Sheet 5: Solutions

Problem 1

We have

$$\omega_{\alpha\beta} = \Pi^{\mu}_{\alpha}\Pi^{\nu}_{\beta}\left(\nabla_\mu u_\nu - \nabla_\nu u_\mu\right) \qquad (5.1.1)$$

with $\Pi^{\mu}_{\alpha} = \delta^{\mu}_{\nu} - \dfrac{u^\mu u_\nu}{c^2}$ $\qquad (5.1.2)$

hence

$$\omega_{\alpha\beta} = \left(\delta^{\mu}_{\alpha} - \frac{u^\mu u_\alpha}{c^2}\right)\nabla_\mu u_\beta$$

$$- \left(\delta^{\nu}_{\beta} - \frac{u^\nu u_\beta}{c^2}\right)\nabla_\nu u_\alpha$$

$$= \nabla_\alpha u_\beta - \nabla_\beta u_\alpha - \frac{1}{c^2}\left(u_\alpha a_\beta - u_\beta a_\alpha\right) \qquad (5.1.3)$$

or $\omega = du^\flat - \frac{1}{c^2}\, u \wedge a$

$$\Rightarrow \quad du^\flat = \omega + \frac{1}{c^2}\, u \wedge a \qquad (5.1.4)$$

$$u^\flat \wedge du^\flat = u \wedge \omega \qquad (5.1.5)$$

$$i_u\left(u^\flat \wedge du^\flat\right) = i_u\left(u \wedge \omega\right)$$

$$= c^2\omega - u\, \underbrace{i_u\omega}_{=0}$$

Hence

$$\omega = \frac{1}{c^2}\, i_u \left(u^\flat \wedge d u^\flat \right) \tag{5.1.6}$$

Note: For any k-form Ω the map

$$\Omega \longmapsto \frac{1}{c^2}\, i_{u^\flat} \left(u^\flat \wedge \Omega \right) \tag{5.1.7}$$

equals the projection $\perp$ to u.

In components the proof is more involved.

Recall that the wedge-product of a p- and a q-form in components is

$$\lambda = \frac{1}{p!}\, \lambda_{a_1 \cdots a_p}\, dx^{a_1} \wedge \cdots \wedge dx^{a_p}$$

$$\eta = \frac{1}{q!}\, \eta_{a_1 \cdots a_q}\, dx^{a_1} \wedge \cdots \wedge dx^{a_q}$$

$$\lambda \wedge \eta = \frac{1}{(p+q)!} (\lambda \wedge \eta)_{a_1 \cdots a_{p+q}}\, dx^{a_1} \wedge \cdots \wedge dx^{a_{p+q}}$$

Where

$$(\lambda \wedge \eta)_{a_1 \cdots a_{p+q}} = \frac{(p+q)!}{p!\, q!}\, \lambda_{[a_1 \cdots}\, \eta_{a_{p+1} \cdots a_{p+q}]}$$

Hence

$$(u^\flat \wedge d u^\flat)_{\alpha\beta\gamma} = \frac{(1+2)!}{1!\, 2!}\, u_{[\alpha} \left(\nabla_\beta u_{\gamma]} - \nabla_\gamma u_{\beta]} \right) \tag{5.1.8}$$

$$(u^\flat \wedge du^\flat)_{\alpha\beta\gamma}$$

$$\begin{aligned} = \; & u_\alpha \nabla_\beta u_\gamma + u_\beta \nabla_\gamma u_\alpha + u_\gamma \nabla_\alpha u_\beta \\ & - u_\beta \nabla_\alpha u_\gamma - u_\alpha \nabla_\gamma u_\beta - u_\gamma \nabla_\beta u_\alpha \end{aligned} \qquad (5.1.9)$$

$$[i_u (u^\flat \wedge du^\flat)]_{\beta\gamma}$$

$$= u^\alpha (u^\flat \wedge du^\flat)_{\alpha\beta\gamma}$$

$$= c^2 (\nabla_\beta u_\gamma - \nabla_\gamma u_\beta) - (u_\beta a_\gamma - u_\gamma a_\beta)$$

$$= c^2 \; \omega_{\beta\gamma} \qquad (5.1.10)$$

according to (5.1.3)

Hence once more

$$\frac{1}{c^2} i_u (u^\flat \wedge du^\flat) = \omega. \qquad (5.1.11)$$

Problem 2

$$g = c^2\, dt \otimes dt - \delta_{ab}\, dx^a \otimes dx^b \qquad (5.2.1)$$

$$k = \partial_t + \varepsilon_{ab}{}^{c}\, \Omega^a X^b \partial_c \qquad (5.2.2)$$

∂_t is itself a killing field. We have

$$L_k g = -\delta_{ab}\, d(L_k X^a) \otimes dX^b - \delta_{ab}\, dX^a \otimes d(L_k X^b) \qquad (5.2.3)$$

But $L_k X^a = \varepsilon_{mn}{}^{a}\, \Omega^m X^n$

$$\rightsquigarrow \quad d(L_k X^a) = \Omega^m \varepsilon_{mn}{}^{a}\, dX^n \qquad (5.2.4)$$

hence

$$L_k g = -\delta_{ab}\Omega^m \left(\varepsilon_{mn}{}^{a}\, dX^n \otimes dX^b + \varepsilon_{mn}{}^{b}\, dX^a \otimes dX^n\right)$$

$$= -\Omega^m \left(\varepsilon_{mnb}\, dX^n \otimes dX^b + \varepsilon_{mna}\, dX^a \otimes dX^n\right)$$

$$= -\Omega^m (\varepsilon_{mab} + \varepsilon_{mba})\, dX^a \otimes dX^b$$

$$= 0 \qquad (5.2.5)$$

The corresponding 1-form to k is

$$k^\downarrow = g(k, \cdot) \qquad (5.2.6)$$

$$K^\downarrow = c^2 dt - \delta_{ab}\, \varepsilon_{mn}{}^a\, \Omega^m X^n\, dX^b$$

$$= c^2 dt - \varepsilon_{abc}\, \Omega^a X^b\, dX^c \qquad (5.2.7)$$

hence

$$g(K, K) = K^\downarrow (K) =$$

$$= c^2 - \varepsilon_{abc}\, \varepsilon_{mn}{}^c\, \Omega^a X^b \Omega^m X^n$$

$$= c^2 - (\Omega \times \vec{X})^2$$

$$= c^2 - \vec{\Omega}^2 \vec{X}^2 - (\vec{\Omega} \cdot \vec{X})^2$$

$$= c^2 - \vec{\Omega}^2 \left(\vec{X} - \vec{n}(\vec{n} \cdot \vec{X}) \right)^2$$

where $\vec{n} = \vec{\Omega} / \|\vec{\Omega}\|$

$$= c^2 - \Omega^2 \vec{X}_\perp^2 \qquad (5.2.8)$$

For this to be ≥ 0 we must have

$$\|\vec{X}_\perp\| \leq c / \|\vec{\Omega}\| \qquad (5.2.9)$$

Hence K is timelike in the space-time cylinder

$$U_K := \{ (X^0, \vec{X}) \in M : \|\vec{X}_\perp\| \leq c / \|\vec{\Omega}\| \} \qquad (5.2.10)$$

E5.6

From (5.2.7) get

$$dK^{\downarrow} = -\varepsilon_{abc}\,\Omega^a\, dx^b \wedge dx^c \qquad (5.2.11)$$

$$K^{\downarrow} \wedge dK^{\downarrow} = -\varepsilon_{abc}\,\Omega^a\, c^2\, dt \wedge dx^b \wedge dx^c$$
$$+\, \varepsilon_{mnd}\,\Omega^m x^n\, \varepsilon_{abc}\,\Omega^a\, dx^d \wedge dx^b \wedge dx^c \qquad (5.2.12)$$

The second term can be rewritten using

$$dx^d \wedge dx^b \wedge dx^c = \varepsilon^{dbc}\, dx^1 \wedge dx^2 \wedge dx^3 \qquad (5.2.13)$$

$$2^{nd}\ \text{Term} = \varepsilon_{mnd}\,\underbrace{\varepsilon_{abc}\,\varepsilon^{dbc}}_{2\delta^d_a}\,\Omega^m\,\Omega^a\, x^n$$

$$= 2\,\varepsilon_{mna}\,\Omega^m\,\Omega^a\, x^n = 0 \qquad (5.2.14)$$

Hence

$$K^{\downarrow} \wedge dK^{\downarrow} = -c\,\Omega^a\,\varepsilon_{abc}\, dx^0 \wedge dx^a \wedge dx^b$$
$$(5.2.15)$$

Problem 3

$$g = c^2\, dt \otimes dt - dr \otimes dr - r^2\, d\varphi \otimes d\varphi \qquad (5.3.1)$$

$$\varphi \mapsto \psi := \varphi - \Omega t \qquad (5.3.2)$$

$$\curvearrowright \quad g = c^2\, dt \otimes dt - dr \otimes dr$$
$$- r^2 (d\psi + \Omega\, dt) \otimes (d\psi + \Omega\, dt)$$

$$= (c^2 - \Omega^2 r^2)\, dt \otimes dt - dr \otimes dr$$
$$- r^2\, d\psi \otimes d\psi - r^2 \Omega (d\psi \otimes dt + dt \otimes d\psi)$$

$$= \left[1 - \left(\frac{\Omega r}{c}\right)^2\right] \left[dx^0 - \frac{r^2 \Omega / c}{1 - \left(\frac{\Omega r}{c}\right)^2}\, d\psi\right]^{\otimes 2}$$
$$- dr \otimes dr - r^2 \left[1 + \frac{(r\Omega/c)^2}{1 - \left(\frac{\Omega r}{c}\right)^2}\right] d\psi \otimes d\psi$$

$$= \left[1 - \left(\frac{\Omega r}{c}\right)^2\right] (dx^0 + A)^{\otimes 2} - dr \otimes dr$$
$$- \frac{r^2}{1 - \left(\frac{\Omega r}{c}\right)^2}\, d\psi \otimes d\psi \qquad (5.3.3)$$

$$= \phi^2(r)\, \theta \otimes \theta - h \qquad (5.3.4)$$

Where

$$\phi(r) = \left[1 - \left(\frac{\Omega r}{c}\right)^2\right]^{1/2} \quad (5.3.5)$$

$$\Theta = dx^0 + A \quad (5.3.6)$$

$$A = \frac{r^2 \Omega / c}{1 - (r\Omega/c)^2} \, d\varphi \quad (5.3.7)$$

$$h = dr \otimes dr + \frac{r^2 \, d\varphi \otimes d\varphi}{1 - (r\Omega/c)^2} \quad (5.3.8)$$

What 2-dim. geometry describes h?
The circle of constant radius r is given by curve $\gamma(\lambda) = (r(\lambda), \varphi(\lambda))$:

$$\gamma(\lambda) = (r, \lambda), \quad \lambda \in [0, 2\pi] \quad (5.3.9)$$

$$\dot{\gamma} = (0, 1)$$

$$u(r) = \int_0^{2\pi} \left[h(\dot{\gamma}, \dot{\gamma})\right]^{1/2} d\lambda \quad (5.3.10)$$

$$= \frac{2\pi r}{\sqrt{1 - \left(\frac{r\Omega}{c}\right)^2}} \quad (5.3.11)$$

The radial distance along the curve

$$\gamma(\lambda) = (\lambda, 0) \qquad \lambda \in [0, r]$$

$$R(r) = \int_0^r \left(h(\dot{\gamma}, \dot{\gamma})\right)^{1/2} d\lambda$$

$$= \int_0^r \left[h\left(\partial/\partial r, \partial/\partial r\right)\right] d\lambda$$

$$= \int_0^r d\lambda = r \qquad (5.3.12)$$

Hence circumference divided by radius is

$$\frac{U(r)}{R(r)} = \frac{2\pi}{\sqrt{1-\left(\frac{r\Omega}{c}\right)^2}} > 2\pi \qquad (5.3.13)$$

Can you visualize a 2-dim. surface embedded in $\mathbb{R}^3$ whose induced geometry has that behaviour <u>and</u> is rotationally symmetric? The answer must be no!, because this is impossible. The surface must be negatively curved, i.e. saddle-like, which cannot be embedded isometrically and be invariant under rotations at the same time.

Here is a calculation of the curvature tensor using Cartan structure equations.

We introduce orthonormal 1-forms

$$\theta^1 = dr \qquad (5.3.14)$$

$$\theta^2 = \frac{r}{\sqrt{1-\left(\frac{\Omega r}{c}\right)^2}}\, d\varphi \qquad (5.3.15)$$

So that

$$h = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 \qquad (5.3.16)$$

Cartan's 1. structure equation is used to calculate the connection 1-form $\omega^a{}_b = \omega^a{}_{cb}\,\theta^c$ for the Levi-Civita connection

$$\boxed{d\theta^a + \omega^a{}_b \wedge \theta^b = 0} \qquad (5.3.17)$$

with $\omega_{ab} = -\omega_{ba}$ (5.3.18)

where $\omega_{ab} = \delta_{ac}\,\omega^c{}_b$ (5.3.19)

Set $a=1$ $\quad \theta^1 = dr \rightsquigarrow d\theta^1 = 0$

$$\rightsquigarrow \omega^1{}_2 \wedge \theta^2 = 0$$

$$\Longleftrightarrow \quad \omega^1{}_2 \sim \theta^2 \tag{5.3.20}$$

Set $a=2$

$$\theta^2 = f(r)\, d\varphi \tag{5.3.21}$$

where we set $f(r) := \frac{1}{\sqrt{1-\left(\frac{\Omega r}{c}\right)^2}}$ (5.3.22)

Then

$$d\theta^2 = f'(r)\, dr \wedge d\varphi$$

$$= (f'/f)\, \theta^1 \wedge \theta^2$$

$$= (-f'/f)\, \theta^2 \wedge \theta^1$$

$$= -\omega^2{}_1 \wedge \theta^1 \tag{5.3.23}$$

$$\Longleftrightarrow \quad \omega^2{}_1 = (f'/f)\, \theta^2 + \sim \theta^1 \tag{5.3.24}$$

Now, since

$$\omega^2{}_1 = \omega_{21} = -\omega_{12} = -\omega^1{}_2 \tag{5.3.25}$$

we get from (5.3.20) and (5.3.24) together that

$$\omega^1{}_2 = -\omega^2{}_1 = -(f'/f)\, \theta^2$$

$$= -f'\, d\varphi \tag{5.3.26}$$

E5.12

Cartan's 2. structure equation is used to calculate the curvature 2-form $\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d$ from the $\omega^a{}_b$'s:

$$\boxed{\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_d} \quad (5.3.27)$$

In 2 dimensions, because $\omega^1{}_2 = -\omega^2{}_1$ is the only non-zero component of ω ($\omega^1{}_1 = \omega^2{}_2 = 0$) the second term $\sim \omega \wedge \omega$ vanishes. Hence

$$\Omega^1{}_2 = d\omega^1{}_2 = -f'' \, dr \wedge d\varphi$$

$$= -\frac{f''}{f} \, \theta^1 \wedge \theta^2 \quad (5.3.28)$$

Therefore the only independent non-zero component of the Riemann-tensor with respect to orthonormal frame is

$$R^1{}_{212} = -f''/f \quad (5.3.29)$$

Ricci-tensor

$$\left.\begin{aligned} R_{11} &= R^a{}_{1a1} = R^2{}_{121} = R^1{}_{212} \\ &= R_{22} = -f''/f \end{aligned}\right\} (5.3.30)$$

Ricci - scalar

$$R = h^{ab} R_{ab} = \delta^{ab} R_{ab}$$

$$= R_{11} + R_{22}$$

$$= -2 f''/f \qquad (5.3.31)$$

In our case

$$f = r \left[1 - \left(\frac{\Omega r}{c} \right)^2 \right]^{-1/2}$$

$$f' = [\;]^{-1/2} - \frac{r}{2} [\cdot\cdot]^{-3/2} (-2) \frac{r \Omega^2}{c^2}$$

$$= [\cdot\cdot]^{-1/2} + [\cdot\cdot]^{-3/2} \left(\frac{r \Omega}{c} \right)^2$$

$$= \left[1 - \left(\frac{\Omega r}{c} \right)^2 \right]^{-3/2} \qquad (5.3.32)$$

$$f'' = -\frac{3}{2} [\cdot\cdot]^{-5/2} (-2) \frac{\Omega^2}{c^2} r$$

$$= 3 \left[1 - \left(\frac{\Omega r}{c} \right)^2 \right]^{-5/2} \left(\frac{\Omega}{c} \right)^2 r \qquad (5.3.33)$$

$$\rightsquigarrow f''/f = 3 \left[\frac{\Omega / c}{1 - \left(\frac{\Omega r}{c} \right)^2} \right]^2 \qquad (5.3.34)$$

Hence the Gaussian Curvature of the (r, φ) - plane is

$$K = R_{1212} = -3 \left[\frac{\Omega / c}{1 - \left(\frac{\Omega r}{c}\right)^2} \right]^2 \qquad (5.3.31)$$

which is negative everywhere, smallest in modulus at the origin $r = 0$

$$K(r=0) = -3\,(\Omega/c)^2 \qquad (5.3.32)$$

with monotonuously increasing modulus for increasing r, diverging at $r = c/\Omega$ where the rotation of the frame becomes lightlike.

Problem 4

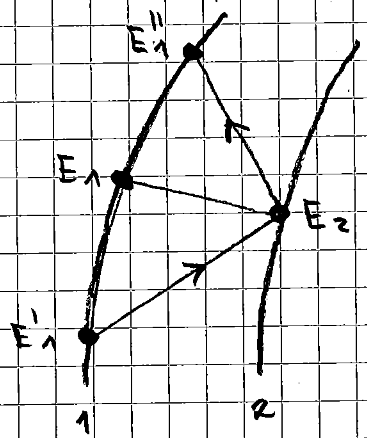
$$g = \phi^2\, \Theta \otimes \Theta - h \qquad (5.4.1)$$

$$i_K g = g(K, \cdot) = K^{\downarrow} = \phi^2 c\, \Theta \qquad (5.4.2)$$

because $\Theta(K) = (dx^0 + A)(\partial/\partial t) = c$

Hence the kernel of $K^{\downarrow}$, which is the orthogonal complement of K, is identical to the kernel of Θ.

Consider two infinitesimally neighbouring orbits of K, called 1 and 2 and an Einstein synchronisation-procedure between them. That is: Send a light signal from 1 to 2 where it is reflected back to 1



Then the event E_1 on 1 that is declared to be "Einstein synchronous" with the reflection event E_2 on 2 is that half way (in terms of proper length) between the event E_1' of sending the signal from 1 and the event E_1'' of receiving the reflected light back at 1.

As is well known this "midpoint-condition" is equivalent to the condition that the vector connecting E_2 with E_1 is perpendicular to the world-line Z.
Hence a curve orthogonal to K, i.e in the kernel of θ, corresponds to a set of successively Einstein-simultaneous events. Hence, if γ is such a curve,

$$\left.\begin{aligned} &\theta(\dot{\gamma}) = 0 \iff \\ &dX^0(\dot{\gamma}) + A(\dot{\gamma}) = 0 \end{aligned}\right\} \quad (5.4.3)$$

i.e.

$$\int dX^0(\dot{\gamma})\,d\lambda = \Delta X^0 = -\int_\gamma A \qquad (5.4.4)$$

So the integral of A along the path γ equals minus the difference in the X^0-coordinate for successive Einstein-synchronised events. But

$$A = \frac{(r\Omega/c)}{1-(r\Omega/c)^2}\, r\,d\varphi \qquad (5.4.5)$$

Integrated along a path whose spatial projection is a circle of radius r, We get

$$\Delta t = -\frac{\Omega}{c^2}\frac{2\pi r^2}{1-\left(\frac{\Omega r}{c}\right)^2} \qquad (5.4.6)$$

or, since $\Delta\tau = \phi\,\Delta t$

$$\Delta\tau = -\frac{\Omega}{c^2}\frac{2\pi r^2}{\sqrt{1-\left(\frac{\Omega r}{c}\right)^2}} \qquad (5.4.7)$$

This lapse in Eigenzeit between the last and the first synchronised clock on a rotating frame in Minkowski space shows the failure of transitivity of Einstein synchronisation if $K\wedge dK \neq 0$, i.e. if the world lines of the frame have non-zero vorticity. Note that the area of the spatial disk $r = \text{const}$ is

$$A(r) = \int_r \theta^1\wedge\theta^2 = \int \frac{r}{\sqrt{1-\left(\frac{\Omega r}{c}\right)^2}}\,dr\wedge d\varphi$$

$$= 2\pi\left(-\frac{c^2}{\Omega^2}\right)\left[1-\left(\frac{\Omega r}{c}\right)^2\right]^{1/2}\Bigg|_0^r$$

$$= 2\pi\left(\frac{c}{\Omega}\right)^2\left\{1-\left[1-\left(\frac{\Omega r}{c}\right)^2\right]^{1/2}\right\} \qquad (5.4.8)$$

$$= \pi r^2 + \text{Terms} \sim \left(\frac{\Omega r}{c}\right)^{n\geq 2} \qquad (5.4.9)$$

Note $A(r) > \pi r^2 \rightarrow$ neg. curvature

(5.4.7) and (5.4.9) give to leading order of Ω:

$$\Delta\tau = -\frac{\Omega}{c^2}\, 2\, A(r) \qquad (5.4.10)$$

Multiplied by the frequency $\nu = \frac{c}{\lambda}$ of an electromagnetic wave and multiplied by a factor of 2 for the difference between a clock- and anticlockwise synchronisation using light, this gives a phase difference of cycles

$$\Delta n = \frac{4\,\Omega A}{\lambda c} \qquad (5.4.10)$$

corresponding to a phase difference

$$\Delta\varphi = 2\pi\,\Delta n = \frac{8\pi\Omega A}{\lambda c} \qquad (5.4.11)$$

This is just the phase shift observed in the <u>Sagnac-Effect</u>

Problem 5

$$g = \phi^2 c^2 dt \otimes dt - \bar{g}_{ab}\, dx^a \otimes dx^b \qquad (5.5.1)$$

ϕ and $\bar{g}_{ab}$ do not depend on t

Hence

$$\Gamma^0_{00} = \tfrac{1}{2} g^{0\lambda}(-g_{00,\lambda} + \cancel{g_{\lambda 0,0}} + \cancel{g_{0\lambda,0}}) \qquad (5.5.2)$$

$$= \tfrac{1}{2} g^{0l}(-g_{00,l}) = 0 \quad \text{since } g^{0l} = 0$$

$$\Gamma^0_{0a} = \Gamma^0_{a0} = \tfrac{1}{2} g^{0\lambda}(-\cancel{g_{0a,\lambda}} + g_{\lambda 0,a} + \cancel{g_{0\lambda,0}})$$

$$= \tfrac{1}{2} g^{00} g_{00,a} = \phi_{,a}/\phi = [\ln\phi]_{,a} \qquad (5.5.3)$$

$$\Gamma^a_{00} = \tfrac{1}{2} g^{a\lambda}(-g_{00,\lambda} + \cancel{g_{\lambda 0,0}} + \cancel{g_{0\lambda,0}})$$

$$= \tfrac{1}{2} g^{ab}(-g_{00,b})$$

$$= \tfrac{1}{2} \bar{g}^{ab}\, 2\phi\phi_{,b} = \bar{g}^{ab}\phi\phi_{,b} \qquad (5.5.4)$$

$$\Gamma^0_{ab} = \tfrac{1}{2} g^{0\lambda}(-g_{ab,\lambda} + g_{\lambda a,b} + g_{b\lambda,a})$$

$$= \tfrac{1}{2} g^{00}(-\cancel{g_{ab,0}} + \cancel{g_{0a,b}} + \cancel{g_{b0,a}})$$

$$= 0 \qquad (5.5.5)$$

$$\Gamma^{a}_{0b} = \Gamma^{a}_{b0}$$

$$= \frac{1}{2} g^{a\lambda} \left(- g_{\cancel{b0},\lambda} + g_{\cancel{\lambda b},0} + g_{0\lambda,b}\right)$$

$$= \frac{1}{2} g^{a0} g_{00,b} = 0 \qquad (5.5.6)$$

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{a\lambda} \left(- g_{bc,\lambda} + g_{\lambda a,b} + g_{b\lambda,a}\right)$$

$$= \frac{1}{2} g^{al} \left(- g_{bc,l} + g_{la,b} + g_{bl,a}\right)$$

$$= \frac{1}{2} \bar{g}^{al} \left(- \bar{g}_{bc,l} + \bar{g}_{bl,a} + \bar{g}_{ab,l}\right)$$

$$= \bar{\Gamma}^{a}_{bc} \qquad (5.5.7)$$

Riemann tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma_\nu{}^{\alpha}{}_{\beta} - \partial_\nu \Gamma_\mu{}^{\alpha}{}_{\beta}$$

$$\Gamma_\mu{}^{\alpha}{}_{\gamma} \Gamma_\nu{}^{\gamma}{}_{\beta} - \Gamma_\nu{}^{\alpha}{}_{\gamma} \Gamma_\mu{}^{\gamma}{}_{\beta} \qquad (5.5.8)$$

Ricci - tensor

$$R_{\alpha\beta} = \partial_\lambda \Gamma^{\lambda}_{\alpha\beta} - \partial_\beta \Gamma_\alpha{}^{\lambda}{}_{\lambda}$$

$$+ \Gamma_\lambda{}^{\lambda}{}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \Gamma_\alpha{}^{\lambda}{}_{\gamma} \Gamma_\beta{}^{\gamma}{}_{\lambda} \qquad (5.5.9)$$

Calculation of R_{00}

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \cancel{\Gamma_0{}^\lambda{}_\lambda} + \Gamma^\lambda_{\lambda\gamma}\Gamma^\gamma_{00} - \Gamma_0{}^\lambda{}_\gamma \Gamma_0{}^\gamma{}_\lambda \qquad (5.5.10)$$

$$\partial_\lambda \Gamma^\lambda_{00} = \partial_a \Gamma^a_{00} = \partial_a \left(\bar{g}^{ab}\, \phi\, \phi_{,b}\right)$$

$$\begin{aligned}
\Gamma^\lambda_{\lambda\gamma}\Gamma^\gamma_{00} &= \Gamma^\lambda_{\lambda a}\Gamma^a_{00} \\
&= \Gamma^l_{la}\Gamma^a_{00} + \Gamma^0_{0a}\Gamma^a_{00} \\
&= \bar{\Gamma}^l_{la}\, \bar{g}^{ab}\, \phi\, \phi_{,b} \\
&\quad + (\phi_{,a}/\phi)\, \bar{g}^{ab}\, \phi\, \phi_{,b} \\
&= \bar{\Gamma}^l_{la}\, \bar{g}^{ab}\, \phi\, \phi_{,b} + \bar{g}^{ab}\, \phi_{,a}\, \phi_{,b}
\end{aligned} \qquad (5.5.11)$$

$$\begin{aligned}
\Gamma_0{}^\lambda{}_\gamma \Gamma_0{}^\gamma{}_\lambda &= \Gamma_0{}^0{}_a \Gamma_0{}^a{}_0 + \Gamma_0{}^a{}_0 \Gamma_0{}^0{}_a \\
&= 2\,\Gamma^0_{0a}\Gamma^a_{00} = 2\, \bar{g}^{ab}\, \phi_{,a}\, \phi_{,b}
\end{aligned} \qquad (5.5.12)$$

$$\Rightarrow \quad \begin{aligned}
R_{00} &= \partial_a \left(\bar{g}^{ab}\, \phi\, \phi_{,b}\right) \\
&\quad + \bar{\Gamma}^l_{la}\, g^{ab}\, \phi\, \phi_{,b} \\
&\quad - \bar{g}^{ab}\, \phi_{,a}\, \phi_{,b}
\end{aligned} \qquad (5.5.13)$$

Introducing the covariant derivative on the $t = \text{const.}$ submanifolds with respect to $\bar{g}$:

$$\bar{\nabla}_a V^b = \partial_a V^b + \bar{\Gamma}_a{}^b{}_c V^c \qquad (5.5.14)$$

applied to $V^a = \bar{g}^{ab}\phi\phi_{,b}$ we see that

$$R_{00} = \bar{\nabla}_a(\bar{g}^{ab}\phi\phi_{,b}) - \bar{g}^{ab}\bar{\nabla}_a\phi\bar{\nabla}_b\phi$$

$$= \phi\bar{g}^{ab}\bar{\nabla}_a\bar{\nabla}_b\phi$$

$$= \phi\bar{\Delta}\phi \qquad (5.5.15)$$

where $\bar{\Delta} = \bar{g}^{ab}\bar{\nabla}_a\bar{\nabla}_b$

is the Laplacian w.r.t. $\bar{g}$

Further components of the Ricci-tensor are

$$R_{0a} = R_{a0} = \partial_\lambda\Gamma^\lambda_{a0} - \cancel{\partial_0\Gamma_a{}^\lambda{}_\lambda}$$

$$+ \Gamma^\lambda_{\lambda\gamma}\Gamma^\gamma_{a0} - \Gamma_a{}^\lambda{}_\gamma\Gamma_0{}^\gamma{}_\lambda \qquad (5.5.16)$$

$$\partial_\lambda\Gamma^\lambda_{a0} = \partial_\ell\cancel{\Gamma^\ell_{a0}} + \partial_0\cancel{\Gamma^0_{a0}} = 0$$

$$\Gamma^\lambda_{\lambda\gamma}\Gamma^\gamma_{a0} = \Gamma^\lambda_{\lambda 0}\Gamma^0_{a0} + \Gamma^\lambda_{\lambda c}\cancel{\Gamma^c_{a0}}$$

$$= \cancel{\Gamma^0_{00}}\Gamma^0_{a0} + \cancel{\Gamma^\ell_{\ell 0}}\Gamma^0_{a0} = 0$$

$$\Gamma_a{}^\lambda{}_\gamma \Gamma_0{}^\gamma{}_\lambda = \Gamma_a{}^l{}_\gamma \Gamma_0{}^\gamma{}_l + \Gamma_a{}^0{}_\gamma \Gamma_0{}^\gamma{}_0$$

$$= \cancel{\Gamma_a{}^l{}_0} \Gamma_0{}^0{}_l + \Gamma_a{}^l{}_c \cancel{\Gamma_0{}^c{}_l}$$

$$+ \Gamma_a{}^0{}_0 \cancel{\Gamma_0{}^0{}_0} + \cancel{\Gamma_a{}^0{}_c} \Gamma_0{}^c{}_0$$

$$= 0$$

Hence

$$R_{a0} = 0 \qquad (5.5.17)$$

Finally

$$R_{ab} = \partial_\lambda \Gamma^\lambda{}_{ab} - \partial_b \Gamma_a{}^\lambda{}_\lambda$$

$$+ \Gamma^\lambda{}_{\lambda\gamma} \Gamma^\gamma{}_{ab} - \Gamma_a{}^\lambda{}_\gamma \Gamma_b{}^\gamma{}_\lambda$$

$$= \bar{R}_{ab}$$

$$+ \partial_0 \cancel{\Gamma^0{}_{ab}} - \partial_b \Gamma_a{}^0{}_0$$

$$+ \Gamma^l{}_{l0} \cancel{\Gamma^0{}_{ab}} + \Gamma^0{}_{0c} \Gamma^c{}_{ab} + \Gamma^0{}_{\cancel{00}} \Gamma^0{}_{ab}$$

$$- \Gamma_a{}^l{}_{\cancel{0}} \cancel{\Gamma_b{}^0{}_l} - \Gamma_{\cancel{a}}{}^0{}_{\cancel{c}} \cancel{\Gamma_b{}^c{}_0} - \Gamma_a{}^0{}_0 \Gamma_b{}^0{}_0$$

$$= \bar{R}_{ab} - \partial_b (\phi_{,a}/\phi)$$

$$+ (\phi_{,c}/\phi)\, \bar{\Gamma}^c{}_{ab} - \phi_{,a}\phi_{,b}/\phi^2$$

$$= \bar{R}_{ab} - \frac{\bar{\nabla}_a \bar{\nabla}_b \phi}{\phi^2} \qquad (5.5.18)$$

Einstein's equations for $\Lambda = 0$, $T_{\alpha\beta} = 0$ are equivalent to

$$R_{\alpha\beta} = 0 \qquad (5.5.19)$$

which in the static case hence become

$$\bar{\Delta}\phi = 0 \qquad (5.5.20)$$

$$\bar{R}_{ab} = \bar{\nabla}_a\phi\,\bar{\nabla}_b\phi \,/\, \phi \qquad (5.5.21)$$

If ϕ is regular everywhere and $\phi(\vec{x} \to \infty) = 1$, then $\phi \equiv 1$ since a harmonic function assumes its extrema on the boundary. Another proof is this: If ϕ is harmonic and asymptotically 1 then $\psi := \phi - 1$ is harmonic and asymp. $= 0$. Then on $\Sigma = \{x \in M : t = \text{const}\}$

$$\int_\Sigma \bar{g}^{ab}\,\bar{\nabla}_a\psi\,\bar{\nabla}_b\psi \; d\mu_\Sigma(\bar{g})$$

$$= \int \bar{\nabla}_a(\bar{g}^{ab}\,\psi\,\bar{\nabla}_b\psi)\, d\mu_\Sigma(\bar{g})$$

$$- \int \psi\,\bar{\Delta}\psi\; d\mu_\Sigma(\bar{g})$$

$$\overset{\text{Gauß}}{=} \int_{\partial\Sigma} \underbrace{(\psi\,\vec{\nabla}_b\psi)}_{r^{-3}}\, n^b\, dO_\Sigma(\bar{g}) = 0 \qquad (5.5.22)$$

Hence

$$0 = \int \underbrace{\bar{g}^{ab} \bar{\nabla}_a \psi \bar{\nabla}_b \psi}_{\text{pos. def.}} \, d\mu_\Sigma(\bar{g})$$

$$\Leftrightarrow \bar{\nabla}_a \psi = 0 \Leftrightarrow \psi \equiv 0$$

$$\Leftrightarrow \phi \equiv 1. \qquad 5.5.23$$

If ϕ is constant $\equiv 1$, from (5.5.21)

$$\Rightarrow \overline{Ric} = 0 \Rightarrow \overline{Riem} = 0$$

$$\Rightarrow \bar{g} \text{ is flat}$$

$$\Rightarrow g = dx^0 \otimes dx^0 - d\vec{x} \otimes d\vec{x} \qquad (5.5.24)$$

$$= \text{Minkowski}$$

Problem 6

$$g = \phi^2 (dx^0 \otimes dx^0 - \hat{g}_{ab}\, dx^a \otimes dx^b)$$ (5.6.1)

where ϕ and $\hat{g}_{ab}$ only depend on $\vec{x}$.

$$L = \frac{1}{2} \phi^2 ((\dot{z}^0)^2 - \hat{g}_{ab} \dot{z}^a \dot{z}^b) \quad (5.6.2)$$

Euler Lagrange Equation

$$\frac{\partial L}{\partial \dot{z}^0} = \phi^2 \dot{z}^0 , \quad \frac{\partial L}{\partial z^0} = 0 \quad (5.6.3)$$

$$\Rightarrow \quad \phi^2 \dot{z}^0 = k = \text{const} \quad (5.6.4)$$

or $$\phi^2 \left[\ddot{z}^0 + 2 \frac{\dot{\phi} \dot{z}^0}{\phi} \right] = 0 \quad (5.6.5)$$

$$\frac{\partial L}{\partial \dot{z}^a} = - \phi^2 \hat{g}_{ab} \dot{z}^b \quad (5.6.7)$$

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{z}^a} = - \phi^2 \hat{g}_{ab} \ddot{z}^b - \phi^2 \hat{g}_{an,b} \dot{z}^n \dot{z}^b$$
$$- 2 \phi \dot{\phi} \hat{g}_{ab} \dot{z}^b \quad (5.6.8)$$

where $\dot{\phi} = \phi_{,n} \dot{z}^n$

$$\frac{\partial L}{\partial z^a} = \frac{\phi_{,a}}{\phi} \phi^2 ((\dot{z}^0)^2 - \hat{g}_{ab} \dot{z}^a \dot{z}^b)$$
$$- \frac{1}{2} \phi^2 \hat{g}_{nb,a} \dot{z}^n \dot{z}^b \quad (5.6.9)$$

Hence

$$\frac{d}{d\lambda}\left[\frac{\partial L}{\partial \dot{z}^a}\right] - \frac{\partial L}{\partial z^a}$$

$$-\phi^2 \hat{g}_{ab} \ddot{z}^b - \phi^2 \frac{1}{2}\left(-\hat{g}_{nb,a} + \hat{g}_{an,b} + \hat{g}_{ab,n}\right)\dot{z}^n \dot{z}^b$$

$$-2\phi\dot{\phi}\,\hat{g}_{ab}\dot{z}^b - \sigma\,\phi_{,a}/\phi \qquad = 0 \qquad (5.6.10)$$

Where we used

$$\phi^2\left((\dot{z}^0)^2 - \hat{g}_{ab}\dot{z}^a \dot{z}^b\right) = \sigma \qquad (5.6.11)$$

with $\sigma = \text{const}$.

This is equivalent to

$$\left.\begin{aligned} &\ddot{z}^a + \hat{\Gamma}^a_{bc}\dot{z}^b \dot{z}^c + 2\frac{\dot{\phi}}{\phi}\dot{z}^a \\ &\qquad = -\sigma\,\hat{g}^{ab}\frac{\phi_{,b}}{\phi^3} \end{aligned}\right\} (5.6.12)$$

Written in terms of a new parameter $\lambda' = f(\lambda)$ this becomes $({}' = d/d\lambda')$

$$\dot{f}^2\left[z''^a + \hat{\Gamma}^a_{bc} z'^b z'^c\right] + \ddot{f} z'^a$$

$$+ 2(\dot{\phi}/\phi)\dot{z}^a = -\sigma\,\hat{g}^{ab}\phi_{,b}/\phi^3 \qquad (5.6.13)$$

Choosing the new parameter s.t.

$$\ddot{f}\, z'^a + 2\,\frac{\dot\phi}{\phi}\,\dot z^a$$

$$= \left(\ddot f + 2\,\dot\phi \dot f/\phi\right) \dot z^a = 0$$

i.e. such that

$$\ddot f + 2\,\frac{\dot\phi \dot f}{\phi} = 0 \qquad (5.6.14)$$

From (5.6.5) we see that this equation is satisfied by

$$f(\lambda) = z^0(\lambda) \qquad (5.6.15)$$

Hence we can choose $f(\lambda) = z^0(\lambda)$. From (5.6.4), i.e. $\phi \dot z^0 = k = \text{const}$ we get

$$z''^a + \hat\Gamma^a_{bc} z'^b z'^c = -\frac{\sigma}{k^2}\,\phi\, \hat g^{ab} \phi_{,b}$$

$$= -\frac{\sigma}{2k^2}\,\hat g^{ab}\, \phi^2{}_{,b} \qquad (5.6.16)$$

or, writing $z^0 = ct$

$$\frac{d^2 z^a}{dt^2} + \hat\Gamma^a_{bc}\frac{dz^b}{dt}\frac{dz^c}{dt} = -\frac{\sigma c^2}{2k^2}\,\hat g^{ab}\,(\phi^2)_{,b}$$

$$= -C\,\hat g^{ab}\,\phi^2{}_{,b} \qquad (5.6.17)$$

For lightlike geodesics we have

$$\phi^2\left((\dot{z}^0)^2 - \hat{g}_{ab}\dot{z}^a\dot{z}^b\right) = \sigma = 0 \qquad 5.6.18$$

Then

$$\frac{d^2 z^a}{dt^2} + \hat{\Gamma}^a_{bc}\frac{dz^a}{dt}\frac{dz^b}{dt} = 0 \qquad (5.6.19)$$

Hence we have proven that the spatial projection of a lightlike geodesic is a spacelike geodesic in the optical metric

$$\hat{g}_{ab} := \frac{-g_{ab}}{g_{00}} \qquad (5.6.20)$$

with static time t as affine parameter proportional to optical arc length:
Note that $\hat{g}_{ab}$ is a Riemannian (i.e. positive definite) metric. Hence the spatial path minimizes the optical arc length:

$$\delta\int \underbrace{\left[\hat{g}_{ab}(z(\lambda))\,\dot{z}^a(\lambda)\,\dot{z}^b(\lambda)\right]^{1/2} d\lambda}_{dt} = 0 \qquad (5.6.21)$$

$\Leftrightarrow$ Fermat's principle of "quickest arrival" with respect to static-time t.