Sheet 6: Solutions

Problem 1

$$\left.\begin{aligned} g &= \left(1+\frac{2\phi}{c^2}\right) c\,dt \otimes c\,dt \\ &\quad - \left(1-\frac{2\phi}{c^2}\right) d\vec{x} \,\dot{\otimes}\, d\vec{x} \\ \phi(\vec{x}) &= -G \int \frac{\rho(\vec{x}\,')}{\|\vec{x}-\vec{x}\,'\|}\, d^3x' \end{aligned}\right\} \quad (6.1.1)$$

We use spherical polar-coordinates for the $\vec{x}\,'$- integration, such that $\vec{x}$ points along the pole-axis:

$$\left.\begin{aligned} \vec{x} &= r\,\vec{e}_z \\ \vec{x}\,' &= r'\,(\sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y \\ &\qquad + \cos\theta\,\vec{e}_z) \end{aligned}\right\} \quad (6.1.2)$$

$$\rightsquigarrow \quad \|\vec{x}-\vec{x}\,'\| = \left(r^2 + r'^2 - 2rr'\cos\theta\right)^{1/2} \qquad (6.1.3)$$

$$\rightsquigarrow \quad \phi(\vec{x}) = -G\rho_0 \int_{R_1}^{R_2} dr'\,r'^2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} \frac{d\sigma}{\left[r^2+r'^2-2rr'\sigma\right]^{1/2}} \qquad (6.1.4)$$

where $\sigma = \cos\theta$

$$\int_{-1}^{1} \frac{d\sigma}{\left[r^2 + r'^2 - 2rr'\sigma\right]^{1/2}}$$

$$= \left(-\frac{1}{rr'}\right) \left[r^2 + r'^2 - 2rr'\sigma\right]^{1/2} \Big|_{-1}^{+1}$$

$$= \left(-\frac{1}{rr'}\right) \left\{ \left[(r-r')^2\right]^{1/2} - \left[(r+r')^2\right]^{1/2} \right\}$$

$$= \frac{1}{rr'} \left[|r+r'| - |r-r'| \right] \qquad (6.1.5)$$

Since $r < R_1 < r' < R_2$ this is

$$= \frac{1}{rr'} \left[r + r' - (r' - r) \right]$$

$$= \frac{2}{r'} \quad \text{(indep. of } r\text{!)} \qquad (6.1.6)$$

Hence (6.1.4) becomes

$$\phi(\vec{x}) = -2\pi G \varrho_0 \int_{R_1}^{R_2} dr' \, \frac{2}{r'} r'^2$$

$$= -4\pi G \varrho_0 \, \frac{1}{2}(R_2^2 - R_1^2)$$

$$= -\frac{4\pi}{3}(R_2^3 - R_1^3) \varrho_0 \, G \frac{1}{2} \, \frac{3\, R_2^2 - R_1^2}{R_2^3 - R_1^3}$$

$$= -GM \, \frac{3}{2} \, \frac{R_2^2 - R_1^2}{R_2^3 - R_1^3} \qquad (6.1.7)$$

The essential observation is that this is a constant. (→ Faraday cage!) The value is not really relevant for us.

Hence inside the inner sphere at $r = R_1$ the metric becomes

$$g = A^2 \, c\,dt \otimes c\,dt - B^2 \, d\vec{x} \otimes d\vec{x} \qquad (6.1.8)$$

where

$$\left.\begin{aligned} A &= \left(1 + \frac{2\phi}{c^2}\right)^{1/2} \\ B &= \left(1 - \frac{2\phi}{c^2}\right)^{1/2} \end{aligned}\right\} \qquad (6.1.9)$$

and ϕ is the constant on the r.h.s of (6.1.7)

Rescaling coordinates

$$\left.\begin{aligned} t &\mapsto t' := A t \\ \vec{x} &\mapsto \vec{x}' := B \vec{x} \end{aligned}\right\} \qquad (6.1.10)$$

gives

$$g = c\,dt' \otimes c\,dt' - d\vec{x}' \otimes d\vec{x}' \qquad (6.1.11)$$

= Minkowski metric.

Equation (6.1.8) shows

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad (6.1.12)$$

where the constant $h_{\alpha\beta}$ are

$$\left.\begin{aligned} h_{00} &= \frac{2\phi}{c^2} \\ h_{0a} &= 0 \\ h_{ab} &= \delta_{ab}\frac{2\phi}{c^2} \end{aligned}\right\} (6.1.13)$$

Under a gauge-transformation

$$h_{\alpha\beta} \mapsto h'_{\alpha\beta} := h_{\alpha\beta} + \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha \qquad (6.1.14)$$

If we choose

$$\Lambda_\alpha = -\frac{\phi}{c^2} X^\alpha \qquad (6.1.15)$$

$$\rightsquigarrow \quad \partial_\alpha \Lambda_\beta + \partial_\beta \Lambda_\alpha = -\frac{2\phi}{c^2}\delta_{\alpha\beta} \qquad (6.1.16)$$

Hence

$$h'_{\alpha\beta} = 0 \qquad (6.1.17)$$

so that the metric is gauge equivalent to the Minkowski metric.

Problem 2

$$\vec{h}(\vec{x}) = \frac{4G\rho_0}{c^3}\, \vec{\Omega} \times \underbrace{\int d^3x' \frac{\vec{x}'}{\|\vec{x}-\vec{x}'\|}}_{\vec{x}\,\frac{4\pi}{3}\int dr'\, r'} \qquad (6.2.1)$$

Since we consider the case $r < r'$

Hence

$$\vec{h}(\vec{x}) = \frac{4G\rho_0}{c^3}\left(\vec{\Omega}\times\vec{x}\right)\frac{4\pi}{3}\frac{1}{2}\left(R_2^2 - R_1^2\right) \qquad (6.2.2)$$

If we set $R_1 = 0$ and $R_2 = R$,
this becomes

$$\vec{h}(\vec{x}) = \underbrace{\frac{4\pi}{3}R^3\rho_0}_{M}\,\frac{2G}{c^2R}\,\frac{1}{c}\left(\vec{\Omega}\times\vec{x}\right)$$

$$= \underbrace{\frac{2GM}{c^2R}}_{\substack{\text{dimensionless} \\ =:\ \text{so-called "gravitational radius"}}}\,\frac{1}{c}\left(\vec{\Omega}\times\vec{x}\right) \qquad (6.2.3)$$

The gravitomagnetic field is

$$\vec{B}(\vec{x}) = -c(\vec{\nabla} \times \vec{h})$$

$$= \frac{2GM}{c^2 R} \left(- \vec{\nabla} \times (\Omega \times \vec{x}) \right)$$

$$= \frac{2GM}{c^2 R} \left[- \vec{\Omega} \underbrace{(\vec{\nabla} \cdot \vec{x})}_{3} + (\vec{\Omega} \cdot \vec{\nabla}) \vec{x} \right]$$

$$= -2 \, \frac{2GM}{c^2 R} \vec{\Omega} \qquad (6.2.4)$$

which is constant

[Note electromagnetic analogue : Magnetic field inside a homogeneously charged sphere that is rotating.]

In Lecture we have seen that a gravitomagnetic field gives rise to a term in the geodesic equation of the form

$$\ddot{\vec{z}} = \dots + \dot{\vec{z}} \times \vec{B} \qquad (6.2.5)$$

Corresponding to a coriolis term $-2 \vec{\omega} \times \dot{\vec{z}}$ with

$$\vec{\omega} = -\frac{1}{2} \vec{B} = \frac{2GM}{c^2 R} \vec{\Omega} \qquad (6.2.6)$$

$$\ddot{\vec{z}} = \cdots + \frac{2GM}{c^2 R} \, 2\vec{\Omega} \times \dot{\vec{z}} \qquad (6.2.7)$$

For that to equal $2\vec{\Omega} \times \dot{\vec{z}}$ we must have

$$\frac{2GM}{c^2 R} = 1 \qquad (6.2.7)$$

We will later see that this is a very suggestive equation. It says that the gravitational radius of our mass distribution equals its Schwarzschild Radius.

$$R = R_S(M) := \frac{2GM}{c^2} \qquad (6.2.8)$$

i.e. our mass-distribution is sufficiently compact to form a black hole (or: a closed universe).

For

$$\begin{aligned} \rho_0 &= 10^{-29} \, g \cdot cm^{-3} \\ &= 10^{-26} \, kg \cdot m^{-3} \end{aligned} \qquad (6.2.9)$$

We get from (6.2.7)

$$\frac{2G\rho_0 \frac{4\pi}{3} R^3}{c^2 R} = 1 \qquad (6.2.10)$$

or

$$R = \frac{c}{2} \cdot \left[\frac{3}{2\pi G \rho_0} \right]^{1/2} \qquad (6.2.11)$$

In SI-units (m, s, kg) have

$$R = 1.5 \cdot 10^{8} \left[\frac{3}{2\pi \cdot 6.67 \cdot 10^{-11} \cdot 10^{-26}} \right]^{1/2} \text{m}$$

$$= 1{,}27 \cdot 10^{26} \text{ m}$$

$$= 13{,}41 \cdot 10^{9} \text{ ly} \qquad (6.2.12)$$

[ly = light-year = $9{,}46 \times 10^{15}$ m.]

The radius (6.2.12) is surprisingly close to the Hubble-Radius or

c x Age of universe

$= c \cdot 13{,}7 \cdot 10^{9}$ years

$= 13{,}7$ ly.

Problem 3

Formula (11.76) of the lecture-notes reads

$$\vec{\omega}_{TL}(\vec{x}) = \frac{2}{5}\,\frac{G}{c^2}\,MR^2\,\frac{3(\vec{\Omega}\cdot\vec{n})\vec{n}-\vec{\Omega}}{r^3} \qquad (6.3.1)$$

Here M = total mass of body
R = radius of body
$\vec{\Omega}$ = angular velocity

Applied to Earth

$$\left.\begin{aligned} M &= 5.972 \times 10^{24}\ \text{kg} \\ R &= 6.371 \times 10^{6}\ \text{m} \\ \Omega &= 2\pi / \text{sideral period} \\ &= 2\pi / 23\text{h}\,56\text{m}\,4.1\text{s} \\ &= 2\pi / 86'164.1\ \text{s} \\ &= 7.29\ 10^{-5}\ \text{s}^{-1} \end{aligned}\right\} (6.3.2)$$

We choose spherical polar coordinates with $\vec{\Omega} = \Omega\vec{e}_z$ pointing along polar axis. Since (6.3.1) only depends on r and $(\Omega\cdot\vec{n}) = \Omega\cos\theta$ but not on azimuth φ we can evaluate it for $\varphi = 0$ i.e. in the $y = 0$ plane, i.e. the xz-plane. Then $\vec{n} = \sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_z$. Also, on the surface of the Earth $r = R$, so that

$$\vec{\omega}_{TL}(\theta) = \frac{2GM}{c^2 R} \cdot \frac{\Omega}{5} \cdot \left[3 (\vec{e}_z \cdot \vec{n}) \vec{n} - \vec{e}_z \right] \tag{6.3.3}$$

Introducing the "gravitational radius"

$$R_g := \frac{2GM}{c^2} \tag{6.3.4}$$

also known as "Schwarzschild-radius", which for the Earth is

$$R_g^{♀} = \frac{2GM_{♀}}{c^2} = \frac{2 \cdot 6.674 \cdot 10^{-11} \cdot 5.972 \cdot 10^{24}}{(2.99792)^2 \cdot 10^{16}} \, m$$

$$= \frac{2 \cdot 6.674 \cdot 5.972}{(2.99792)^2} \cdot 10^{-3} \, m$$

$$= 8.87 \cdot 10^{-3} \, m = 8.87 \, mm \tag{6.3.5}$$

so that (R = geom. radius of Earth)

$$\frac{R_g^{♀}}{R} = \frac{8.87 \cdot 10^{-3}}{6.371 \cdot 10^{6}} = 1.4 \times 10^{-9} . \tag{6.3.6}$$

Hence the prefactor of (6.3.3) is

$$\frac{R_g^{♀}}{R} \frac{\Omega}{5} = 2 \times 10^{-14} \, s^{-1} \tag{6.3.7}$$

The latitude dependent part of (6.3.3) is

$$3(\vec{e}_z \cdot \vec{n})\vec{n} - \vec{e}_z$$

$$= 3\cos\theta(\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_z) - e_z$$

$$= 3\sin\theta\cos\theta\,\vec{e}_x + (3\cos^2\theta - 1)\,\vec{e}_z \qquad (6.3.8)$$

Note that

$$3\sin\theta\cos\theta = \frac{3}{2}\sin(2\theta) \qquad (6.3.9)$$

and

$$3\cos^2\theta - 1 = \frac{3}{2}\left[(\cos^2\theta - \sin^2\theta) + \cos^2\theta + \sin^2\theta)\right] - 1$$

$$= \frac{3}{2}\cos(2\theta) + \frac{1}{2} \qquad (6.3.10)$$

Hence

$$3(e_z \cdot \vec{n})\vec{n} - \vec{e}_z$$

$$= \frac{3}{2}\left[\sin(2\theta)\,e_x + \cos(2\theta)\,e_z\right] + \frac{1}{2}\vec{e}_z \qquad (6.3.11)$$

which consists of a constant part $\frac{1}{2}\vec{e}_z$ and a θ-dependent part which rotates orthogonally with twice the polar angle θ from $\frac{3}{2}\vec{e}_z$ (for $\theta = 0$) to $-\frac{3}{2}\vec{e}_z$ (for $\theta = \frac{\pi}{2}$) and back to $\frac{3}{2}\vec{e}_z$ (for $\theta = \pi$).

For the surface of the Earth we get

$$\vec{\omega}_{TL}(\theta) = 10^{-14} \cdot s^{-1} \times$$

$$\left\{ 3 \left[\sin(2\theta)\vec{e}_x + \cos(2\theta)\vec{e}_z \right] + \vec{e}_z \right\} \tag{6.3.12}$$

Wettzell is located at 49° northern latitude $\Rightarrow 90^\circ - \theta = 49^\circ \Rightarrow \theta = 41^\circ$. Hence

$$\|\vec{\omega}_{TL}(\theta = 41^\circ)\| \cong 3.3 \times 10^{-14} \, s^{-1} \tag{6.3.13}$$

At Wettzell variations of the Earth rotation rate, i.e. variations of the length of a day can be detected with accuracy of 0.1 milliseconds per day. That is

$$\frac{\Delta T}{T} = \frac{10^{-4} \, s}{86400 \, s} = 1{,}16 \cdot 10^{-9} \tag{6.3.14}$$

As $\Omega = 2\pi / T$, have also

$$\left| \frac{\Delta \Omega}{\Omega} \right| = 1{,}16 \cdot 10^{-9} \tag{6.3.15}$$

or $\Delta \Omega = 7{,}29 \cdot 10^{-5} \, s^{-1} \cdot 1{,}16 \cdot 10^{-9}$

$$= 8{,}46 \cdot 10^{-14} \, s^{-1} \tag{6.3.16}$$

According to

$$\Delta\varphi = \frac{8\pi A}{\lambda c}\,\Delta\Omega \qquad (6.3.17)$$

with $A = 16\ m^2$ and $\lambda = 632\ nm$ this corresponds to amazing accuracy of:

$$\Delta\varphi = 2\pi\ \frac{4A}{\lambda c}\ \Delta\Omega$$

$$= 2\pi\ \frac{64\ m^2}{6{,}3\cdot 10^{-7}\cdot 3\cdot 10^{8}}\ \Delta\Omega$$

$$\cong 2\pi\cdot\frac{1}{3}\ \Delta\Omega$$

$$= 2\pi\ 3\cdot 10^{-14} \qquad (6.3.18)$$

Given that, and is already obvious from comparing (6.3.16) and (6.3.13), the ring-laser at Wettzell is about a factor

$$\frac{3{,}3}{8{,}46} \cong 0{,}4 \qquad (6.3.19)$$

away from being able to detect $\vec{\omega}_{TL}$. and hence the gravito-magnetic field of the Earth.

Problem 4

Let $s \mapsto z(s) \in M$ be the timelike world-line in Minkowski-space, parametrised by proper length

$$s = c\tau \,, \quad \tau = \text{proper time} \tag{6.4.1}$$

so that for $\dot{z} = d/ds\, z$ we have

$$\dot{z} \cdot \dot{z} = 1 \tag{6.4.2}$$

where "$\cdot$" stands for the Minkowski inner-product. Clearly

$$\dot{z} \cdot \ddot{z} = 0$$

Let $S \in ST_{\gamma}{}^{1}{}_{0}(M)$ be a vector field along the curve $z(s)$ satisfying the constraint

$$S(s)\, \dot{z}(s) = 0 \tag{6.4.3}$$

for all s.

The Fermi-derivative of a vector field S over z is

$$F_z S = \nabla_{\dot{z}} S + \dot{z}\,(\ddot{z} \cdot S) - \ddot{z}\,(\dot{z} \cdot S) \tag{6.4.4}$$

If (6.4.3) holds then $F_z S = 0$ is equivalent to

$$\nabla_{\dot{z}} S = -\dot{z}\,(\ddot{z} \cdot S) \tag{6.4.5}$$

Where, in general, we have in comp.

$$(\nabla_{\dot{z}} S)^\alpha = \dot{S}^\alpha + \dot{z}^\lambda \Gamma_\lambda{}^\alpha{}_\beta S^\beta \qquad (6.4.6)$$

If we use an affine chart on Minkowski space we have $\Gamma_\lambda{}^\alpha{}_\beta = 0$ and (6.4.5) turns into

$$\dot{S}^\alpha = -\dot{z}^\alpha (\ddot{z}_\beta S^\beta) . \qquad (6.4.7)$$

Since $S \cdot \dot{z} = 0$ there are only 3 independent components of S. Let K and K' denote the instantaneous rest-frame of the particle and the laboratory frame, respectively, so that the four-velocity $\dot{z}$ of the particle has components

$$\left.\begin{aligned} \dot{z}^\alpha &= (1, \vec{0}) \qquad (\text{w.r.t. } K) \\ \dot{z}'^\alpha &= \gamma (1, \vec{\beta}) \qquad (\text{w.r.t. } K') \end{aligned}\right\} (6.4.8)$$

where $\vec{\beta} = \vec{v}/c$ is the velocity/c of K with respect to K' and $\gamma = (1-\beta^2)^{-1/2}$.

Let $(S^0, \vec{S})$ and $(S'^0, \vec{S}')$ be the components of S with respect to K and K', respectively. The same pure boost (i.e. "rotation free") Lorentz transformation that maps $\dot{z}^\alpha = (1, \vec{0})$ to $\dot{z}'^\alpha = \gamma(1, \vec{\beta})$ also maps $(S^0, \vec{S})$ to $(S'^0, \vec{S}')$.

Now

$$\Lambda^{\alpha}{}_{\beta}\, \dot{z}^{\beta} = \dot{z}'^{\alpha} \tag{6.4.9}$$

where

$$\Lambda^{\alpha}{}_{\beta} = \begin{bmatrix} \gamma, & \gamma\vec{\beta}^{T} \\ \gamma\vec{\beta}, & E_3 + (\gamma-1)\,\vec{n}\otimes\vec{n}^{T} \end{bmatrix} \tag{6.4.10}$$

is a pure boost in $\vec{\beta}$-direction, where $\vec{n} = \vec{\beta}/\beta$. Since $\dot{z}\cdot S = 0$ have in K

$$\dot{z}^{\alpha} S_{\alpha} = S_0 = 0 \implies S^0 = 0. \tag{6.4.11}$$

Hence

$$S^{\alpha} = (0, \vec{S}) \tag{6.4.12}$$

and

$$S'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, S^{\beta} \tag{6.4.13}$$

$$\begin{aligned} \leadsto \quad S'^{0} &= \Lambda^{0}{}_{\beta} S^{\beta} = \Lambda^{0}{}_{a} S^{a} \\ &= \gamma(\vec{\beta}\cdot\vec{S}) \end{aligned} \tag{6.4.14}$$

$$\begin{aligned} S'^{a} &= \Lambda^{a}{}_{\beta} S^{\beta} = \Lambda^{a}{}_{b} S^{b} \\ &= S^{a} + (\gamma-1)\, n^{a}\, (\vec{n}\cdot\vec{S}) \end{aligned} \tag{6.4.15}$$

or

$$\vec{S}' = \vec{S} + (\gamma - 1)\,(\vec{S}\cdot\vec{n})\,\vec{n} \qquad (6.4.16)$$

with $\vec{n} = \vec{\beta}/\beta$

Alternatively, using $\gamma^2 = (1-\beta^2)^{-1}$, hence

$$\beta^2 = 1 - \gamma^{-2}$$

$$\vec{S}' = \vec{S} + \frac{\gamma - 1}{\beta^2}\,(\vec{S}\cdot\vec{\beta})\,\vec{\beta}$$

$$= \vec{S} + \frac{\gamma^2}{\gamma + 1}\,(\vec{S}\cdot\vec{\beta})\,\vec{\beta} \qquad (6.4.17)$$

This shows that

$$S'^{\alpha} = \left(\gamma\,(\vec{\beta}\cdot\vec{S}),\ \vec{S} + \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\,\vec{\beta}\right) \qquad (6.4.18)$$

are the components of the spin-vector in the laboratory frame K' in terms of its three independent components $\vec{S}$ in the instantaneous rest frame of the particle that is not rotated relative to the laboratory frame, i.e. related to it by a pure boost.

[Recall definition of a "pure boost" relative to a "frame" or "observer" (the notion of a "pure boost as such" does not exist) : Let v be a timelike vector, "the observer", a pure boost relative to v is a Lorentz transformation that only moves points in the timelike plane $\mathrm{Span}\{v, n\}$, where n is some spacelike vector that w.l.o.g we can choose s.t. $v \cdot n = 0$, and pointwise fixes all points in the spacelike plane orthogonal to $\mathrm{Span}\{v, n\}$. Such a pure boost is always of the form $\exp(n \otimes v_\downarrow - v \otimes n_\downarrow)$.]

Using the parametrization (6.4.8) and (6.4.18) we evaluate (6.4.7) in the laboratory frame

$$\dot{S}'^{\alpha} = - \dot{Z}'^{\alpha} \, (\ddot{Z}'_{\beta} \, S'^{\beta}) \qquad (6.4.19)$$

Note

$$\dot{\gamma} = \frac{d}{ds} \left(1 - \vec{\beta}^2(s)\right)^{-1/2}$$

$$= \frac{1}{2} \left(1 - \vec{\beta}^2\right)^{-3/2} \, 2 \, \vec{\beta} \cdot \dot{\vec{\beta}}$$

$$= \gamma^3 \, (\vec{\beta} \cdot \dot{\vec{\beta}}) \qquad (6.4.20)$$

and

$$\ddot{z}'^{\alpha} = (\dot{\gamma},\ \dot{\gamma}\vec{\beta} + \gamma\dot{\vec{\beta}}) \qquad (6.4.21)$$

$$\rightsquigarrow\ \ddot{z}\cdot S = \eta_{\alpha\beta}\,\ddot{z}'^{\alpha} S'^{\beta} = \ddot{z}'_{\beta} S'^{\beta}$$

$$= \left(\dot{\gamma}\,\gamma\,(\vec{\beta}\cdot\vec{S}) - (\dot{\gamma}\vec{\beta} + \gamma\dot{\vec{\beta}})\left(\vec{S} + \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\vec{\beta}\right)\right)$$

$$= \dot{\gamma}\gamma(\vec{\beta}\cdot\vec{S}) - \dot{\gamma}(\vec{\beta}\cdot\vec{S}) - \frac{\dot{\gamma}\gamma^2}{\gamma+1}\beta^2(\vec{\beta}\cdot\vec{S})$$
$$- \gamma(\dot{\vec{\beta}}\cdot\vec{S}) - \frac{\gamma^3}{\gamma+1}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})$$

[using $\frac{\gamma^2}{\gamma+1}\beta^2 = \frac{\gamma^2-1}{\gamma+1} = \gamma - 1$ we see that the 3rd term cancels the first two]

$$= -\gamma\left[(\dot{\vec{\beta}}\cdot\vec{S}) + \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})\right] \qquad (6.4.22)$$

Hence (6.4.19) reads in full glory:

$$\left.\begin{aligned} &\left(\gamma(\vec{\beta}\cdot\vec{S}),\ \vec{S} + \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\vec{\beta}\right)^{\bullet} \\ &= (1,\vec{\beta})\ \gamma^2\left[\dot{\vec{\beta}}\cdot\vec{S} + \frac{\gamma^2}{1+\gamma}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})\right]\end{aligned}\right\} (6.4.23)$$

which constitutes one equation for the time component and three equations for the space components. They are

$$\left(\gamma\,(\vec\beta\cdot\vec S)\right)^{\circ} = \gamma^2\left[\dot{\vec\beta}\cdot\vec S + \frac{\gamma^2}{\gamma+1}(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta)\right] \quad (6.4.24)$$

$$\left(\vec S + \frac{\gamma^2}{\gamma+1}(\vec S\cdot\vec\beta)\,\vec\beta\right)^{\cdot} = \vec\beta\;\gamma^2\,[\,\cdots\,] \quad (6.4.25)$$

$$= \vec\beta\;\left(\gamma\,(\vec\beta\cdot\vec S)\right)^{\cdot} \quad (6.4.26)$$

Here we used (6.4.24) to replace the right-hand side of (6.4.25) by a simpler expression (6.4.26). We will evaluate these equations in turn.

Evaluation of (6.4.24)

Using (6.4.20) for $\dot\gamma$, we have

$$\left(\gamma\,(\vec\beta\cdot\vec S)\right)^{\cdot} = \gamma\,\vec\beta\cdot\dot{\vec S} + \gamma\,(\dot{\vec\beta}\cdot\vec S) + \gamma^3\,(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta) \quad (6.4.27)$$

and (6.4.24) is equivalent to

$$\gamma(\vec\beta\cdot\dot{\vec S}) + \gamma(\dot{\vec\beta}\cdot\vec S) + \gamma^3\,(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta)$$

$$= \gamma^2\left[\dot{\vec\beta}\cdot\vec S + \frac{\gamma^2}{\gamma+1}(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta)\right]$$

$$\Leftrightarrow\quad (\vec\beta\cdot\dot{\vec S}) = (\gamma-1)(\dot{\vec\beta}\cdot\vec S) + \left(\frac{\gamma^3}{\gamma+1} - \gamma^2\right)(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta)$$

$$= (\gamma-1)(\dot{\vec\beta}\cdot\vec S) - \frac{\gamma^2}{\gamma+1}(\vec S\cdot\vec\beta)(\dot{\vec\beta}\cdot\vec\beta) \quad (6.4.28)$$

Evaluation of (6.4.26)

$$\dot{\vec{S}} = \vec{\beta}\ [\gamma(\vec{S}\cdot\vec{\beta})]^{\bullet} - \left[\frac{\gamma}{\gamma+1}\ \gamma(\vec{S}\cdot\vec{\beta})\,\vec{\beta}\right]^{\bullet}$$

$$\left.\begin{aligned} &= \left[1 - \frac{\gamma}{\gamma+1}\right]\ (\gamma(\vec{S}\cdot\vec{\beta}))^{\bullet}\ \vec{\beta} \\ &\quad - \left(\frac{\gamma}{\gamma+1}\right)^{\bullet}\ \gamma(\vec{S}\cdot\vec{\beta})\,\vec{\beta} \\ &\quad - \frac{\gamma^2}{\gamma+1}\ (\vec{S}\cdot\vec{\beta})\ \dot{\vec{\beta}} \end{aligned}\right\} (6.4.29)$$

Using

$$\frac{\gamma}{\gamma+1} = \dot{\gamma}\ \frac{d}{d\gamma}\left[\frac{\gamma}{\gamma+1}\right]$$

$$= \gamma^3(\dot{\vec{\beta}}\cdot\vec{\beta})\underbrace{\left[\frac{1}{\gamma+1} - \frac{\gamma}{(\gamma+1)^2}\right]}_{1/(\gamma+1)^2}$$

$$= \frac{\gamma^3}{(\gamma+1)^2}(\dot{\vec{\beta}}\cdot\vec{\beta}) \qquad (6.4.30)$$

We get

$$\dot{\vec{S}} = \frac{\gamma}{\gamma+1}(\dot{\vec{S}}\cdot\vec{\beta})\vec{\beta} + \frac{\gamma}{\gamma+1}(\vec{S}\cdot\dot{\vec{\beta}})\vec{\beta}$$

$$\left.\begin{aligned} &+ \frac{\gamma^3}{\gamma+1}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})\vec{\beta} \\ &- \frac{\gamma^4}{(\gamma+1)^2}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})\vec{\beta} \end{aligned}\right\} \frac{\gamma^3}{(\gamma+1)^2}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\beta)\vec{\beta}$$

$$- \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\dot{\vec{\beta}} \qquad (6.4.31)$$

Elimination of $(\dot{\vec{S}}\cdot\vec{\beta})$ via (6.4.28) in the first term gives

$$\dot{\vec{S}} = \frac{\gamma}{\gamma+1}\left[(\gamma-1)(\dot{\vec{\beta}}\cdot\vec{S}) - \cancel{\frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})}\right]\vec{\beta}$$

$$+ \frac{\gamma}{\gamma+1}(\vec{S}\cdot\dot{\vec{\beta}})\vec{\beta} + \cancel{\frac{\gamma^3}{(\gamma+1)^2}(\vec{S}\cdot\vec{\beta})(\dot{\vec{\beta}}\cdot\vec{\beta})\vec{\beta}}$$

$$- \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\dot{\vec{\beta}}$$

$$= \frac{\gamma^2}{\gamma+1}\left[(\vec{S}\cdot\dot{\vec{\beta}})\vec{\beta} - (\vec{S}\cdot\vec{\beta})\dot{\vec{\beta}}\right]$$

$$= \frac{\gamma^2}{\gamma+1}\vec{S}\times(\vec{\beta}\times\dot{\vec{\beta}})$$

$$\left.\begin{aligned} &= \vec{\omega}_T \times \vec{S} \\ &\text{where } \vec{\omega}_T = \frac{\gamma^2}{\gamma+1}(\dot{\vec{\beta}}\times\vec{\beta}) \end{aligned}\right\} (6.4.32)$$

Note that we may replace $\dot{} = \frac{d}{ds} = \frac{1}{c}\frac{d}{d\tau}$ by $d/d\tau$ so that

$$\left.\begin{aligned} \frac{d\vec{S}}{d\tau} &= \omega_T \times \vec{S} \\ \vec{\omega}_T &= \frac{\gamma^2}{\gamma+1}\left(\frac{d\vec{\beta}}{d\tau} \times \vec{\beta}\right) \end{aligned}\right\} \quad (6.4.33)$$

This $\vec{\omega}_T$ is called the "Thomas frequency". It describes the proper-time angular frequency with which the spin vector precesses in a local rest frame that is non-rotating with respect to a fixed global inertial system (the "laboratory frame"), i.e. instantaneously related to the latter by a pure boost (at each instant).

Note that (6.4.32) gives back (6.4.28):

$$\begin{aligned} (\vec{\beta}\cdot\dot{\vec{S}}) &= \frac{\gamma^2}{\gamma+1}\left((\vec{S}\cdot\dot{\vec{\beta}})\,\beta^2 - (\vec{S}\cdot\vec{\beta})\,\dot{\vec{\beta}}\right) \\ &= (\gamma-1)\,(\vec{S}\cdot\dot{\vec{\beta}}) - \frac{\gamma^2}{\gamma+1}(\vec{S}\cdot\vec{\beta})\,\dot{\vec{\beta}} \end{aligned} \quad (6.4.34)$$

Hence (6.4.32) is, in fact, equivalent to (6.4.23).