Sheet 8 : Solutions

Problem 1

$$h_+(z-ct) = A\cos\left(\frac{\omega}{c}(z-ct)\right) \tag{8.1.1}$$

$$\omega = 2\pi\nu = 2\pi \cdot 100\ s^{-1} \tag{8.1.2}$$

$$A = 10^{-21} \tag{8.1.3}$$

Equation (13.41) reads

$$\langle t_{\mu\nu}\rangle = \frac{c^4}{32\pi G} k_\mu k_\nu A^2 \tag{8.1.4}$$

$$\left.\begin{aligned} &k_0 = \frac{\omega}{c}, \quad k_3 = -\frac{\omega}{c} \\ &\text{or} \quad k^\mu = \frac{\omega}{c}(1,0,0,1) \end{aligned}\right\} \tag{8.1.5}$$

$$\langle t^{03}\rangle = \frac{c^4}{32\pi G}\left(\frac{\omega}{c}\right)^2 A^2$$

$$= \frac{\pi c^2}{8G}\nu^2 A^2$$

$$= \frac{1}{c} S^3$$

$$\rightsquigarrow S = S^3 = \frac{\pi c^3}{8G}\nu^2 A^2 = 1{,}6\cdot 10^{35}\ \frac{W}{m^2} \times \nu^2[Hz]\cdot A^2$$

$$= 1{,}6\cdot 10^{35}\ \frac{W}{m^2}\left(10^{-19}\right)^2 \tag{8.1.6}$$

Therefore, the energy-current density for $\nu = 100\ \mathrm{Hz}$ and $A = 10^{-21}$ is

$$S = 1.6\ \frac{\mathrm{mW}}{\mathrm{m}^2} \tag{8.1.7}$$

Compare with solar constant

$$S_\odot = 1{,}37\ 10^3\ \frac{\mathrm{W}}{\mathrm{m}^2} \tag{8.1.8}$$

A 100 W - light bulb at a distance of 70 m (easily visible)

$$\frac{100\ \mathrm{W}}{4\pi(70\ \mathrm{m})^2} \cong 8 \cdot 10^{-4}\ \frac{\mathrm{W}}{\mathrm{m}^2} = 1{,}6\ \frac{\mathrm{mW}}{\mathrm{m}^2} \tag{8.1.9}$$

$\Rightarrow$ The plane gravitational wave at frequency $\nu = 100\ \mathrm{Hz}$ and amplitude $A = 10^{-21}$ has an energy current density like a 100 W light-bulb seen from a distance of 70 m.

Problem 2

$$L_{GW}^{(\text{rod})} = \frac{2}{45} \cdot \frac{G}{c^5} \cdot \omega^6 \cdot M^2 \cdot L^4 \qquad (8.2.1)$$

The centrifugal force of a mass element dm at distance r from the midpoint is

$$dF = \omega^2 r \, dm(r) \, , \qquad (8.2.2)$$

The sum of one half of the rod is

$$\left.\begin{aligned} F &= \int_0^{L/2} \omega^2 r \, \frac{dm}{dr} \, dr \\ \frac{dm}{dr} &= \frac{M}{L} = \varrho q \end{aligned}\right\} (8.2.3)$$

$$\curvearrowright \quad F = \varrho q \, \omega^2 \, \frac{1}{2} \left(\frac{L}{2}\right)^2 = \frac{1}{2} q \, \varrho \left(\frac{L\omega}{2}\right)^2$$

$$\curvearrowright \quad F/q = \sigma = \frac{1}{2} \varrho \, V^2 \qquad (8.2.4)$$

where $V = \dfrac{L\omega}{2}$ = velocity of rod end.

Each half of the rod pulls with force $F = (q/2) \, \varrho \, V^2$ away from the axis of rotation. The total tension at cross section is F/q (not $2F/q$!).

If σ_{max} is the breaking-tension of some material, we get an upper bound for V:

$$V \leq \left[\frac{2\,\sigma_{max}}{\rho}\right]^{1/2} \qquad (8.2.5)$$

$$L_{GW}^{(rod)} = \frac{2}{45} \cdot \frac{G}{c^5} \cdot \omega^6 \cdot M^2 \cdot L^4$$

$$= \frac{128}{45} \cdot \frac{G}{c^5} \left(\frac{M}{L}\right)^2 V^6$$

$$\leq \frac{1024}{45} \cdot \frac{G}{c^5} \cdot \frac{q^2 \cdot \sigma_{max}^3}{\rho} \qquad (8.2.6)$$

	ρ	σ_{max}
Steel	$7.85 \frac{g}{cm^3}$	$2100\ N/mm^2$
fibreglass	$2.5 \frac{g}{cm^3}$	$4800\ N/mm^2$

(8.2.7)

From that we get for steel

$$\left.\begin{aligned} \rho^{(steel)} &= 7.85 \cdot 10^3 \ \frac{kg}{m^3} \\ \sigma_{max}^{(steel)} &= 2.1 \cdot 10^9 \ \frac{N}{m^2} \end{aligned}\right\} \qquad (8.2.8)$$

$$L_{GW}^{(steel)} \leq 7{,}4 \cdot 10^{-28}\ W \times q^2\,[m^2] \qquad (8.2.9)$$

Likewise for fibreglass

$$\left.\begin{aligned} \rho^{(fg)} &= 2.5 \cdot 10^{3} \ \frac{kg}{m^3} \\ \sigma^{(fg)}_{max} &= 4.8 \cdot 10^{9} \ \frac{kg}{m^2} \end{aligned}\right\} \quad (8.2.10)$$

$$L^{(fg)}_{GW} \leq 2.8 \cdot 10^{-26} \ W \times q^2 \ [m^2] \qquad (8.2.11)$$

The maximal velocities for the rod's ends are from (8.2.5) and (8.2.7):

$$V^{(steel)}_{max} = 731,5 \ \frac{m}{s} = 2633.3 \ \frac{km}{h} \qquad (8.2.12)$$

$$V^{(fg)}_{max} = 1959,6 \ \frac{m}{s} = 7054.5 \ \frac{km}{h} \qquad (8.2.13)$$

Problem 3

The TT amplitude is, according to (14.49)

$$h^{TT}(t,\vec{x}) = \underbrace{-\frac{4G}{c^4}\frac{\omega^2 \varepsilon\Theta}{r}}_{A(r)} \left\{ \cos(2\omega t)\,\psi_+^{TT} + \sin(2\omega t)\,\psi_\times^{TT} \right\} \qquad (8.3.1)$$

where for a rod we have (14.50):

$$\varepsilon\Theta = -\frac{1}{12} M L^2 \qquad (8.3.2)$$

Hence

$$A(r) = \frac{4G}{c^4}\,\frac{ML^2}{12}\,\frac{\omega^2}{r}$$

$$= \frac{2}{3}\cdot\frac{2GM}{c^2 r}\cdot\left(\frac{L\omega}{2c}\right)^2$$

$$= \frac{2}{3}\left(\frac{R_g}{r}\right)\left(\frac{v}{c}\right)^2 \qquad (8.3.3)$$

where R_g = gravitational radius for M

$$= 2GM/c^2 \qquad (8.3.4)$$

v = velocity of rod's ends.

$$= L\omega/2 \qquad (8.3.5)$$

The amplitude is bounded above by

$$v \leq v_{max} = \frac{2\,\sigma_{max}}{\varrho} \qquad (8.3.6)$$

Using $R_g = \frac{2GM}{c^2} = \frac{2G}{c^2}\,\varrho\, L\, q \qquad (8.3.7)$

we get

$$A(r) \leq \frac{4}{3}\,\frac{G}{c^2 r}\,\varrho\, L\, q \cdot \frac{2\,\sigma_{max}}{\varrho\, c^2} \qquad (8.3.8)$$

Cancel!

$$= \frac{8}{3} \cdot \frac{L}{r} \cdot \frac{G \cdot \sigma_{max}}{c^4}\, q \qquad (8.3.9)$$

For fibreglass : $\sigma_{max} = 4.8 \cdot 10^9\ N/m^2$
we get

$$A(r) \leq \left(\frac{L}{r}\right) \cdot q\,[m] \cdot 10^{-35} \qquad (8.3.10)$$

$$\leq q\,[m] \cdot 10^{-35} \qquad (\text{since } r > L)$$

A "strain" of 10^{-35} is at best 12 orders of magnitude away from current detection technology (Geo 600).

Problem 4

See Lecture 14, pages L14.14-19