Sheet 7: Solutions

Problem 1:

See solution to Problem 6 of Sheet 5.

Problem 2

$$k^\alpha \tilde{h}_{\alpha\beta}(k) = 0 \qquad (7.2.1)$$

$$v^\alpha \tilde{h}_{\alpha\beta}(k) = 0 \qquad (7.2.2)$$

$$\eta^{\alpha\beta} \tilde{h}_{\alpha\beta}(k) = 0 \qquad (7.2.3)$$

(7.2.1-2) imply that the contraction of $\tilde{h}_{\alpha\beta}(k)$ with λ^α vanishes for any $\lambda \in \mathrm{Span}\{k, v\}$. For a plane wave with $k_* = (\omega/c)(e_0 + e_3)$ and the choice $v = e_0$ this means that $\tilde{h}(k_*)$ has no component in e_0 and e_3 direction, that is, the only non-vanishing components are those with indices $\in \{1,2\}$. (7.2.3) then implies $\tilde{h}_{11} = -\tilde{h}_{22} =: \tilde{h}_+$ and symmetry implies $\tilde{h}_{12} = \tilde{h}_{21} =: \tilde{h}_\times$. Hence

$$\hat{h}(k_*) = \tilde{h}_{\alpha\beta}(k)\, \theta^\alpha \otimes \theta^\beta$$
$$= \tilde{h}_+ (\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2) + \tilde{h}_\times (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1). \qquad (7.2.4)$$

The two polarisation degrees of freedom corresponding to the amplitudes $\tilde{h}_+$ and $\tilde{h}_\times$ are given by the vectors

$$\left.\begin{aligned} \varphi_+ &:= (\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2) \\ \varphi_\times &:= (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \end{aligned}\right\} \quad (7.2.5)$$

Note that $\{\theta^\alpha : \alpha = 0, 1, 2, 3\}$ is an orthonormal Basis of V^* and hence

$$\{\theta^{\alpha\beta} := \theta^\alpha \otimes \theta^\beta \mid 0 \leq \alpha \leq \beta \leq 4\}$$

is a bases of $V^* \vee V^*$.

$\vee$ ↳ symmetrised $\otimes$.

which is also orthonormal w.r.t. $\eta^\otimes = \eta^{-1} \otimes \eta^{-1}$:

$$\eta^\otimes (\theta^{\alpha\beta}, \theta^{\mu\nu}) = \eta^{\alpha\mu} \eta^{\beta\nu} \quad (7.2.6)$$

Hence

$$\left.\begin{aligned} \eta^\otimes (\varphi_+, \varphi_+) &= \eta^{11}\eta^{11} + \eta^{22}\eta^{22} = 2 \\ \eta^\otimes (\varphi_\times, \varphi_\times) &= \eta^{11}\eta^{22} + \eta^{22}\eta^{11} = 2 \\ \eta^\otimes (\varphi_+, \varphi_\times) &= \eta^{11}\eta^{12} + \eta^{12}\eta^{11} - \eta^{21}\eta^{22} - \eta^{22}\eta^{21} \\ &= 0. \end{aligned}\right\} \quad (7.2.7)$$

Hence the basis $\{\varphi_+, \varphi_\times\}$ for the 2-dimensional subspace of symmetric trace-free tensors orthogonal to Span$\{k_*, v\}$ is orthogonal, but not orthonormal; they are normalised to $\sqrt{2}$. That is not really important for what is to follow, as long as φ_+ and $\varphi_\times$ have the same norm. We could just as well have defined $\tilde{h}'_+ = \sqrt{2}\,\tilde{h}_+$ and $\tilde{h}'_\times = \sqrt{2}\,\tilde{h}_\times$ to be our amplitude function.

A rotation in the oriented plane

$$E_{12} := \text{Span}\{e_1, e_2\} \tag{7.2.8}$$

is given by

$$\left.\begin{aligned} R_\varphi(e_1) &= \cos(\varphi)\, e_1 + \sin(\varphi)\, e_2 \\ R_\varphi(e_2) &= -\sin(\varphi)\, e_1 + \cos(\varphi)\, e_2 \end{aligned}\right\} \tag{7.2.9}$$

where φ is counted as positive for $e_1 \to e_2$, i.e. positive in the given orientation. Hence

$$\left.\begin{aligned} R_\varphi = {} & \cos(\varphi)\,(e_1 \otimes \theta^1 + e_2 \otimes \theta^2) \\ & + \sin(\varphi)\,(e_2 \otimes \theta^1 - e_1 \otimes \theta^2)\,. \end{aligned}\right\} \tag{7.2.10}$$

That rotation is represented on the dual space by its inverse-transposed

$$R^*_\varphi(\theta^\alpha) = (R_\varphi^{-1})^T(\theta^\alpha) := \theta^\alpha \circ R^{-1}_\varphi$$

$$= \theta^\alpha \circ R_{-\varphi} \quad (7.2.11)$$

Hence

$$R^*_\varphi(\theta^1) = \theta^1 \circ R_{-\varphi}$$

$$= \theta^1 \Big[\cos(\varphi)\,(e_1 \otimes \theta^1 + e_2 \otimes \theta^2)$$

$$- \sin(\varphi)\,(e_2 \otimes \theta^1 - e_1 \otimes \theta^2)\Big]$$

$$= \cos(\varphi)\,\theta^1 + \sin(\varphi)\,\theta^2 \quad (7.2.12)$$

$$R^*_\varphi(\theta^2) = \theta^2 \circ R_{-\varphi}$$

$$= \theta^2 \Big[\cos(\varphi)\,(e_1 \otimes \theta^1 + e_2 \otimes \theta^2)$$

$$- \sin(\varphi)\,(e_2 \otimes \theta^1 - e_1 \otimes \theta^2)\Big]$$

$$= \cos(\varphi)\,\theta^2 - \sin(\varphi)\,\theta^1 \quad (7.2.13)$$

It is no surprise that these are just the same coefficient as for the dual basis, since for orthogonal transformations transposition and inversion cancel each other on the level of matrices that refer to mutually dual bases (the transposed map acts on V^*, not V, i.e. on a different vector space.).

The representation of rotations on $V^*\otimes V^*$ is by the tensor product of the inverse transposed:

$$\varphi \longmapsto (R_\varphi^{-1})^T \otimes (R_\varphi^{-1})^T =: R_\varphi^{*\otimes} \qquad (7.2.14)$$

$$R_\varphi^{*\otimes}(\theta^\alpha \otimes \theta^\beta) = (\theta^\alpha \circ R_{-\varphi}) \otimes (\theta^\beta \circ R_{-\varphi}) \qquad (7.2.15)$$

This implies via

$$R_\varphi^{*\otimes}(\theta^1 \otimes \theta^1)$$

$$= \cos^2(\varphi)\, \theta^1 \otimes \theta^1 + \sin^2(\varphi)\, \theta^2 \otimes \theta^2$$

$$+ \sin(\varphi)\cos(\varphi)\, (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \qquad (7.2.16)$$

$$R_\varphi^{*\otimes}(\theta^2 \times \theta^2)$$

$$= \cos^2(\varphi)\, \theta^2 \otimes \theta^2 + \sin^2(\varphi)\, \theta^1 \otimes \theta^1$$

$$- \sin(\varphi)\cos(\varphi)\, (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \qquad (7.2.17)$$

$$R_\varphi^{*\otimes}(\theta^1 \otimes \theta^2)$$

$$= -\sin(\varphi)\cos(\varphi)(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2)$$

$$+ \cos^2(\varphi)\, \theta^1 \otimes \theta^2 - \sin^2(\varphi)\, \theta^2 \otimes \theta^1 \qquad (7.2.18)$$

$$R_\varphi^{*\otimes}(\theta^2 \otimes \theta^1)$$

$$= -\sin(\varphi)\cos(\varphi)(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2)$$

$$+ \cos^2(\varphi)\, \theta^2 \otimes \theta^1 - \sin^2(\varphi)\, \theta^1 \otimes \theta^2 \qquad (7.2.19)$$

Hence

$$R_\varphi^{*\otimes}(\varphi_+) = R_\varphi^{*\otimes}(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2)$$

$$= (\cos^2(\varphi) - \sin^2(\varphi))(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2)$$

$$+ 2\sin(\varphi)\cos(\varphi)(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1)$$

$$= \cos(2\varphi)\, \varphi_+ + \sin(2\varphi)\, \varphi_\times \qquad (7.2.20)$$

$$R_\varphi^{*\otimes}(\varphi_\times) = R_\varphi^{*\otimes}(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1)$$

$$= -2\sin(\varphi)\cos(\varphi)(\theta^1 \otimes \theta^1 - \theta^2 \otimes \theta^2)$$

$$+ \cos^2(\varphi) - \sin^2(\varphi)(\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1)$$

$$= -\sin(2\varphi)\, \varphi_+ + \cos(2\varphi)\, \varphi_\times \qquad (7.2.21)$$

So if

$$\tilde{h} = h_+ \varphi_+ + h_\times \varphi_\times \qquad (7.2.22)$$

then

$$R^{*\otimes}_\varphi \tilde{h} = h_+ R^{*\otimes}_\varphi (\varphi_+) + h_\times R^{*\otimes}_\varphi (\varphi_\times)$$

$$= h_+ (\cos(2\varphi)\varphi_+ + \sin(2\varphi)\varphi_\times)$$

$$+ h_\times (-\sin(2\varphi)\varphi_+ + \cos(2\varphi)\varphi_\times)$$

$$= [h_+ \cos(2\varphi) - h_\times \sin(2\varphi)]\varphi_+$$

$$+ [h_+ \sin(2\varphi) + h_\times \cos(2\varphi)]\varphi_-$$

$$=: h'_+ \varphi_+ + h'_\times \varphi_- \qquad (7.2.23)$$

where

$$\begin{pmatrix} h'_+ \\ h'_\times \end{pmatrix} = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix} \qquad (7.2.24)$$

This transformation is again orthogonal; hence $\varphi \mapsto R^{*\otimes}_\varphi$ is an orthogonal representation of $SO(2)$ on the 2-dim subspace of $V^* \otimes V^*$ given by transverse (w.r.t. k_* and v) traceless tensors with inner product $\eta^{-1} \otimes \eta^{-1}$.

Problem 3

Since we use the TT-gauge only $\tilde{h}_+ = \tilde{h}_{11} = -\tilde{h}_{22}$ and $\tilde{h}_\times = \tilde{h}_{12} = \tilde{h}_{21}$ are non-zero.
Since

$$\Box h_+ = \Box h_\times = 0 \tag{7.3.1}$$

and h_+ and $h_\times$ only depend on a single

$$k = k_* = \frac{\omega}{c}(e_0 + e_3) \tag{7.3.2}$$

which means that

$$k \cdot x = \frac{\omega}{c}(ct - z) \tag{7.3.3}$$

h_+ and $h_\times$ only depend on the combination $ct - z$. For the wave to propagate in positive z-direction for increasing t the compination is $z - ct$. Because then the phase that at time t was at z will at time $t + \Delta t$ be at $z + c\Delta t$. Hence

$$\left.\begin{aligned} g = c\,dt \otimes c\,dt &- (1 - h_+(z-ct))\, dx \otimes dx \\ &- (1 + h_+(z-ct))\, dy \otimes dy \\ &- dz \otimes dz \\ &+ h_\times(z-ct)(dx \otimes dy + dy \otimes dx) \end{aligned}\right\} \tag{7.3.4}$$

The energy functional is

$$
\begin{aligned}
E(\dot z(\lambda)) &= \frac{1}{2}\int g_{\alpha\beta}(z(\lambda))\,\dot z^{\alpha}(\lambda)\,\dot z^{\beta}(\lambda)\,d\lambda \\
&= \frac{1}{2}\int d\lambda\,\Big\{ c^2\dot t^2 - \left[1-h_+(z(\lambda)-ct(\lambda))\right]\dot x^2(\lambda) \\
&\qquad - \left[1+h_+(z(\lambda)-ct(\lambda)\right]\dot y^2(\lambda) \\
&\qquad - \dot z^2(\lambda) \\
&\qquad + 2\,h_\times(z(\lambda)-ct(\lambda))\,\dot x(\lambda)\,\dot y(\lambda)
\end{aligned}
\tag{7.3.5}
$$

$$= \int d\lambda\ L(z^{\alpha}(\lambda), \dot z^{\alpha}(\lambda))$$

The Euler-Lagrange equations for (t, x, y, z) are as follows (we write h_+' and $h_\times'$ for the derivatives with respect to the argument, so that

$$\left.\begin{aligned}
&\frac{\partial}{\partial z} h_+ = h_+', \quad \frac{\partial}{\partial t} h_+ = -c\,h_+' \\
&\text{and similarly for } h_\times
\end{aligned}\right\}\tag{7.3.6}$$

We now evaluate the Euler-Lagrange equations for $\alpha = 0, 1, 2, 3$ (t, x, y, z):

$$\frac{\partial L}{\partial z^{\alpha}} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot z^{\alpha}} = 0 \tag{7.3.7}$$

t)
$$\frac{\partial L}{\partial t} = -\frac{c}{2} h'_{+} (\dot{x}^2 - \dot{y}^2)$$
$$- c\, h'_{x} \dot{x} \dot{y}$$

$$\frac{\partial L}{\partial \dot{t}} = c^2 \dot{t}$$

$$\Rightarrow \quad c\ddot{t} = -\frac{h'_{+}}{2} (\dot{x}^2 - \dot{y}^2) - h'_{x} \dot{x} \dot{y} \qquad (7.3.8)$$

x)
$$\frac{\partial L}{\partial x} = 0$$
$$\frac{\partial L}{\partial \dot{x}} = -(1 - h_{+}) \dot{x}$$

$$\Rightarrow \quad \dot{x}(1 + h_{+}) = A = \text{const} \qquad (7.3.9)$$

y)
$$\frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = -(1 + h_{+}) \dot{y}$$

$$\Rightarrow \quad \dot{y}(1 + h_{+}) = B = \text{const.} \qquad (7.3.10)$$

z)
$$\frac{\partial L}{\partial z} = \frac{1}{2} h'_{+} (\dot{x}^2 - \dot{y}^2) + h'_{x} \dot{x} \dot{y}$$

$$\frac{\partial L}{\partial \dot{z}} = -\dot{z}$$

$$\Rightarrow \quad \ddot{z} = \frac{h'_{+}}{2} (\dot{x}^2 - \dot{y}^2) + h'_{x} \dot{x} \dot{y} \qquad (7.3.11)$$

From (7.3. 8-11) we see that

$$x = x_0 , \quad y = y_0 , \quad z = z_0 \qquad (7.3.12)$$

where x_0, y_0, z_0 are constants, are solutions, and

$$\ddot{t} = 0 \Leftrightarrow t(\lambda) = a\lambda + b \qquad (7.3.13)$$

This is true for all h_+ and $h_\times$

Therefore, if h_+ and $h_\times$ has support on the negative real axis, so that, e.g.,

$$h_+(z-ct) = 0 \quad \forall \; z-ct \leq 0 \qquad (7.3.14)$$

then at $z=0$, if for $t \leq 0$, all particles had been at fixed coordinate values, they stay at these fixed values for all $t \geq 0$. If they are distributed on a circle

$$x^2 + y^2 = R^2 \qquad (7.3.15)$$

in the $z = 0$ plane, their mutual instantaneous distance for $t < 0$, where $h_+ = h_\times = 0$, is that of euclidean space. For $t \geq 0$, however, when h_+ and $h_\times$ start to become $\neq 0$ at $z=0$, their spacelike separations start to change.

If $h_\times = 0$ and $h_+ \neq 0$ at time t, the spatial separation of two points a coordinate intervall ΔX apart on the x-axis (i.e. $\Delta y = \Delta z = 0$) is

$$\Delta_x S = \int_x^{x+\Delta x} dx \, \left[1 - h_+(-ct)\right]^{1/2}$$

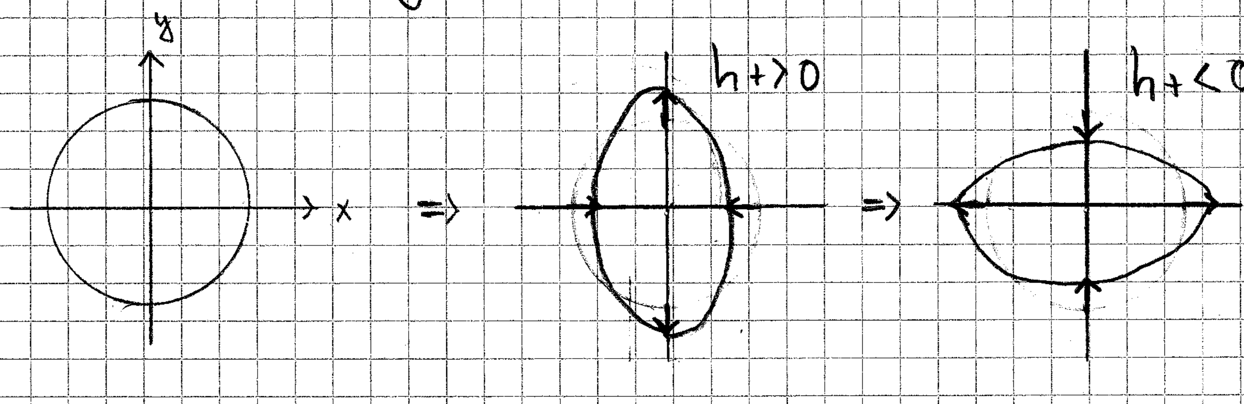
$$\overset{(1)}{=} \left[1 - \frac{1}{2} h_+(-ct)\right] \Delta X \qquad (7.3.16)$$

Their distance before $h_+ \neq 0$ set in was simply $\Delta S = \Delta X$. So the relative change in distance is

$$\frac{\Delta_x S - \Delta X}{\Delta X} = -\frac{1}{2} h_+ \qquad (7.3.17)$$

Likewise for points along the y-axis:

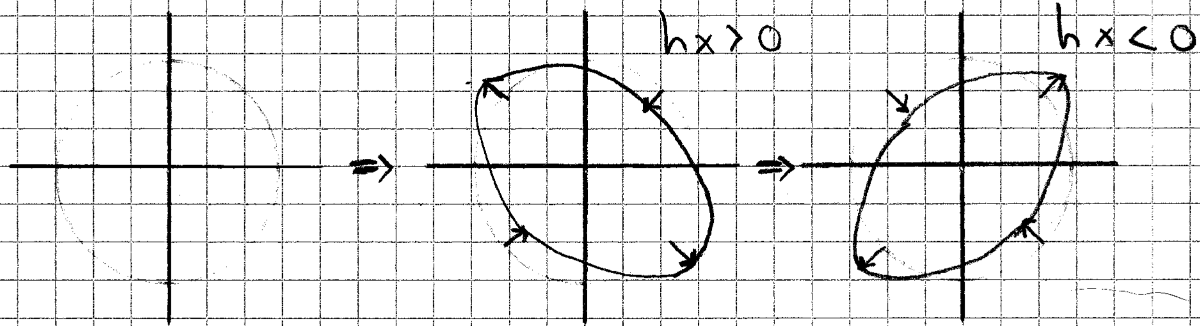
$$\frac{\Delta_y S - \Delta y}{\Delta y} = \frac{1}{2} h_+ \qquad (7.3.18)$$



From equation (7.2.24) of the previous problem we see that for $\varphi = \frac{\pi}{4}$

$$\left.\begin{aligned} h'_+ &= -h_\times \\ h'_\times &= h_+ \end{aligned}\right\} \quad (7.3.19)$$

This means that the $h_\times$-mode is obtained from the h_+ mode by a $45°$-rotation



The particles at fixed coordinates "move" relative to each other, in the sense that their mutual spatial distances change. The coordinate-system is "geodesically comoving" since the world-lines $\vec{X} = \vec{X}_0$ are geodesics.

Problem 4

Let $\{e_\alpha : \alpha = 0,1,2,3\}$ be orthonormal

$$g(e_\alpha, e_\beta) = \mathrm{diag}(1,-1,-1,-1) \qquad (7.4.1)$$

adapted

$$e_0 = K / \sqrt{g(K,K)} \qquad (7.4.2)$$

and stationary

$$L_K e_\alpha = 0 \qquad (7.4.3)$$

Since K is Killing

$$L_K g = 0 \qquad (7.4.4)$$

We have, due to (7.4.4) and $L_K K = [K,K] = 0$:

$$\left.\begin{aligned} K(g(K,K)) &= L_K(g(K,K)) \\ &= (L_K g)(K,K) + 2g(L_K K, K) = 0 \end{aligned}\right\} \quad (7.4.5)$$

Therefore

$$L_K e_0 = L_K\left(\frac{K}{\sqrt{g(K,K)}}\right) = 0 \qquad (7.4.6)$$

So (7.4.3) is satisfied automatically if e_0 is chosen according to (7.4.2).

If the spatial e_a, $a = 1, 2, 3$, are chosen to be propagated along the integral curves of K by (7.4.3) they maintain (7.4.1), since

$$\begin{aligned} K(g(e_\alpha, e_\beta)) &= L_K(g(e_\alpha, e_\beta)) \\ &= (L_K g)(e_\alpha, e_\beta) + g(L_K e_\alpha, e_\beta) \\ &\qquad + g(e_\alpha, L_K e_\beta) = 0 . \end{aligned} \tag{7.4.7}$$

Let γ be an integral curve of K, reparametrised to proper length so that

$$\dot\gamma = K \,/\, \sqrt{g(K,K)} \tag{7.4.8}$$

Let $S \in ST_\gamma M$ obeying

$$\begin{aligned} F_\gamma S &\equiv \nabla_{\dot\gamma} S + g(\ddot\gamma, S)\dot\gamma - g(\dot\gamma, S)\ddot\gamma \\ &= 0 \end{aligned} \tag{7.4.9}$$

If $S \perp \dot\gamma$ at a point p on γ then this will be true for all points on γ. Indeed, $F_\gamma g = 0$ and $F_\gamma \dot\gamma = 0$ give

$$\begin{aligned} \dot\gamma(g(\dot\gamma, S)) &= F_\gamma(g(\dot\gamma, S)) \\ &= (\cancel{F_\gamma g})(\dot\gamma, S) + g(\cancel{F_\gamma \dot\gamma}, S) + g(\dot\gamma, \cancel{F_\gamma S}) \\ &= 0 . \end{aligned} \tag{7.4.10}$$

A field S that satisfies (7.4.9) and $g(\dot\gamma, S) = 0$ satisfies (7.4.9) without the last term:

$$\nabla_{\dot\gamma} S = - g(\ddot\gamma, S)\,\dot\gamma \qquad (7.4.11)$$

Conversely, if (7.4.11) holds, then

$$\dot\gamma\,(g(\dot\gamma, S)) = \nabla_{\dot\gamma}\,(g(\dot\gamma, S))$$

$$= \cancel{(\nabla_{\dot\gamma} g)}(\dot\gamma, S) + g(\ddot\gamma, S) + g(\dot\gamma, \nabla_{\dot\gamma} S)$$

$$= g(\ddot\gamma, S) - g(\dot\gamma, \dot\gamma)\, g(\ddot\gamma, S) = 0 \qquad (7.4.12)$$

So for a field S that is initially orthogonal to $\dot\gamma$, (7.4.9) and (7.4.11) are equivalent.

If $\{e_\alpha : \alpha = 0, 1, 2, 3\}$ is any orthonormal and adapted basis, not necessarily stationary. That is, we have (7.4.1) and (7.4.2) but not necessarily (7.4.3). We set

$$S = S^a e_a \qquad (7.4.13)$$

(no $S^0 e_0$-term since $g(e_0, S) = 0$).

We recall the definition of the connection coefficients ω

$$\nabla_{e_\alpha} e_\beta = \omega_{\alpha\beta}^{\gamma} e_\gamma \qquad (7.4.14)$$

then

$$\left.\begin{aligned} \nabla_{\dot\gamma}(S^a e_a) &= \dot\gamma(S^a) e_a + S^a \nabla_{e_0} e_a \\ &= \dot S^a e_a + S^b \omega_{0b}^{0} e_0 + S^b \omega_{0b}^{a} e_a \end{aligned}\right\} (7.4.15)$$

and

$$\begin{aligned} -g(\ddot\gamma, S)\dot\gamma &= -g(\nabla_{\dot\gamma}\dot\gamma, S)\dot\gamma \\ &= -g(\nabla_{e_0} e_0, S) e_0 \\ &= -\omega_{00}^{a} g(e_a, S) e_0 \\ &= -\omega_{00}^{a} S_a e_0 \qquad (7.4.16) \end{aligned}$$

Hence (7.4.11) gives

$$\begin{aligned} &(\dot S^a e_a + \omega_{0b}^{a} S^b) e_b \\ &+ (S^b \omega_{0b}^{0} + S_b \omega_{00}^{b}) e_0 = 0 \qquad (7.4.17) \end{aligned}$$

But since the ω's are the connection coefficients with respect to an orthonormal basis of a metric connection, we have

$$\omega_{\alpha\beta\gamma} = \eta_{\beta\lambda}\, \omega_\alpha{}^\lambda{}_\gamma$$

$$= \varepsilon_{(\beta)}\, \omega_\alpha{}^\beta{}_\gamma \qquad (7.4.18)$$

$$\text{where } \varepsilon_\beta = \left\{ \begin{array}{ll} 1 & \text{for } \beta = 0 \\ -1 & \text{for } \beta = 1,2,3 \end{array} \right\} \qquad (7.4.19)$$

and an index in brackets indicates that it is not summed over.

Metricity hence implies

$$\omega_{\alpha\beta\gamma} = -\,\omega_{\alpha\gamma\beta} \qquad (7.4.20)$$

$$\Longleftrightarrow \quad \omega_\alpha{}^\beta{}_\gamma = \varepsilon_{(\beta)}\, \varepsilon_{(\gamma)}\, \omega_\alpha{}^\gamma{}_\beta \qquad (7.4.21)$$

In particular

$$\omega_0{}^0{}_b = \varepsilon_{(0)}\, \varepsilon_{(b)}\, \omega_0{}^b{}_0$$

$$= -\,\omega_0{}^b{}_0 \qquad (7.4.22)$$

which included into (7.4.17) shows that the coefficients $\sim e_0$ vanish, as it must be due to the very definition of the Fermi derivative

$$F_\gamma = P_\perp \circ \nabla_{\dot\gamma} \circ P_\perp + P_\parallel \circ \nabla_{\dot\gamma} \circ P_\parallel \qquad (7.4.23)$$

$P_\perp$ and $P_\parallel$ refer to $\dot\gamma = e_0$

Applied to S, where $P_{\parallel} S = 0$, $P_{\perp} S = S$, we have

$$F_\gamma S = P_\perp \nabla_{\dot\gamma} S$$
$$= P_\perp (\nabla_{\dot\gamma} S^a e_a)$$
$$= P_\perp (\dot S^a e_a + S^b \omega_0{}^a{}_b e_a + S^b \omega_0{}^0{}_b e_0)$$
$$= (\dot S^a + \omega_0{}^a{}_b S^b) e_a \qquad (7.4.24)$$

So, in any case, we have

$$F_\gamma S = 0 \iff \dot S^a = -\omega_0{}^a{}_b S^b \qquad (7.4.25)$$

or

$$\left.\begin{aligned} \dot S^1 &= -\omega_0{}^1{}_2 S^2 - \omega_0{}^1{}_3 S^3 \\ \dot S^2 &= -\omega_0{}^2{}_1 S^1 - \omega_0{}^2{}_3 S^3 \\ \dot S^3 &= -\omega_0{}^3{}_1 S^1 - \omega_0{}^3{}_2 S^2 \end{aligned}\right\} \quad (7.4.26)$$

This may be written like

$$\dot{\vec S} = \vec\omega_T \times \vec S \qquad (7.4.27)$$

with

$$\vec\omega_T = (\omega_0{}^2{}_3, \omega_0{}^3{}_1, \omega_0{}^1{}_2) \qquad (7.4.28)$$

Note that again we used (7.4.21), i.e.

$$\left.\begin{aligned} \omega_0{}^1{}_2 &= -\omega_0{}^2{}_1 \\ \omega_0{}^2{}_3 &= -\omega_0{}^3{}_2 \\ \omega_0{}^3{}_1 &= -\omega_0{}^1{}_3 \end{aligned}\right\} \quad (7.4.29)$$

So far (7.4.27) is totally general. It merely expresses the spin precession rate $\vec{\omega}_T$ in terms of the connection 1-forms for any chosen orthonormal frame field $\{e_0, e_1, e_2, e_3\}$ along the curve γ, where $\dot{\gamma} = e_0$.

Now we specialize to stationary spacetimes with timelike Killing vector K, $e_0 = K/[g(K,K)]^{1/2}$ and static basis $L_K e_a = 0$. Then, with

$$K^{\downarrow} = g(K, \cdot) \qquad (7.4.30)$$

$$\left.\begin{aligned} dK^{\downarrow}(e_b, e_c) = \; & e_b(K(e_c)) \\ & - e_a(K(e_b)) \\ & - K([e_a, e_b]) \end{aligned}\right\} \quad (7.4.31)$$

by the general formula for d: See (4.66a) in the DiffGeom-notes.

Hence

$$dK^\flat(e_b, e_c)$$

$$= (\nabla_{e_b} K^\flat)(e_c) - (\nabla_{e_c} K^\flat)(e_b)$$

$$+ K^\flat(\underbrace{\nabla_{e_b} e_c - \nabla_{e_c} e_b - [e_b, e_c]}_{T(e_a, e_b) = 0}) \qquad (7.4.32)$$

(since ∇ is torsion free)

Now, since $e_a \perp e_0 = K/\sqrt{\cdots}$ we have

$$K^\flat(e_a) = 0 \qquad (7.4.33)$$

hence

$$(\nabla_{e_b} K^\flat)(e_c) = - K^\flat(\nabla_{e_b} e_c)$$

$$= - g(K, \nabla_{e_b} e_c) = g(\nabla_{e_b} K, e_c)$$

$$\overset{(*)}{=} g(\nabla_K e_b, e_c) \qquad (7.4.34)$$

In the last step we used two things

1.) vanishing torsion:

$$\Rightarrow \nabla_{e_b} K - \nabla_K e_b - [e_b, K] = 0 \qquad (7.4.35)$$

2.) that e_a is stationary:

$$L_K e_b = [K, e_b] = 0 \qquad (7.4.36)$$

Both together imply

$$\nabla_{e_b} K = \nabla_K e_b \qquad (7.4.37)$$

It is that equality that we used in $\overset{(*)}{=}$ of (7.4.34).

Hence (7.4.32) becomes

$$dK^\downarrow(e_b, e_c) = g(\nabla_K e_b, e_c) - g(\nabla_K e_c, e_b)$$

$$= \|K\| \left[g(\nabla_{e_0} e_b, e_c) - g(\nabla_{e_0} e_c, e_b) \right]$$

$$= \|K\| \left[\omega_0{}^\lambda{}_b \, \eta_{\lambda a} - \omega_0{}^\lambda{}_c \, \eta_{\lambda b} \right]$$

$$= \|K\| \left[-\omega_0{}^a{}_b + \omega_0{}^b{}_c \right]$$

$$= 2 \|K\| \, \omega_0{}^b{}_c \qquad (7.4.38)$$

Using (7.4.28) this reads

$$dK^\downarrow(e_b, e_c) = 2 \|K\| \, \varepsilon_{abc} \, \omega_r{}^a \qquad (7.4.39)$$

or

$$\omega_r{}^a = \frac{1}{4} \varepsilon^{abc} \, dK^\downarrow(e_b, e_c) / \|K\| \qquad (7.4.40)$$

where $\varepsilon_{123} = \varepsilon^{123} = 1$ and indices on ε are raised and lowered with $\delta_{ab} = \delta^{ab}$.

Note that in spacetime

$$\mathcal{E} = \frac{1}{4!}\, \varepsilon_{\alpha\beta\gamma\delta}\, \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta$$

$$= \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$$

i.e. $\varepsilon_{0123} = 1$ (7.4.41)

and

$$\varepsilon^{0123} = \eta^{0\alpha}\eta^{1\beta}\eta^{2\gamma}\eta^{3\delta}\,\varepsilon_{\alpha\beta\gamma\delta}$$

$$= \det\{\eta^{\alpha\beta}\} = -1 \qquad (7.4.42)$$

So that, writing $K/\|K\| = u$

$$\varepsilon^{abc} = -u_\lambda\, \varepsilon^{\lambda abc} = -\frac{K^\downarrow_\lambda}{\|K\|}\,\varepsilon^{\lambda abc} \qquad (7.4.43)$$

Hence the covariant form of (7.4.40) is

$$\omega^\alpha_\tau = -\frac{1}{4\|K\|^2}\, K^\downarrow_\lambda\, \varepsilon^{\lambda\alpha\beta\gamma}\, dK^\downarrow(e_\beta, e_\gamma)$$

$$= \frac{-1}{4\|K\|^2}\, \varepsilon^{\lambda\beta\gamma\alpha}\, K^\downarrow_\lambda\, dK^\downarrow(e_\beta, e_\gamma).$$

$$= \frac{-1}{4\|K\|^2}\, \varepsilon^{\lambda\beta\gamma\alpha}\, \underbrace{K^\downarrow_{[\lambda}\, dK^\downarrow_{\beta\gamma]}}_{\text{antisymmetrisation}} \qquad (7.4.44)$$

BW

$$(K^\downarrow \wedge dK^\downarrow)_{\lambda\beta\gamma} = \frac{3!}{2!} K^\downarrow_{[\lambda} \, dK_{\beta\gamma]} \quad (7.4.45)$$

(see (7.15b) of Diff Geom notes)

Hence

$$\omega^\alpha{}_T = \frac{-1}{4\|K\|^2} \frac{1}{3} \varepsilon^{\lambda\beta\gamma\alpha} (K^\downarrow \wedge dK^\downarrow)_{\lambda\beta\gamma}$$

$$- \frac{1}{2\|K\|^2} \left\{ \left[* (K^\downarrow \wedge dK^\downarrow) \right]^\uparrow \right\}^\alpha \quad (7.4.46)$$

Where we used

$$\left[* (K^\downarrow \wedge dK^\downarrow) \right]_\alpha$$

$$= \frac{1}{3!} \varepsilon^{\lambda\beta\gamma}{}_\alpha \, K_\lambda \, dK_{\beta\gamma} \quad (7.4.47)$$

(see formula 7.44 of Diff Geom notes)

In total for $\omega_T \in ST_\gamma M$:

$$\omega_T = - \frac{1}{2\|K\|^2} \left[* (K^\downarrow \wedge dK^\downarrow) \right]^\uparrow \quad (7.4.48)$$

Where $\uparrow$ stand for "index raising" map.

However

$$\frac{1}{\|K\|^2} \; K^{\downarrow} \wedge dK^{\downarrow}$$

$$= \frac{K^{\downarrow}}{\|K\|} \wedge d\left(\frac{K^{\downarrow}}{\|K\|}\right)$$

$$= u^{\downarrow} \wedge du^{\downarrow} \qquad (7.4.49)$$

since the term $K^{\downarrow} \wedge K^{\downarrow} \; d\left(\frac{1}{\|K\|}\right)$ is zero.

Hence, finally,

$$\omega_T = -\frac{1}{2} \left[* \left(u^{\downarrow} \wedge du^{\downarrow} \right) \right]^{\uparrow} \qquad (7.4.50)$$

Applying first $\downarrow$ on both sides and then $*$, using that on 3-forms in 4 dimensions with signature $(+,-,-,-)$ we have

$$* \circ * = (-1)^{3(4-3)} (-1)^{n_-} = +1 \qquad (7.4.51)$$

$$(* \, \omega_T^{\downarrow}) = -\frac{1}{2} \; u^{\downarrow} \wedge du^{\downarrow} \qquad (7.4.52)$$

$$\Rightarrow i_u * \omega_T^{\downarrow} = -\frac{1}{2} \; i_u (u^{\downarrow} \wedge du^{\downarrow}) \qquad (7.4.53)$$

$$= -\frac{1}{2} P_{\perp} (du^{\downarrow}) \qquad (7.4.54)$$

$P_{\perp} = \pi \otimes \pi$ projection $\perp$ to u