Sheet 4: Solutions

Problem 1

$$g = \left(1 + \frac{2\phi(\vec{x})}{c^2}\right) c^2 dt^2 - d\vec{x}^2 \qquad (4.1.1)$$

Energy-functional has Lagrangian

$$L = \frac{1}{2}\left[\left(1 + \frac{2\phi(\vec{z}(\lambda))}{c^2}\right) c^2 \dot{t}^2(\lambda) - \dot{\vec{z}}(\lambda) \cdot \dot{\vec{z}}(\lambda)\right] \qquad (4.1.2)$$

$$\frac{\partial L}{\partial \dot{t}} = c^2 \left(1 + \frac{2\phi}{c^2}\right) \dot{t} = k = \text{const} \qquad (4.1.3)$$

since variable t is cyclic, i.e. $\partial L / \partial t = 0$.

$$\frac{\partial L}{\partial \vec{z}} = \dot{t}^2 \vec{\nabla} \phi \qquad (4.1.4)$$

$$\frac{\partial L}{\partial \dot{\vec{z}}} = - \dot{\vec{z}} \qquad (4.1.5)$$

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{\vec{z}}} = - \ddot{\vec{z}} \qquad (4.1.6)$$

Hence spatial Euler-Lagrange equation

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{\vec{z}}} - \frac{\partial L}{\partial \vec{z}} = 0 \qquad (4.1.7)$$

is equivalent to

$$\ddot{\vec{z}} = -\dot{t}^2 \vec{\nabla}\phi \qquad (4.1.8)$$

If we replace the parameter λ by t, we have

$$\left.\begin{aligned} \frac{d}{d\lambda} &= \dot{t}\,\frac{d}{dt} \\ \frac{d^2}{d\lambda^2} &= \ddot{t}\,\frac{d}{dt} + \dot{t}^2\,\frac{d^2}{dt^2} \end{aligned}\right\} \quad (4.1.9)$$

Writing $\vec{z}\,' = d\vec{z}/dt$ etc., (4.1.8) turns into

$$\ddot{t}\,\vec{z}\,' + \dot{t}^2\,\vec{z}\,'' = -\dot{t}^2\,\nabla\phi \qquad (4.1.10)$$

$$\Rightarrow \quad \vec{z}\,'' = -\nabla\phi - \frac{\ddot{t}}{\dot{t}^2}\,\vec{z}\,'$$

$$= -\vec{\nabla}\phi + \left(\frac{1}{\dot{t}}\right)^{\bullet}\vec{z}\,' \qquad (4.1.11)$$

From (4.1.3):

$$\frac{1}{\dot{t}} = \frac{c^2}{k}\left(1 + \frac{2\phi}{c^2}\right) \qquad (4.1.12)$$

$$\Rightarrow \quad \left(\frac{1}{\dot{t}}\right)^{\bullet} = \frac{2}{k}(\vec{\nabla}\phi)\cdot\dot{\vec{z}} = \frac{2}{k}(\vec{\nabla}\phi)\,\dot{t}\,\vec{z}\,'$$

$$= \frac{2}{k}(\vec{\nabla}\phi)\,\frac{k}{c^2}\left(1 + \frac{2\phi}{c^2}\right)^{-1}\vec{z}\,'$$

using (4.1.3) once more $\qquad = \vec{\nabla}\ln\left[1 + \frac{2\phi}{c^2}\right]\cdot\vec{z}\,' \qquad (4.1.13)$

Inserting (4.1.12) into (4.1.10) gives

$$\vec{Z}'' = -\vec{\nabla}\phi + \vec{Z}'\cdot\vec{\nabla}\ln\left(1+\frac{2\phi}{c^2}\right)\vec{Z}'$$

$$= -\vec{\nabla}\phi + \left(1+\frac{2\phi}{c^2}\right)^{-1} 2\,\frac{(\vec{Z}'\cdot\vec{\nabla}\phi)\vec{Z}'}{c^2} \qquad (4.1.14)$$

which is just eq. (B) of the problem sheet

The spatial projection of the worldline deviates from being "straight" (in the sense of the affine structure of $\mathbb{R}^3$) not because of any spatial curvature, but because of the space dependent rescaling of Newtonian time given by the factor $(1+2\phi/c^2)$ multiplying dt^2. In the sense of the stationarity of the length-functional, according to which timelike geodesics are the "longest" curves in spacetime, the spacetime trajectory wants to "bend" as much as possible towards large values of $(1+2\phi/c^2)$, i.e. into those spatial regions where ϕ is least negative (respecting boundary conditions).

This will be seen at work in the next exercise.

Problem 2

Let now $\vec{x} = (x, y, z)$ and

$$\phi(\vec{x}) = g z \qquad (4.2.1)$$

Hence eq. of motion (4.1.13) become for $\vec{z}(t) = (x(t), y(t), z(t))$ in $\frac{1}{c^2} \to 0$ limit

$$\left.\begin{aligned} x'' &= y'' = 0 \\ z'' &= -g \end{aligned}\right\} \quad (4.2.2)$$

The initial conditions are $\vec{z}(t=0) = 0$ and $\vec{z}\,'(0) = v\vec{e}_z$. Hence

$$x(t) = y(t) = 0 \qquad (4.2.3)$$

$$z'(t) = -gt + v$$

$$z(t) = vt - \frac{1}{2} g t^2 \qquad (4.2.4)$$

$z(t)$ has zeros for $t(v - \frac{1}{2} g t) = 0 \iff$

$$t = t_1 = 0 \;, \quad t = t_2 = \frac{2v}{g} \qquad (4.2.5)$$

Hence Newtonian time t at second encounter is

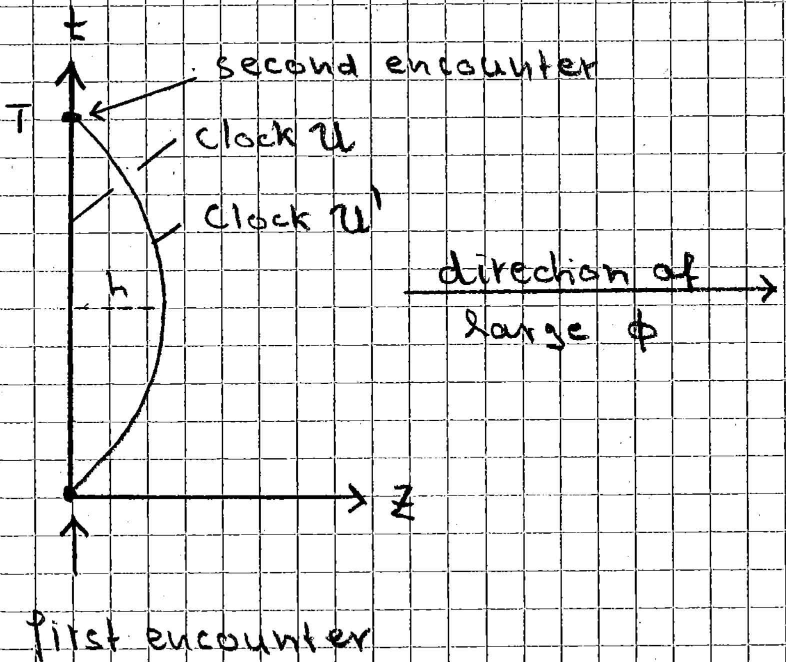
$$T = \frac{2v}{g} \qquad (4.2.6)$$

The maximal height is reached for

$$z'(t) = 0 \Leftrightarrow -gt + V = 0 \Leftrightarrow t = t_{max} = V/g\,,$$

$$h := z_{max} = z(t_{max}) = z(V/g)$$

$$= \frac{1}{2}\frac{V^2}{g} \qquad (4.2.7)$$



Note: In SR the world-line of $\mathfrak{U}'$ would be shorter than that of $\mathfrak{U}$. Here it is longer because it runs in regions of large ϕ.

At second encounter clock $\mathfrak{U}$ shows

$$T = \frac{2V}{g} \qquad (4.2.8)$$

This is because clock $\mathfrak{U}$ stays at $\vec{x} = 0$ where $\phi = 0$ and the eigentime

$$d\tau = \left(1 + \frac{2\phi}{c^2}\right)\Big|_{\vec{x}=0} dt = dt$$

equals Newtonian time (4.2.6).

In contrast, the eigentime of clock U' differs from Newtonian time

$$T' = \int_0^T dt \left[1 + \frac{2 g Z(t)}{c^2} - \frac{[Z'(t)]^2}{c^2}\right]^{1/2}$$

$$= \int_0^T dt \left[1 + \frac{g Z(t)}{c^2} - \frac{1}{2}\frac{[Z'(t)]^2}{c^2}\right] + O(c^{-4})$$

$$= T + \frac{g}{c^2}\int_0^T Z(t)\, dt - \frac{1}{2c^2}\int_0^T [Z'(t)]^2\, dt + O(c^{-4}) \qquad (4.2.9)$$

The first correction term $\sim \frac{g}{c^2}$ results from the gravitational field and tries to <u>increase</u> the proper time to values larger than T; it is sometimes called the GR-correction. The second term $\sim -\frac{1}{2c^2}$ results from the motion against the static coord. system and tries to <u>decrease</u> the proper time to values smaller than T, it is sometimes called the SR-correction (SR = Special Relativity). It will be of interest to calculate both terms separately.

The "GR-contribution" follows from

$$\int_0^T dt\, Z(t) = \int_0^T \left(Vt - \frac{1}{2} g t^2\right) dt$$

$$= \frac{1}{2} V T^2 - \frac{1}{6} g T^3 = \frac{T^2}{2}\left(V - \frac{1}{3} g T\right)$$

$$= \frac{T^2}{6} V = \frac{2}{3} \frac{V^3}{g^2} \qquad (4.2.10)$$

using $T = 2V/g$

Hence

$$(\Delta T)_{GR} := \frac{g}{c^2} \int_0^T Z(t)\, dt$$

$$= \frac{2}{3} \cdot \frac{V}{g} \cdot \left(\frac{V}{c}\right)^2 \qquad (4.2.11)$$

The "SR-contribution" follows from

$$\int_0^T [Z'(t)]^2 dt = \int_0^T \left(V^2 - 2gVt + g^2 t^2\right) dt$$

$$= V^2 T - gVT^2 + \frac{1}{3} g^2 T^3$$

$$= \frac{2V^3}{g} - \frac{4V^3}{g} + \frac{8}{3} \frac{V^3}{g} = \frac{2}{3} \frac{V^3}{g} \qquad (4.2.12)$$

using $T = 2V/g$

Hence

$$(\Delta T)_{SR} := -\frac{1}{2c^2}\int_0^T [Z'(t)]^2\, dt$$

$$= -\frac{1}{3}\cdot\frac{V}{g}\cdot\left(\frac{V}{c}\right)^2 \qquad (4.2.13)$$

Thus we see that "GR wins over SR" and

$$\Delta T = (\Delta T)_{GR} + (\Delta T)_{GR}$$

$$= \frac{1}{3}\cdot\frac{V}{g}\cdot\left(\frac{V}{c}\right)^2$$

$$= \frac{1}{6}\cdot T\cdot\left(\frac{V}{c}\right)^2$$

$$= \frac{1}{3}\cdot T\cdot\frac{2gh}{c^2} \qquad (4.2.14)$$

This is the eigentime by which the freely falling clock (Observer) $\mathcal{U}'$ has aged more than the static one $\mathcal{U}$. This is because $\mathcal{U}$ spent parts of its journey in regions of higher ϕ. Turning the argument around one may say that "staying at locations of small ϕ, i.e. close to a black hole, keeps you young" (→ Interstellar!).

Problem 3

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \qquad (4.3.1)$$

$$= g_{00}\, dx^0 \otimes dx^0$$

$$+ g_{0a}\left(dx^0 \otimes dx^a + dx^a \otimes dx^0 \right)$$

$$+ g_{ab}\, dx^a \otimes dx^b \qquad (4.3.2)$$

$$= g_{00}\left(dx^0 + \frac{g_{0a}}{g_{00}} dx^a \right) \otimes \left(dx^0 + \frac{g_{0b}}{g_{00}} dx^b \right)$$

$$+ \left(g_{ab} - \frac{g_{0a}\, g_{0b}}{g_{00}} \right) dx^a \otimes dx^b \qquad (4.3.3)$$

$$= \phi^2\, \Theta \otimes \Theta - h_{ab}\, dx^a \otimes dx^b \qquad (4.3.4)$$

Where $\phi := \sqrt{g_{00}}$

$$\Theta := dx^0 + A \qquad (4.3.5)$$

$$A := A_a\, dx^a = \frac{g_{0a}}{g_{00}}\, dx^a \qquad (4.3.6)$$

$$h_{ab} := -g_{ab} + \frac{g_{0a}\, g_{0b}}{g_{00}} \qquad (4.3.7)$$

If $g_{\mu\nu}$ - koefficients do not depend on t, have with $K = \partial / \partial x^0$

$$L_K\, g_{\mu\nu} = K(g_{\mu\nu}) = 0 \qquad (4.3.8)$$

Also

$$L_K \, dx^0 = d(L_K X^0)$$
$$= d\left(\frac{\partial x^0}{\partial x^0}\right) = d(1) = 0$$

$$L_K \, dx^a = d \, L_K X^a$$
$$= d\left(\frac{\partial x^a}{\partial x^0}\right) = d(0) = 0. \qquad (4.3.9)$$

Hence

$$L_K g = L_K (g_{\mu\nu} \, dx^\mu \otimes dx^\nu)$$
$$= (L_K g_{\mu\nu}) \, dx^\mu \otimes dx^\nu$$
$$+ g_{\mu\nu} \, d(L_K X^\mu) \otimes dx^\nu$$
$$+ g_{\mu\nu} \, dx^\mu \otimes d(L_K X^\nu)$$
$$= 0. \qquad (4.3.10)$$

Define 1-form associated to K

$$K^\flat := g(K, \cdot) = g\left(\frac{\partial}{\partial x^0}, \cdot\right)$$
$$= g_{00} \, dx^0 + g_{0a} \, dx^a$$
$$= g_{00}\left(dx^0 + \frac{g_{0a}}{g_{00}} dx^a\right)$$
$$= \phi^2 \, \theta \qquad (4.3.11)$$

Then

$$K^\flat \wedge dK^\flat = \phi^2 \theta \wedge (d\phi^2 \wedge \theta + \phi^2 d\theta)$$

$= 0$

$$= \phi^4 \; \theta \wedge d\theta \qquad (4.3.12)$$

But $\theta = dx^0 + A \rightsquigarrow d\theta = dA$

$$\Rightarrow \quad K^\flat \wedge dK^\flat = \phi^4 \; \theta \wedge dA \qquad (4.3.13)$$

Now $\quad K^\flat \wedge dK^\flat = 0$

$$\Leftrightarrow \quad \theta \wedge dA = 0 \qquad (4.3.14)$$

$\Leftrightarrow \quad \exists$ 1-form $\psi = \psi_a dx^a$

$$dA = \theta \wedge \psi \qquad (4.3.15)$$

But this implies $\psi = 0$ since

$$dA = d\left(\frac{g_{0b}}{g_{00}} dx^b\right) = \partial_a\left(\frac{g_{0b}}{g_{00}}\right) dx^a \wedge dx^b \qquad (4.3.16)$$

Since $g_{\mu\nu}$ do not depend on x^0, whereas

$$\theta \wedge \psi = dx^0 \wedge \psi + \underbrace{A \wedge \psi}_{\text{spatial}} \qquad (4.3.17)$$

Hence

$$K^{\downarrow} \wedge dK^{\downarrow} = 0 \iff dA = 0 \qquad (4.3.18)$$

$$\iff d\left(\frac{g_{0b}}{g_{00}} dx^b\right) = 0$$

$$\iff \partial_a\left(\frac{g_{0b}}{g_{00}}\right) - \partial_b\left(\frac{g_{0a}}{g_{00}}\right) = 0 \qquad (4.3.19)$$

By Poincaré's Lemma

$$dA = 0 \iff A = df \qquad (4.3.20)$$

for locally defined function f.

$$\Rightarrow \theta = d(x^0 + f) \qquad (4.3.21)$$

Now set $\tilde{x}^0 := x^0 + f \qquad (4.3.22)$

as new time coordinate, while keeping the spatial coordinates x^a. Then

$$g = \phi^2(\vec{x})\, d\tilde{x}^0 \otimes d\tilde{x}^0 - h_{ab}(\vec{x})\, dx^a \otimes dx^b. \qquad (4.3.23)$$

Problem 4

Let $u \in ST^1_0(M)$ timelike and normalised

$$g(u, u) = c^2 \qquad (4.4.1)$$

Define ($u^\flat := g(u, \cdot)$)

$$\Pi = \mathrm{id} - \frac{u \otimes u^\flat}{c^2} \in ST^1_1(M) \qquad (4.4.2)$$

$$\Rightarrow \Pi(u) = u - \frac{u \cdot c^2}{c^2} = 0 \qquad (4.4.3)$$

$$\Pi(v) = v \quad \forall v \in u^\perp \qquad (4.4.4)$$

and

$$\Pi \circ \Pi = \left(\mathrm{id} - \frac{u \otimes u^\flat}{c^2}\right) \circ \left(\mathrm{id} - \frac{u \otimes u^\flat}{c^2}\right)$$

$$= \left(\mathrm{id} - \frac{u \otimes u^\flat}{c^2}\right) \circ \mathrm{id}$$

$$= \Pi . \qquad (4.4.5)$$

$\Rightarrow \Pi_p$ is orthogonal projector onto

$$u_p^\perp := \{ v \in T_p(M) : g_p(v, u) = 0 \}. \qquad (4.4.6)$$

In components

$$\Pi^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{u^\alpha u_\beta}{c^2}. \qquad (4.4.7)$$

Note: The projection map

$$\Pi_p : T_p(M) \longrightarrow \mathcal{U}_p^{\perp} \subset T_p(M) \qquad (4.4.8)$$

extends to one forms

$$\Pi_p^* : T_p^*(M) \to \{\mathcal{U}_p^{\perp}\}^* \subset T_p^*(M), \qquad (4.4.9)$$

where

$$\{\mathcal{U}_p^{\perp}\}^* := \{\alpha \in T_p^* : \alpha(\mathcal{U}) = 0\}, \qquad (4.4.10)$$

by $\Pi_p^* \alpha := \alpha \circ \Pi_p$ (4.4.11)

and naturally to all tensor products of $T_p(M)$ and $T_p^*(M)$ (by acting factor-wise). Hence it also extends to

$$\nabla \mathcal{U}^{\flat} \in ST_2^0(M) \qquad (4.4.12)$$

$$\Pi^* \otimes \Pi^* (\nabla \mathcal{U}^{\flat}) := \nabla \mathcal{U}^{\flat} \circ (\Pi \otimes \Pi) \qquad (4.4.13)$$

In components

$$[\Pi^* \otimes \Pi^* (\nabla \mathcal{U}^{\flat})]_{\alpha\beta} = \nabla_\mu u_\nu \Pi^\mu_\alpha \Pi^\nu_\beta \qquad (4.4.14)$$

It is this tensor field that we now decompose into its antisymmetric part ω, symmetric part θ, trace θ and symmetric-trace-free part σ.

$$\omega_{\alpha\beta} = \frac{1}{2} \Pi^{\mu}{}_{\alpha} \Pi^{\nu}{}_{\beta} (\nabla_\mu u_\nu - \nabla_\nu u_\mu) \qquad (4.4.15)$$

$$\theta_{\alpha\beta} = \frac{1}{2} \Pi^{\mu}{}_{\alpha} \Pi^{\nu}{}_{\beta} (\nabla_\mu u_\nu + \nabla_\nu u_\mu) \qquad (4.4.16)$$

$$\theta = \Pi^{\alpha\beta} \theta_{\alpha\beta} \qquad (4.4.17)$$

$$\sigma_{\alpha\beta} = \theta_{\alpha\beta} - \frac{1}{3} \Pi_{\alpha\beta} \theta \qquad (4.4.18)$$

Obviously

$$\theta_{\alpha\beta} + \omega_{\alpha\beta} = \Pi^{\mu}{}_{\alpha} \Pi^{\nu}{}_{\beta} \nabla_\mu u_\nu \qquad (4.4.19)$$

The left-hand side is

$$\sigma_{\alpha\beta} + \frac{1}{3} \Pi_{\alpha\beta} \theta + \omega_{\alpha\beta} \qquad (4.4.20)$$

and the right-hand side

$$\Pi^{\mu}{}_{\alpha} \Pi^{\nu}{}_{\beta} \nabla_\mu u_\nu$$

$$= \Pi^{\mu}{}_{\alpha} \delta^{\nu}{}_{\beta} \nabla_\mu u_\nu \quad (\text{since } u^\nu \nabla_\mu u_\nu = 0)$$

$$= \nabla_\alpha u_\beta - \frac{1}{c^2} u_\alpha u^\mu \nabla_\mu u_\nu$$

$$= \nabla_\alpha u_\beta - \frac{1}{c^2} u_\alpha a_\beta \qquad (4.4.21)$$

Hence

$$\nabla_\alpha u_\beta = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3} \Pi_{\alpha\beta} \theta + \frac{1}{c^2} u_\alpha a_\beta \qquad (4.4.22)$$

That ω (antisymmetric) is orthogonal to σ and π (symmetric) is obvious. $u \otimes a$ is orthogonal to all terms since their contractions with u is zero by construction. And σ is orthogonal to its trace-part also by construction. So all terms on the right-hand side are pairwise orthogonal

Problem 5

$$\Theta = \Pi^{\alpha\beta} \, \Theta_{\alpha\beta}$$

$$= \Pi^{\alpha\beta} \frac{1}{2} \Pi^{\mu}{}_{\alpha} \Pi^{\nu}{}_{\beta} \left(\nabla_\mu u_\nu + \nabla_\nu u_\mu \right)$$

$$= \Pi^{\mu\alpha} \, \Pi_{\alpha}{}^{\nu} \, \nabla_\mu u_\nu$$

$$= \Pi^{\mu\nu} \, \nabla_\mu u_\nu$$

$$= \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2} \right) \nabla_\mu u_\nu$$

$$= \nabla_\mu u^\mu \qquad (4.5.1)$$

since $u^\nu \nabla_\mu u_\nu = \frac{1}{2} \nabla_\mu (\underbrace{u^\nu u_\nu}_{c^2}) = 0$ (4.5.2)

Now

$$\dot{\Theta} = u^\alpha \nabla_\alpha \Theta$$

$$= u^\alpha \nabla_\alpha \nabla_\beta u^\beta$$

$$= u^\alpha \nabla_\beta \nabla_\alpha u^\beta + u^\alpha R^\beta{}_{\lambda\alpha\beta} u^\lambda$$

[using $[\nabla_\alpha, \nabla_\beta] u^\beta = R^\beta{}_{\lambda\alpha\beta} u^\lambda$]

$$= \nabla_\beta (u^\alpha \nabla_\alpha u^\beta) - (\nabla_\beta u_\alpha) (\nabla^\alpha u^\beta)$$

$$- R_{\alpha\beta} u^\alpha u^\beta \qquad (4.5.3)$$

[using $R^\beta{}_{\lambda\alpha\beta} = - R^\beta{}_{\lambda\beta\alpha} = - R_{\lambda\alpha}$]

From the previous exercise we have

$$(\nabla_\beta u_\alpha)(\nabla^\alpha u^\beta)$$

$$= \left(\omega_{\beta\alpha} + \sigma_{\beta\alpha} + \frac{1}{3}\Pi_{\beta\alpha}\theta + \frac{1}{c^2} u_\beta u_\alpha\right)$$

$$\left(\omega^{\alpha\beta} + \sigma^{\alpha\beta} + \frac{1}{3}\Pi^{\alpha\beta}\theta + \frac{1}{c^2} u^\alpha a^\beta\right)$$

$$= -\omega_{\alpha\beta}\omega^{\alpha\beta} + \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \frac{1}{3}\theta^2 \qquad (4.5.4)$$

using pairwise orthogonality and antisymmetry of ω; and also $u_\beta a^\beta = 0$.

Hence

$$\dot{\theta} = \omega_{\alpha\beta}\omega^{\alpha\beta} + \nabla_\beta a^\beta$$

$$- \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{3}\theta^2 - R_{\alpha\beta}u^\alpha u^\beta \qquad (4.5.5)$$

Problem 6

Let now $a^\beta = 0$ (4.6.1)

$\omega_{\alpha\beta} = 0$ (4.6.2)

and g satisfies

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

$$\Leftrightarrow \quad R_{\alpha\beta} = \kappa \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) \qquad (4.6.3)$$

$$\Rightarrow \quad R_{\alpha\beta} u^\alpha u^\beta = \kappa \underbrace{\left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) u^\alpha u^\beta}_{\geq 0} \qquad (4.6.4)$$

by strong-energy cond.

Then

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\alpha\beta} u^\alpha u^\beta \qquad (4.6.5)$$

But $\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$ (4.6.6)

since the inner product that g induces in $u^\perp$ is negative definite, hence the inner product it induces on $u^\perp \otimes u^\perp$ is positive definite, likewise on $\{u^\perp\}^* \otimes \{u^\perp\}^*$.

Also $R_{\alpha\beta} u^\alpha u^\beta \geq 0$ (4.6.7)

Inserting (4.6.6) and (4.6.7) into (4.6.5) gives

$$\dot{\theta} \leq -\frac{1}{3}\theta^2 \qquad (4.6.8)$$

$$\frac{\dot{\theta}}{\theta^2} \leq -\frac{1}{3} \qquad (4.6.9)$$

$$\left(-\frac{1}{\theta}\right)^{\bullet} \leq -\frac{1}{3} \qquad (4.6.10)$$

Integrating this along an integral curve of u from proper time τ_* to $\tau > \tau_*$

$$\int_{\tau_*}^{\tau} d\tau' \frac{d}{d\tau'}\left(-\frac{1}{\theta(\tau')}\right) \leq -\frac{1}{3}(\tau-\tau_*) \qquad (4.6.11)$$

$$\Rightarrow -\frac{1}{\theta(\tau)} + \frac{1}{\theta_*} \leq -\frac{1}{3}(\tau-\tau_*) \qquad (4.6.12)$$

$$\Leftrightarrow -\frac{1}{\theta(\tau)} \leq -\frac{1}{3}(\tau-\tau_*) - \frac{1}{\theta_*} \qquad (4.6.13)$$

$$\Leftrightarrow \frac{1}{\theta(\tau)} \geq \frac{1}{3}(\tau-\tau_*) + \frac{1}{\theta_*} \qquad (4.6.14)$$

$$\theta(\tau) \leq \frac{1}{\frac{1}{3}(\tau-\tau_*) + \frac{1}{\theta_*}} \qquad (4.6.15)$$

Suppose $\Theta_* < 0$, so that $\Theta_* = -|\Theta_*|$,

then this is equivalent to

$$\Theta(\tau) \leq \frac{3}{(\tau - \tau_*) - 3/|\Theta_*|} \qquad (4.6.16)$$

This says that at $\tau = \tau_*$ we have $\Theta(\tau_*) = -|\Theta_*| < 0$, as required, and that for proper times $\tau > \tau_*$ Θ becomes more negative and diverges to negative infinite at

$$\tau - \tau_* = 3/|\Theta_*| \qquad (4.6.17)$$

the latent