V \cdot TV = \left(\rho + \frac{p}{c^2}\right)(u \cdot V)^2 - p \qquad (3.5.17)$$

$$TV \cdot TV = \left(\rho + \frac{p}{c^2}\right)^2 (u \cdot V)^2 c^2$$

$$- 2p\left(\rho + \frac{p}{c^2}\right)(u \cdot V)^2 + p^2$$

$$= (u \cdot V)^2 c^2 \left(\rho + \frac{p}{c^2}\right)\left(\rho - \frac{p}{c^2}\right) + p^2$$

$$= (u \cdot V)^2 c^2 \left[\rho^2 - \left(\frac{p}{c^2}\right)^2\right] + p^2 \qquad (3.5.18)$$

$$\operatorname{trace}(T) = (\rho c^2 + p) - 4p = \rho c^2 - 3p \qquad (3.5.18)$$

$$\left(T - \tfrac{1}{2}\operatorname{trace}(T)\, id\right) V =$$

$$= \left(\rho + \frac{p}{c^2}\right)(u \cdot V)\, u - pV - \tfrac{1}{2}(\rho c^2 - 3p)\, V$$

$$= \left(\rho + \frac{p}{c^2}\right)(u \cdot V)\, u + \tfrac{1}{2}(p - \rho c^2)\, V \qquad (3.5.19)$$

$$v \cdot \left(T - \tfrac{1}{2}\,\mathrm{trace}(T)\,\mathrm{id}\right) v$$

$$= \left(\varsigma + \frac{p}{c^2}\right)(u \cdot v)^2 + \frac{1}{2}\left(p - \varsigma c^2\right) \qquad (3.5.20)$$

Now, the key observation is this: As $v \in V_+$ ranges over all unit timelike vectors, the product $(u \cdot v)$ ranges over all values in $[c, \infty)$ if v is in the same component as u and all values in $(-\infty, -c]$ if v is in the other component. In any case, $(u \cdot v)^2$ ranges in the intervall $[c^2, \infty)$, which is the only quantity that appears in our expressions so that we can restrict v to range over only the same component as u without loosing any implication. [Note: If v_1 and v_2 are two unit timelike vectors from the same component, then $v_1 \cdot v_2 = \cosh(\alpha)$ when α is the rapidity of v_2 relative to v_1, i.e. the geodesic distance between the points v_1 and v_2 on the unit hyperboloid of unit timelike vectors in one component. Hence $v_1 \cdot v_2 \in [1, \infty)$.]

If we call

$$X := (u \cdot v) \in [c^2, \infty) \qquad (3.5.21)$$

then (3.5.17), (3.5.20) and (3.5.18) can be written, respectively, as

$$F_1(X) := (v \cdot T v)$$
$$= \left(\rho + \frac{p}{c^2}\right) X - p \qquad (3.5.21)$$

$$F_2(X) := v \cdot \left(T - \tfrac{1}{2}\,\mathrm{trace}(T)\,\mathrm{id}\right) v$$
$$= \left(\rho + \frac{p}{c^2}\right) X + \frac{1}{2}\left(p - \rho c^2\right) \qquad (3.5.22)$$

$$F_3(X) := T v \cdot T v$$
$$= c^2\left[\rho^2 - \left(\frac{p}{c}\right)^2\right] X + p^2 \qquad (3.5.23)$$

The energy-conditions are, respectively, involve the conditions that $F_{1,2,3}(X) \geq 0 \quad \forall\ X \in [c^2, \infty)$, which, since all functions are of the form of a straight line $F(X) = mX + b$ is equivalent to $F(c^2) \geq 0$ and $F(X \to \infty) > 0$, i.e to $c^2 m + b \geq 0$ and $m \geq 0$.

For F_1 this means

$$\rho \geq 0 \quad \text{and} \quad p \geq -c^2\rho \qquad (3.5.26)$$

for F_2

$$\rho c^2 + 3p \geq 0 \quad \text{and} \quad \rho c^2 + p \geq 0 \qquad (3.5.27)$$

and for F_3

$$\rho^2 \geq 0 \quad \text{and} \quad \rho^2 \geq \left(\frac{p}{c^2}\right)^2 \qquad (3.5.28)$$

(3.5.26) is already our equivalent to the weak energy condition. (3.5.27) may be put into an equivalent form that distinguishes the cases of positive and negative ρ: If $\rho \geq 0$ then (3.5.27) says that $p \geq -\rho c^2/3$ and $p \geq -\rho c^2$, which, since $-\rho c^2/3 \geq -\rho c^2$ for $\rho \geq 0$ is equivalent to $p \geq -\rho c^2/3$. If $\rho \leq 0$ then $-\rho c^2 \geq -\rho c^2/3$ implies that (3.5.27) is equivalent to $p \geq -\rho c^2$. Hence the strong energy condition is equivalent to

$$p \geq \begin{cases} -\dfrac{\rho c^2}{3} & \text{if } \rho \geq 0 \\ -\rho c^2 & \text{if } \rho \leq 0 \end{cases} \qquad (3.5.29)$$

Finally, the condition $\rho^2 \geq 0$ in (3.5.28) is vacuous, so (3.5.28) is equivalent to

$$|p| \leq c^2 |\rho| \qquad (3.5.30)$$

or

$$-c^2|\rho| \leq p \leq c^2 |\rho| \qquad (3.5.31)$$

The condition of energy dominance is $V \cdot TV \geq 0$ and $TV \cdot TV \geq 0$, i.e. it consists of (3.5.26) and (3.5.31). Together they are equivalent to

$$\rho \geq 0 \quad \text{and} \quad -c^2 \rho \leq p \leq c^2 \rho \qquad (3.5.32)$$

Note that the "strong" energy condition is not strictly stronger than the weak condition, i.e. it does not imply the weak condition. This is only true for $\rho \geq 0$ where the weak condition implies $p \geq -c^2\rho$ and the strong $p \geq -c^2\rho/3$, which implies the weak since $-c^2\rho/3 \geq -c^2\rho$ for $\rho \geq 0$. But the strong condition does allow for $\rho < 0$, in which case it makes the rather unphysical requirement of a <u>very large lower bound</u> for p: $p \geq -\rho c^2$!

Problem 6

The first part has already been done in the solution to problem 5.

As for eq. (8), we recall that the condition of energy dominance is equivalent to

$$T(\overline{V}^{\uparrow}_{+}) \subseteq \overline{V}^{\uparrow}_{+} \qquad (3.6.1)$$

Since scalar products of vectors in $\overline{V}^{\uparrow}_{+}$ are non-negative

$$\eta(v,w) \geq 0 \quad \forall\, v,w \in \overline{V}^{\uparrow}_{+}$$

We have for any orthonormal basis $\{e_0, e_1, e_2, e_3\}$, with e_0 timelike, that e_0 and all $e_0 \pm e_a \in \overline{V}^{\uparrow}_{+}$. Hence

$$\eta(e_0, Te_0) = T_{00} \geq 0 \qquad (3.6.2)$$

$$\eta(e_0, T(e_0 \pm e_a)) \geq 0$$

$$\Leftrightarrow \quad T_{00} \pm T_{0a} \geq 0$$

$$\Rightarrow \quad T_{00} \geq |T_{0a}| \qquad (3.6.3)$$

E3.21

$$\eta(e_0 \pm e_a, T(e_0 \pm e_a)) \geq 0$$

$$\Leftrightarrow \quad T_{00} + T_{ab} \pm T_{0a} \pm T_{b0} \geq 0 \qquad (3.6.4)$$

$$\Rightarrow \quad T_{00} + T_{ab} \geq 0 \qquad (3.6.5)$$

$$\eta(e_0 \pm e_a, T(e_0 \mp e_b)) \geq 0$$

$$\Leftrightarrow \quad T_{00} - T_{ab} \pm T_{a0} \mp T_{0b} \geq 0 \qquad (3.6.6)$$

$$\Rightarrow \quad T_{00} - T_{ab} \geq 0 \qquad (3.6.7)$$

(3.6.5) and (3.6.7) together imply

$$T_{00} \geq |T_{ab}| \qquad (3.6.8)$$

(3.6.3) and (3.6.8) imply

$$T_{00} \geq |T_{\mu\nu}| \qquad (3.6.9)$$