Sheet 9 : Solutions

Problem 1

We use expressions (15.13-14) for L_{GW}

$$L_{GW} = 7.26 \times 10^{51}\,W \times \left(\frac{R_g^{(1)}}{r}\right)^2 \cdot \left(\frac{R_g^{(2)}}{r}\right)^2 \cdot \left(\frac{R_g^{(12)}}{r}\right) \qquad (9.1.1)$$

$$m_1 = m_{Earth} = 5.972 \times 10^{24}\,kg$$

$$R_g^{(1)} = 2G\,m_{Earth}/c^2 = 8.87 \cdot 10^{-3}\,m \qquad (9.1.2)$$

$$m_2 = m_{sun} = 1.989 \times 10^{30}\,kg$$

$$R_g^{(2)} = 2G\,m_{sun}/c^2 = 2.95 \cdot 10^{3}\,m \qquad (9.1.3)$$

$$m_1 + m_2 \simeq m_{sun}$$

$$R_g^{(12)} \simeq R_g^{(2)} = 2.95 \cdot 10^{3}\,m \qquad (9.1.4)$$

r = mean distance Earth-sun

$$= 1\,AU = 1.52 \times 10^{11}\,m \qquad (9.1.5)$$

$$\Rightarrow \underline{\underline{L_{GW} = 181\,W}} \qquad (9.1.6)$$

Problem 2

A homogeneous ball of radius R and mass M has a moment of inertia (for any axis through the centre)

$$\Theta' = \int \rho(\vec{x})\, r_\perp^2\, d^3x$$

$$= \frac{M}{\frac{4\pi}{3}R^3} \int_0^{2\pi} d\varphi \int_0^R dr\, r^2 \int_0^\pi d\theta \sin\theta\, \underbrace{r^2 \sin^2\theta}_{r_\perp^2}$$

$$= \frac{M}{\frac{4\pi}{3}R^3}\; 2\pi \int_0^R dr\, r^4 \int_{-1}^{+1} d\sigma\, (1-\sigma^2)$$

$$= \frac{3M}{2R^3}\; \frac{1}{5} R^5 \cdot \frac{4}{3} = \frac{2}{5} M R^2 \qquad (9.2.1)$$

Hence the kinetic (rotational) energy is

$$E_{rot} = \frac{1}{2}\Theta' \omega^2 = \frac{1}{5} M R^2 \omega^2$$

$$= \frac{4\pi^2}{5} M R^2 T^{-2} \qquad \left(\omega = \frac{2\pi}{T}\right) \qquad (9.2.2)$$

$$\curvearrowright \quad \dot{E}_{rot} = -\frac{8\pi^2}{5} M R^2\, \dot{T} / T^3 \qquad (9.2.3)$$

$$\left.\begin{aligned} \text{For}\quad M &= 1.5\, M_\odot = 2.98 \times 10^{30}\ \text{kg} \\ T &= 3.35 \times 10^{-2}\ \text{s} \\ \dot{T} &= 4.4 \times 10^{-13} \\ R &= 10^4\ \text{m} \end{aligned}\right\} \qquad (9.2.4)$$

We get

$$\dot{E}_{rot} = - 5.51 \times 10^{31} \, W . \qquad (9.2.5)$$

If this is to equal in magnitude

$$L_{GW} = \frac{32}{5} \frac{G}{c^5} \omega^6 (\varepsilon \Theta)^2 , \qquad (9.2.6)$$

where $\omega = 2\pi / T$ and

$$\Theta = I_1' + I_2' = \Theta_3' = \Theta'$$

as in (9.2.1), we must have

$$\left. \begin{aligned} &\left[5.51 \times 10^{31} \, W \Big/ \frac{32}{5} \frac{G}{c^5} \left(\frac{2\pi}{T} \right)^6 \Theta'^2 \right]^{1/2} \\ &= \varepsilon \end{aligned} \right\} (9.2.7)$$

Since

$$\Theta' = \frac{2}{5} M R^2 = \frac{2}{5} \cdot 2{,}98 \cdot 10^{30} \, kg \; 10^8 \, m^2$$

$$= 1.192 \; 10^{38} \, kg \, m^2 \qquad (9.2.8)$$

We have

$$\frac{32}{5} \frac{G}{c^5} \left(\frac{2\pi}{T} \right)^6 \Theta'^2 = 1.1 \cdot 10^{38} \, W \qquad (9.2.9)$$

and

$$\varepsilon = \left\{ \frac{5.51 \times 10^{31}}{1.1 \times 10^{38}} \right\}^{1/2} = 7.1 \times 10^{-4} \qquad (9.2.10)$$

The amplitude on Earth can be estimated by using 14.49

$$|h| = \frac{4G}{c^4} \frac{\omega^2 \varepsilon \Theta}{r} \qquad (9.2.11)$$

where $r = 2 \cdot 10^3$ pc

$$= 2 \cdot 10^3 \cdot 3.086 \times 10^{16} \text{ m}$$

$$= 6.17 \times 10^{19} \text{ m} \qquad (9.2.12)$$

$$\omega = \frac{2\pi}{T} = 1.88 \times 10^2 \text{ s}^{-1} \qquad (9.2.13)$$

Then

$$|h| = 1.6 \times 10^{-24} \qquad (9.2.14)$$

If we model the neutron star by a homogeneous ellipsoid

$$E(a,b,c) := \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} \qquad (9.2.15)$$

which we assume to rotate about the z-axis, then the 2nd mass-moments are not difficult to calculate. The volume of E(a,b,c) is

$$V = \int_{E(a,b,c)} d^3x = \int_{E(a,b,c)} dx \wedge dy \wedge dz$$

$$= a \cdot b \cdot c \int_{B_1(0)} dx' \wedge dy' \wedge dz'$$

$$= abc \int_{B_1(0)} d^3x' = abc \, \frac{4\pi}{3} \qquad (9.2.16)$$

where $\vec{x}' = (x', y', z') := \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$. (9.2.17)

Hence the mass-density is

$$\varrho = \frac{3M}{4\pi abc} \qquad (9.2.18)$$

The 2nd moments of the mass distribution are $I'_{ab} = \mathrm{diag}(I'_1, I'_2, I'_3)$ where

$$I'_1 = \varrho \int_{E(a,b,c)} x^2 d^3x = \varrho \, a^2 (a \cdot b \cdot c) \int_{B_1(0)} x'^2 d^3x'$$

$$= \varrho \, a^2 (a \cdot b \cdot c) \, 2\pi \int_0^1 r'^4 dr' \int_{-1}^1 d\sigma \, \sigma^2$$

↳ use polar coord. with x' as polar axis

$$= \rho \ a^2 \ (a \cdot b \cdot c) \ 2\pi \ \frac{1}{5} \ \frac{2}{3}$$

$$= \frac{M}{\cancel{\frac{4\pi}{3}} \cancel{a \cdot b \cdot c}} \cdot a^2 \cdot \cancel{a \cdot b \cdot c} \ \cancel{\frac{4\pi}{3}} \ \frac{1}{5}$$

$$= \frac{1}{5} \ M a^2 \qquad (9.2.19)$$

Correspondingly

$$I_2' = \frac{1}{5} M b^2 \qquad (9.2.20)$$

$$I_3' = \frac{1}{5} M c^2 \qquad (9.2.21)$$

$$\curvearrowright \quad \Theta = I_1' + I_2' = \frac{1}{5} M (a^2 + b^2) \qquad (9.2.22)$$

$$\varepsilon = (I_1' - I_2') / \Theta = \frac{a^2 - b^2}{a^2 + b^2} \qquad (9.2.23)$$

For $b = a + \delta a$ we get

$$\varepsilon = \frac{a^2 - (a + \delta a)^2}{a + (a + \delta a)^2} = - \frac{\delta a}{a} + O\left[\left(\frac{\delta a}{a}\right)^2\right] \qquad (9.2.24)$$

If the magnitude of ε is to equal (9.2.10), we have

$$\frac{|\delta a|}{a} = 7.1 \times 10^{-4} \qquad (9.2.25)$$

an with $a = 10$ km

$$|\delta a| = 7.1 \text{ m} \qquad (9.2.26)$$

Note: These 7.1 meters are the height of an "equatorial mountain" that would cause an energy loss of the crab-pulsar due to the emission of gravitational waves equal to its total observed energy loss of rotational spin-down. So if, say, we assumed a 1% contribution to that by GW-emission, we would need an ε of 1/10 the just computed value and hence a 70 cm-mountain. (Note L_{GW} is quadratic in ε and hence a 1/100-fold L_{GW} means a 1/10-fold ε.). So if we can exclude a, say, more than 1% contribution of L_{GW} to $\dot{T}$ of a pulsar, e.g. because we understand the other (electromagnetic) contributions to $\dot{T}$, we can limit the height of mountains to, say, below meter scale.

Problem 3

$$E_{rot} \sim \omega^2 \quad (9.3.1)$$

$$\dot{E}_{rot} \sim \omega \dot{\omega} \quad (9.3.2)$$

If $\dot{E}_{rot} \sim \omega^{n+1}$ due to some radiation dissipation, have

$$\omega \dot{\omega} \sim \omega^{n+1} \quad \text{i.e.}$$

$$\dot{\omega} = a \, \omega^n \quad (9.3.3)$$

for some constant a.

The "braking index" in generally defined by

$$n := \frac{\omega \ddot{\omega}}{\dot{\omega}^2} \quad (9.3.4)$$

which coincides with our notation above. Indeed, from (9.3.3) get

$$\ddot{\omega} = a \, n \, \omega^{n-1} \dot{\omega}$$
$$= a^2 n \, \omega^{2n-1}$$

$$\rightsquigarrow \quad \frac{\omega \ddot{\omega}}{\dot{\omega}^2} = \frac{a^2 n \, \omega^{2n}}{a^2 \omega^{2n}} = n$$

For $\omega = \frac{2\pi}{T}$ have

$$\dot{\omega} = -\frac{2\pi}{T^2}\dot{T}$$

$$\ddot{\omega} = 2\pi\left(-\frac{\ddot{T}}{T^2} + 2\frac{\dot{T}^2}{T^3}\right)$$

$$\curvearrowright \quad n = \frac{\omega\ddot{\omega}}{\dot{\omega}^2} = \frac{-\frac{\ddot{T}}{T^3} + 2\frac{\dot{T}^2}{T^4}}{\dot{T}^2 / T^4}$$

$$= 2 - \frac{T\ddot{T}}{\dot{T}^2}$$

or

$$\frac{T\ddot{T}}{\dot{T}^2} = 2 - n$$

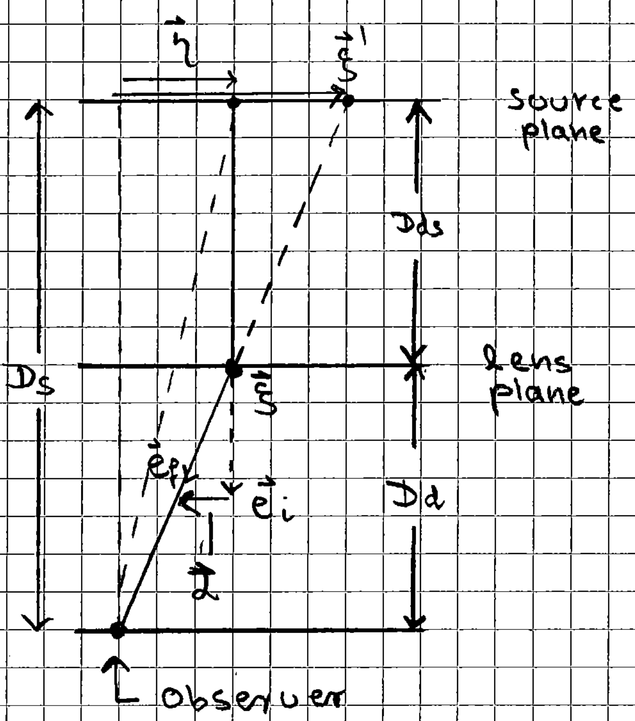
Observing the second derivatives $\ddot{T}$ or $\ddot{\omega}$ would then discriminate between electromagnetic dipole radiation

$$L_{\text{mag-dipole}} = \frac{1}{3}\frac{\mu_0}{4\pi}\frac{1}{c^3}\|\ddot{m}\|^2 \sim \omega^4$$

$$L_{GW} = \frac{1}{5}G \cdot \frac{1}{c^5}\|\dddot{Q}\|^2 \sim \omega^6$$

So braking index $= 3$ for mag. dipole and $n = 5$ for GW. Braking indices can and have been measured on young pulsars; - the ranges are huge (up to almost 3000).
See arXiv: 2003.13303 (astro-ph.HE).

Problem 4



(9.4.1)

From the picture we immediately read off:

$$\vec{\xi}' = \frac{D_s}{D_d}\,\vec{\xi} \tag{9.4.2}$$

$$\vec{\eta} = \vec{\xi}' + (\vec{\eta} - \vec{\xi}') \tag{9.4.3}$$

$$\vec{\eta} - \vec{\xi}' = D_{ds}\,\vec{\alpha} \tag{9.4.4}$$

Hence $$\vec{\eta}(\vec{\xi}) = \frac{D_s}{D_d}\,\vec{\xi} + D_{ds}\,\vec{\alpha}(\vec{\xi}) \tag{9.4.5}$$

Note that this is only valid in leading-order approximation for small $\alpha := \|\vec{\alpha}\|$.

Let ξ_0 be a scale on the lens-plane and

$$\eta_0 = \frac{D_s}{D_d}\,\xi_0 \tag{9.4.6}$$

be the corresponding scale on the source plane. We set

$$\left.\begin{aligned} \vec{x} &= \vec{\xi}/\xi_0 \\ \vec{y} &= \vec{\eta}/\eta_0 \end{aligned}\right\} \tag{9.4.7}$$

and rewrite (9.4.5) as

$$\vec{y}(\vec{x}) = \vec{x} + \frac{D_{ds}D_d}{D_s\,\xi_0}\,\vec{\alpha}(\vec{\xi}). \tag{9.4.8}$$

Using

$$\vec{\alpha}(\vec{\xi}) = -\frac{4G}{c^2}\int_{\mathbb{R}^2} d^2\xi'\,\Sigma(\vec{\xi}')\,\frac{\vec{\xi}-\vec{\xi}'}{\|\vec{\xi}-\vec{\xi}'\|^2}$$

$$= -\frac{4G}{c^2}\,\xi_0\int_{\mathbb{R}^2} d^2x'\,\Sigma(\xi_0\vec{x}')\,\frac{\vec{x}-\vec{x}'}{\|\vec{x}-\vec{x}'\|^2} \tag{9.4.9}$$

this becomes

$$\vec{y}(\vec{x}) = \vec{x} - \frac{4G}{c^2}\,\frac{D_{ds}D_d}{D_s}\int_{\mathbb{R}^2} d^2x'\,\Sigma(\xi_0\vec{x}')\,\frac{\vec{x}-\vec{x}'}{\|\vec{x}-\vec{x}'\|^2}$$

(9.4.10)

Since

$$\vec{\nabla}_x \ln(\|\vec{x}-\vec{x}'\|) = \frac{1}{\|\vec{x}-\vec{x}'\|}\frac{\vec{x}-\vec{x}'}{\|\vec{x}-\vec{x}'\|} = \frac{\vec{x}-\vec{x}'}{\|\vec{x}-\vec{x}'\|^2} \qquad (9.4.11)$$

We have

$$\left.\begin{aligned}
\vec{y}(\vec{x}) &= \vec{\nabla}\varphi(\vec{x}) \\
\varphi(\vec{x}) &= \tfrac{1}{2}\|\vec{x}\|^2 - \psi(\vec{x}) \\
\psi(\vec{x}) &= \frac{1}{\pi}\int_{\mathbb{R}^2} K(\vec{x}')\ln(\|\vec{x}-\vec{x}'\|)\, d^2x' \\
K(\vec{x}) &= \frac{4\pi G}{c^2}\frac{D_d D_{ds}}{D_s}\Sigma(\xi_0\vec{x})
\end{aligned}\right\} \quad (9.4.12)$$

The trace of the Hessian of $\psi(\vec{x})$ is just the Laplacian. Now, just as we have

$$\Delta \frac{1}{\|\vec{x}-\vec{x}'\|} = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}') \qquad (9.4.13)$$

in $\mathbb{R}^3$, we have in $\mathbb{R}^2$:

$$\Delta \ln(\|\vec{x}-\vec{x}'\|) = 2\pi\,\delta^{(2)}(\vec{x}-\vec{x}') \qquad (9.4.14)$$

The proof is entirely analogous to that in 3-dimensions; see below.

Indeed, if we use polar coordinates based at $\vec{x}'$, $\|\vec{x}-\vec{x}'\| = r$, we have

$$\left.\begin{aligned} &\Delta \ln(\|\vec{x}-\vec{x}'\|) = \Delta \ln r = \\ &= \frac{1}{r}\partial_r(r\,\partial_r \ln r) \\ &= 0 \quad \text{for } r \neq 0 \end{aligned}\right\} \quad (9.4.15)$$

Now, if $f \in C_c^{\infty}(\mathbb{R}^2, \mathbb{R})$ is a test function with compact support inside disk $D_R \subset \mathbb{R}^2$ of radius R, then

$$\langle f, \Delta \ln(r)\rangle = \langle \Delta f, \ln(r)\rangle$$

$$= \lim_{\varepsilon\to 0} \int_{\varepsilon < \|\vec{x}\| < R} d^2x \; \ln(r)\, \Delta f(\vec{x})$$

$$= \lim_{\varepsilon \to 0} \Big\{ \int d^2x \; \Delta \cancel{\ln(r)}^{\,0} \; f(\vec{x})$$

$$+ \int_{r=\varepsilon} (\vec{n}\cdot\vec{\nabla}) f \; \underbrace{\ln(r)\; r}_{\to 0 \text{ for } \varepsilon \to 0} d\varphi$$

$$- \int_{r=\varepsilon} \Big(\frac{\vec{x}}{r^2} \cdot \vec{n}\Big) f \; r\, d\varphi \Big\}$$

$\vec{n}$ $\downarrow$ outward pointing normal of integration domain $= -\vec{x}/r$ at $r = \varepsilon$

$$= 2\pi f(0). \quad (9.4.16)$$

Hence $\Delta \varphi = 2K$ (9.4.17)

The Jacobian of the map $\vec{x} \to \vec{y}(\vec{x})$ is

$$\frac{\partial y^a}{\partial x^b} = \partial_a \partial_b \varphi = \delta_{ab} - \partial_a \partial_b \psi \qquad (9.4.18)$$

which for small $\vec{\alpha}$ deviates from the identity only by the small quantity $\partial_a \partial_b \psi$, the Hessian of ψ. The 2 dim volume forms transform as

$$d^2 y = \det\left\{\frac{\partial y^a}{\partial x^b}\right\} d^2 x$$

$$\cong (1 - \Delta\psi)\, d^2 x$$

$$= (1 - 2K)\, d^2 x \qquad (9.4.19)$$

$\Rightarrow$ Outside supp$\{K\}$ the lens-map is area preserving (to leading order).

Problem 5

If $\quad \Sigma(\vec{\xi}) = m\, \delta^{(2)}(\vec{\xi}) \qquad (9.5.1)$

then

$$\psi(\vec{x}) = \frac{1}{\pi} \int_{\mathbb{R}^2} \ln(\|\vec{x} - \vec{x}'\|)$$

$$\times \frac{4\pi G}{c^2} \frac{D_d D_{ds}}{D_s} m\, \delta^{(2)}(\xi_0 \vec{x}')\, d^2x'$$

$$= \frac{4G}{c^2} m \frac{D_d D_{ds}}{D_s\, \xi_0^2} \ln(\|\vec{x}\|)$$

$$= \ln(\|\vec{x}\|) \qquad (9.5.2)$$

if we choose

$$\xi_0 = \left(2 \cdot \frac{2Gm}{c^2} \frac{D_d D_{ds}}{D_s}\right)^{1/2} \qquad (9.5.3)$$

Then

$$\vec{y}(\vec{x}) = \vec{x} - \vec{\nabla} \ln(\|\vec{x}\|)$$

$$= \vec{x} - \frac{\vec{x}}{\|\vec{x}\|^2}$$

$$= \vec{x}\left(1 - \|\vec{x}\|^{-2}\right) \qquad (9.5.4)$$

The map consists of two parts

$$\left.\begin{aligned} \vec{y}(\vec{x}) &= \vec{y}_1(\vec{x}) + \vec{y}_2(\vec{x}) \\ \vec{y}_1(\vec{x}) &= \vec{x} \\ y_2(\vec{x}) &= -\vec{x} / \|x\|^2 \end{aligned}\right\} \quad (9.5.5)$$

Hence it deviates from the identity $\vec{y}_1$ by a radial inward focussing $\vec{y}_2$, which is however area-preserving

$$\vec{\nabla}\vec{y}_2(\vec{x}) = -\frac{2}{\|x\|^2} + 2\,\frac{\vec{x}}{\|x\|^3}\,\frac{\vec{x}}{\|x\|} = 0. \quad (9.5.6)$$

A point $\vec{y}$ on the source plane corresponds to one – or many – pre-image points $\vec{x}$ the lens plane. Write $r = \|\vec{x}\|$; then

$$\vec{y} = \vec{x}(1 - r^{-2}) \quad (9.5.7)$$

$$\rightsquigarrow \quad \|\vec{y}\| = r - r^{-1}$$

$$\rightsquigarrow \quad r^2 - r\|\vec{y}\| - 1 = 0$$

$$\Rightarrow \quad r_{1,2} = \frac{\|\vec{y}\|}{2} \pm \left(\frac{\|\vec{y}\|^2}{4} + 1\right)^{1/2}$$

only positive value of r counts

$$\|\vec{x}\| = r = \frac{\|\vec{y}\|}{2} + \left(\frac{\|\vec{y}\|^2}{4} + 1\right)^{1/2} \quad (9.5.8)$$

If $r \neq 1$ then $(1 - r^{-2}) \neq 0$ and we can invert (9.5.7)

$$\vec{x} = \frac{\vec{y}}{1 - r^{-2}} = \frac{r\,\vec{y}}{r - r^{-1}}$$

$$= r \frac{\vec{y}}{\|\vec{y}\|}$$

$$= \vec{y}\left[\frac{1}{2} + \left(\frac{1}{4} + \|\vec{y}\|^{-1}\right)^{1/2}\right] \quad (9.5.9)$$

An exception are the points on the circle $r = 1$, all of which are the preimages of $\vec{y} = 0$. That means that the point object at $\vec{y} = 0$ is seen in the sky smeared over the circle

$$\|\vec{x}\| = 1 \quad \Leftrightarrow$$

$$\left.\begin{aligned} \|\vec{\xi}\| = R_E &:= \left(2 \cdot \frac{2Gm}{c^2} \frac{D_d D_{ds}}{D_s}\right)^{1/2} \\ &= 610\, R_\odot \left[\frac{m}{M_\odot} \frac{D_s}{\text{kpc}} \frac{D_d}{D_s}\left(1 - \frac{D_d}{D_s}\right)\right]^{1/2} \end{aligned}\right\} \quad (9.5.10)$$

solar radius and mass.

R_E is called the <u>Einstein radius.</u>

Problem 6

$$\nabla^{\mu} F_{\mu\nu} = \nabla^{\mu} (\nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu})$$

$$= \nabla^{\mu} \nabla_{\mu} A_{\nu} - \nabla^{\mu} \nabla_{\nu} A_{\mu} \qquad (9.6.1)$$

Have

$$\nabla_{\alpha} \nabla_{\beta} A_{\mu} = \nabla_{\beta} \nabla_{\alpha} A_{\mu} - R^{\lambda}{}_{\mu\alpha\beta} A_{\lambda} \qquad (9.6.2)$$

hence

$$\nabla^{\mu} \nabla_{\nu} A_{\mu} = \nabla_{\nu} \nabla^{\mu} A_{\mu} - R^{\lambda}{}_{\mu}{}^{\mu}{}_{\nu} A_{\lambda}$$

$$= \nabla_{\nu} \nabla^{\mu} A_{\mu} + R^{\lambda}{}_{\nu} A_{\lambda} \qquad (9.6.3)$$

$$\rightsquigarrow \quad \nabla^{\mu} F_{\mu\nu} = \Box A_{\nu} - \nabla_{\nu} \nabla^{\mu} A_{\mu} - R^{\lambda}{}_{\nu} A_{\lambda} \qquad (9.6.4)$$

Imposing covariant Lorentz gauge

$$\nabla^{\lambda} A_{\lambda} = 0 \qquad (9.6.5)$$

leaves us with

$$\Box A_{\mu} - R^{\lambda}{}_{\mu} A_{\lambda} = 0 \qquad (9.6.6)$$

Set

$$A_\mu = (a_\mu + \varepsilon b_\mu + O(2)) \exp(\tfrac{i}{\varepsilon}\psi) \qquad (9.6.7)$$

with

$$k_\lambda := \nabla_\lambda \psi \qquad (9.6.8)$$

$$a := |g^{\mu\nu} a_\mu a_\nu|^{1/2} \qquad (9.6.9)$$

$$f_\mu := a_\mu / a \qquad (9.6.10)$$

For any function of the form

$$f = g \exp(\tfrac{i}{\varepsilon}\psi) \qquad (9.6.11)$$

have

$$\nabla_\mu f = \left[(\nabla_\mu + \tfrac{i}{\varepsilon} k_\mu) g\right] \exp(\tfrac{i}{\varepsilon}\psi) \qquad (9.6.12)$$

in particular:

$$\nabla_\lambda A_\mu = \Big[\tfrac{i}{\varepsilon} k_\lambda a_\mu + \nabla_\lambda a_\mu + i k_\lambda b_\mu + \text{Terms} \sim \varepsilon^{n \geq 1} \Big] \exp(\cdot\cdot) \qquad (9.6.13)$$

$$\rightsquigarrow \quad \nabla^\mu A_\mu = \Big[\frac{i}{\varepsilon} k^\lambda a_\lambda + \nabla^\lambda a_\lambda + i k^\lambda b_\lambda$$

$$+ \text{Terms } \varepsilon^{n \geq 1} \Big] \exp(\cdot\cdot) \qquad (9.6.14)$$

$$\nabla^\lambda \nabla_\lambda A_\mu = \Big[(\nabla^\lambda + \frac{i}{\varepsilon} k^\lambda) \big(\frac{i}{\varepsilon} k_\lambda a_\mu + \nabla_\lambda a_\mu + i k_\lambda b_\mu$$

$$+ \text{Terms} \sim \varepsilon^{n \geq 1} \big) \Big] \exp(\cdot\cdot)$$

$$(9.6.15)$$

$$= \Big\{ \varepsilon^{-2} (- k^\lambda k_\lambda a_\mu)$$

$$+ \varepsilon^{-1} i \big(\nabla^\lambda (k_\lambda a_\mu) + k^\lambda \nabla_\lambda a_\mu + i k^\lambda k_\lambda b_\mu \big)$$

$$+ \text{Terms} \sim \varepsilon^{n \geq 0} \Big\} \exp\big(\frac{i}{\varepsilon} \varphi\big) \qquad (9.6.16)$$

Hence get from ε^{-2} and ε^{-1} terms of (9.6.14)

ε^{-2} : no condition

ε^{-1} : $k^\lambda a_\lambda = 0$ $\qquad$ (9.6.17)

$\rightsquigarrow$ $a \perp$ to lightlike vector (see below)

$\Rightarrow \quad g(k,k) \leq 0$ $\qquad$ (9.6.18)

And from (9.6.16):

$$\varepsilon^{-2}: \quad k^\lambda k_\lambda = 0 \qquad (9.6.19)$$

$$\varepsilon^{-1}: \quad \nabla^\lambda (k_\lambda a_\mu) + k^\lambda \nabla_\lambda a_\mu$$
$$+ i\, k^\lambda k_\lambda b_\mu = 0 \qquad (9.6.20)$$

Using (9.6.19)

$$2 k^\lambda \nabla_\lambda a_\mu + a_\mu \nabla^\lambda k_\lambda = 0$$

or

$$k^\lambda \nabla_\lambda a_\mu = -\frac{1}{2} a_\mu \nabla^\lambda k_\lambda \qquad (9.6.21)$$

Since $k_\alpha = \nabla_\alpha \psi$ have

$$\nabla_\alpha k_\beta = \nabla_\alpha \nabla_\beta \psi = \nabla_\beta \nabla_\alpha \psi$$
$$= \nabla_\beta k_\alpha \qquad (9.6.22)$$

Hence, from (9.6.19)

$$0 = \nabla_\nu (k^\mu k_\mu) = 2 k^\mu \nabla_\nu k_\mu$$
$$= 2 k^\mu \nabla_\mu k_\nu$$

which tells us that any lightlike gradient vector field $k = \nabla \psi$ is geodesic:

$$\nabla_k k = 0 \qquad (9.6.23)$$

In fact: Any lightlike, hypersurface orthogonal vector field k is auto-parallel; i.e.

$$k \cdot k = 0 \quad \text{and}$$

$$k^{\downarrow} \wedge dk^{\downarrow} = 0 \qquad (9.6.24)$$

implies $\quad \nabla_k k \sim k \qquad (9.6.25)$

Indeed, (9.6.24) is equivalent to

$$k_\alpha (\nabla_\beta k_\gamma - \nabla_\gamma k_\beta)$$

$$+ k_\beta (\nabla_\gamma k_\alpha - \nabla_\alpha k_\gamma)$$

$$+ k_\gamma (\nabla_\alpha k_\beta - \nabla_\beta k_\alpha) = 0 \qquad (9.6.26)$$

Multiplikation with k^β gives, because of $k^\beta k_\beta = 0$ and $k^\beta \nabla k_\beta = 0$

$$k_\alpha k^\beta \nabla_\beta k_\gamma - k_\gamma k^\beta \nabla_\beta k_\alpha = 0$$

or $$k^{\downarrow} \wedge (\nabla_k k)^{\downarrow} = 0 \qquad (9.6.27)$$

$\Longleftrightarrow \quad \nabla_k k \sim k$

For the amplitude we have (9.6.21), i.e.

$$\nabla_k a = -\tfrac{1}{2} a (\nabla \cdot k) \qquad (9.6.28)$$

Hence

$$\nabla_k (g(a,a)) = 2 g(\nabla_k a, a)$$

$$= -(\nabla \cdot k)\, g(a,a) \qquad (9.6.29)$$

Hence for $|a| := \sqrt{-g(a,a)}$

$$\nabla_k |a| = \frac{1}{2|a|} \nabla_k (-g(a,a))$$

$$= -(\nabla \cdot k)(-g(a,a)) / 2|a|$$

$$= -\frac{1}{2} (\nabla \cdot k)\, |a| \qquad (9.6.30)$$

Finally, for the polarization vector

$f = a / |a|$ have

$$\nabla_k f = \frac{1}{|a|} \nabla_k a - \frac{a}{|a|^2} \nabla_k |a|$$

$$= \frac{1}{|a|}\left(-\frac{1}{2} a (\nabla \cdot k)\right) - \frac{a}{|a|^2}\left(-\frac{1}{2}(\nabla \cdot k)\right)|a|$$

$$= 0 \qquad (9.6.31)$$